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First and Second Laws of Information Processing by Nonequilibrium Dynamical States

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The averaged steady-state surprisal links a driven stochastic system's information processing to its nonequilibrium thermodynamic response. By explicitly accounting for the effects of nonequilibrium steady states, a decomposition of the surprisal results in an information processing First Law that extends and tightens—to strict equalities—various information processing Second Laws. Applying stochastic thermodynamics' integral fluctuation theorems then shows that the decomposition reduces to the Second Laws under appropriate limits. In unifying them, the First Law paves the way to identifying the mechanisms by which nonequilibrium steady-state systems leverage information-bearing degrees of freedom to extract heat. To illustrate, we analyze an autonomous Maxwellian information ratchet that tunably violates detailed balance in its effective dynamics. This demonstrates how the presence of nonequilibrium steady states qualitatively alters an information engine's allowed functionality.

I. INTRODUCTION

In 1861, Maxwell introduced a thought experiment in which a "very neat-fingered being" leveraged observations to control a system that violated the Second Law of thermodynamics [1]. A century later, attempting to resolve the paradox, Landauer quantitatively bounded the requisite thermodynamic resources for erasing a single bit of information in a physical information-bearing degree of freedom [2]. These results have since stimulated many explorations of the fundamental physics tying a system's thermodynamic behavior to its functioning as an information processor [3].

One particular line of inquiry focused on *autonomous Maxwellian ratchets*. In this, a ratchet embedded in a thermal environment moves along an *information tape*, interacting with a single tape symbol at a time. The information in the tape's cells modifies the ratchet's statistical properties while the ratchet absorbs and dissipates energy [4]. Recent results introduced an *information processing Second Law* (IPSL) for such systems that bounds the asymptotic rate $\langle \dot{W} \rangle$ of extracted work [5]:

$$\beta \langle \dot{W} \rangle \leq \Delta h_\mu, \tag{1}$$

where $\Delta h_\mu = h'_\mu - h_\mu$, h'_μ is the Shannon entropy rate of the statistical process generating the output tape, h_μ is the same for the input tape, and β is the inverse temperature of the thermal environment.

Notably, the IPSL bound corrected previous "single-symbol" relations by accounting explicitly for arbitrary-order temporal correlations in the input and output symbol strings. This, then, led to the discovery that removing such correlation increases the system's capacity to produce work—despite the ratchet interacting with only a single symbol at a time.

More recently, Ref. [6] developed a similar IPSL not for the asymptotic rate of extracted work from an infinite tape, but for the finite-time ensemble-averaged work extracted when operating on a finite tape:

$$\beta \langle W \rangle \leq \Delta H[Z], \tag{2}$$

where Z is the random variable associated with the joint space of the ratchet and tape and $H[Z]$ is its Shannon entropy.

The following first derives a simple information-thermodynamic equality by considering the averaged steady-state surprisal of a general driven stochastic process:

$$\Delta H[Z] = \langle \mathcal{W}_{\mathrm{ex}} \rangle - \langle \mathcal{Q}_{\mathrm{ex}} \rangle - \Delta D_{\mathrm{KL}}[Z \,\|\, \Lambda]. \tag{3}$$

Here, Z is the random variable associated with a system's state, Λ that associated with an environmentally-induced steady state, and $\langle \mathcal{W}_{\mathrm{ex}} \rangle$ and $\langle \mathcal{Q}_{\mathrm{ex}} \rangle$ are (in entropic units) the average excess work and heat of nonequilibrium steady state thermodynamics, respectively [7, 8]. The Kullback-Leibler divergence [9] $D_{\mathrm{KL}}[Z \,\|\, \Lambda]$ monitors the difference in information between the system's state and the would-be steady state.

We refer to Eq. (3) as the *information processing First Law* (IPFL) since, beyond the obvious change in the joint system's information content, the lefthand side acts as a kernel for describing a ratchet's information processing—discussed in detail in Sec. IV. Additionally, the righthand side expresses a generalized First Law used to define excess heat and work—discussed further in Secs. II A and II D. In essence, Eq. (3) expresses a First Law for the system's information content in the same way the original equilibrium First Law does for a system's energy (non)conservation.

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Subsequently, we show that the IPFL together with stochastic thermodynamics' integral fluctuation theorems—particularly those presented in Refs. [8, 10]—generalize and modify the two preceding asymptotic and finite-tape IPSLs. Identifying the role of the average dissipated housekeeping heat $\langle Q_{\mathrm{hk}} \rangle$ and the divergence from final steady-state conditions $D_{\mathrm{KL}}[Z_N \,\|\, \Lambda_N]$, it shows that for finite-tape systems:

$$-\beta \langle Q \rangle \leq \Delta H[Z] + D_{\mathrm{KL}}[Z_N \,\|\, \Lambda_N] - \beta \langle Q_{\mathrm{hk}} \rangle \,, \quad (4)$$

where $-\langle Q \rangle$ corresponds to the averaged heat extracted from the thermal environment.

With *equilibrium steady states* (ESSs), this is the work W done by the ratchet-tape system: $-\langle Q \rangle = W$, recalling previous bounds. However, as Sec. II E discusses, invoking this equivalence is not generally possible in the case of *nonequilibrium steady states* (NESSs). Instead, we give the bounds in terms of heat extracted from the thermal environment.

Compared to the extracted-heat form of Eq. (2), which gives $-\beta \langle Q \rangle \leq \Delta H[Z]$, this explicit accounting for the effects of NESSs and nonequilibrium *dynamical* (nonsteady) start and end configurations gives a strictly tighter bound for finite- and even-state (defined below) systems.

In short, for NESS systems an increase in randomness—as measured by $\Delta H[Z]$—must additionally compensate for persistent housekeeping costs—as measured by $\beta \langle Q_{\mathrm{hk}} \rangle$—to leverage the thermal environment as a reservoir of extractable energy.

Finally, we demonstrate that for infinite-tape, finite-ratchet systems the asymptotic bound is similarly tightened:

$$-\beta \langle \dot{Q} \rangle \leq \Delta h_\mu - \beta \langle \dot{Q_{\mathrm{hk}}} \rangle, \quad (5)$$

where $\beta \langle \dot{Q_{\mathrm{hk}}} \rangle$ is the asymptotic rate of housekeeping dissipation.

To summarize, fluctuation theorems take the IPFL directly to a suite of simultaneously-true Second Laws for information processing. This, once again, mirrors informational generalization of the familiar equilibrium Second Law.

Overall, this clarifies and unifies derivations of these IPSLs. More importantly, it extends their domains to explicitly include the effects of initial- and final-state dependence as well as nonequilibrium steady states. Practically, this opens the door to considering detailed information-energy tradeoffs for systems that arbitrarily violate detailed balance in their effective dynamics. The following demonstrates this via an example ratchet designed to tunably violate detailed balance while remaining tractable for analysis. This uncovers qualitative corrections to a ratchet's ability to extract heat from its environment.

The development proceeds as follows. First, Sec. II sets out the preliminary notation and introduces the relevant stochastic dynamical functionals. Section II D maps the general stochastic dynamical picture to an explicitly thermodynamic one, immersed in a single-temperature environment. This points toward concrete example realizations of the general stochastic theory. Section II E reviews autonomous Maxwellian information ratchets, which comprise our example system class.

With the preliminaries in hand, Sec. III derives Eq. (3)'s IPFL. It applies the IPFL to the information ratchet picture, revealing strict equalities relating a ratchet's thermodynamic dissipation with its information processing in transforming an input tape to an output.

Section IV then specializes the IPFL in two ways. First, Sec. IV A introduces and uses integral fluctuation theorems to take the equality to an inequality. This arrives at the kernel of previous IPSLs, explicitly generalizing and tightening that of Ref. [6]. Then, Sec. IV B considers the asymptotic rate limit of an infinite tape, similarly generalizing the previous asymptotic IPSL to include the effects of nonequilibrium dynamical state-dependence and potentially infinite-state ratchets. The restriction to finite ratchets in Sec. IV C rounds out our derivations, revealing a simple correction tightening Ref. [5]'s asymptotic IPSL.

Finally, Sec. V applies the developed theory to a finite-state information ratchet that arbitrarily violates detailed balance and so exhibits NESSs. We find that even for simple cases, NESSs have dramatic effects on a ratchet's ability to extract heat, qualitatively changing its landscape of allowed behaviors.

Taken together, the results (i) unify both previously-reported IPSLs for ratchets by deriving them explicitly from the underlying IPFL and integral fluctuation theorems, (ii) place the specific application of autonomous ratchet function in the broader context of the exchange between energy and information in complex systems, including generally nondetailed-balanced ratchets, and (iii) demonstrate severe restrictions nonequilibrium dynamical states place on allowed ratchet functionality—restrictions critical to understanding the thermodynamics of information processing by complex systems.

II. PRELIMINARIES

Consider a system under study (SUS) that stochastically realizes states z in a countable space $\mathcal{Z}$. It is driven in discrete time by a *protocol* written as a sequence of parameter values λ in a parameter space $\mathcal{A}$, denoted by $\lambda_{0:N} \doteq \lambda_0 \lambda_1 \ldots \lambda_N$ for a positive integer N. The resulting driven stochastic process $Z_{0:N}$ is not stationary. However, we assume it is *conditionally stationary*: for any protocol indefinitely fixed at λ, there is a unique corresponding stationary state distribution $\boldsymbol{\pi}_\lambda$.

Initially, we place no further restrictions on our system: it need not have a particular dynamical structure—Markov, master equation, Langevin, detailed balanced, and so on. And, we make no claim about the scale of its state space or time scale. The protocol itself may be a realization of a separate stochastic process, and the state space may be a joint one with meaningfully decomposable parts. In point of fact, we treat the latter as an example later. First, though, we derive our main result in greater generality, requiring only the conditional stationarity assumption and involving only the functionals of trajectory-protocol pairs we now define.

A. Dynamical Functionals

With a would-be stationary distribution $\boldsymbol{\pi}_\lambda$ associated to each driving parameter λ, denote its elements by $\pi_\lambda(z)$. Without loss of generality we define the *steady-state surprisal* [7, 8, 10, 11]:

$$\phi_\lambda(z) \doteq -\ln \pi_\lambda(z), \tag{6}$$

so called as it is the Shannon self-information [9] of the system state being z under the distribution $\boldsymbol{\pi}_\lambda$. Hereafter, we take all logarithms to the natural base and, following information-theoretic convention, refer to the unit of entropy and surprisal as a *nat* [12].

For notational uniformity, we cast the sequence of stationary distributions during a protocol as a stochastic process over random variables $\Lambda_t \sim \boldsymbol{\pi}_{\lambda_t}$. ("$\sim$" reads "distributed as" [9].) Upon averaging:

$$\begin{aligned} \langle \boldsymbol{\pi}_\lambda | \boldsymbol{\phi}_\lambda \rangle &\doteq \sum_{z \in \mathcal{Z}} \pi_\lambda(z)\, \phi_\lambda(z) \\ &= H[\Lambda], \end{aligned} \tag{7}$$

the Shannon entropy of the distribution $\boldsymbol{\pi}_\lambda$.

We now define *stochastic excess work* $\mathcal{W}_{\mathrm{ex}}$ and *stochastic excess heat* $\mathcal{Q}_{\mathrm{ex}}$ as distinct contributions to a system's change in steady-state surprisal:

$$\begin{aligned} \Delta\phi &= \underbrace{\Delta_\lambda \phi} + \underbrace{\Delta_z \phi} \\ &\doteq \;\mathcal{W}_{\mathrm{ex}} - \;\mathcal{Q}_{\mathrm{ex}}, \end{aligned} \tag{8}$$

where, for N time steps:

$$\Delta_\lambda \phi \doteq \sum_{n=0}^{N-1} \phi_{\lambda_{n+1}}(z_n) - \phi_{\lambda_n}(z_n) \qquad \text{and} \tag{9}$$

$$\Delta_z \phi \doteq \sum_{n=0}^{N-1} \phi_{\lambda_{n+1}}(z_{n+1}) - \phi_{\lambda_{n+1}}(z_n). \tag{10}$$

That is, by stochastic excess work $\mathcal{W}_{\mathrm{ex}}$ we refer to the change in steady-state surprisal owing to a changing environmental drive. And, by stochastic excess heat $\mathcal{Q}_{\mathrm{ex}}$ we identify the change in steady-state surprisal owing to the system's state change—its response or *adaptation* to environmental conditions.

Next, we denote the *conditional path irreversibility* by $\mathcal{Q}$:

$$\mathcal{Q} \doteq \ln \frac{\Pr\left(Z_{1:N} = z_{1:N} \quad \middle| \; Z_0 = z_0 \;\; ; \lambda_{0:N}\right)}{\Pr\left(Z_{1:N} = \tilde{z}_{N-1:0} \;\middle|\; Z_0 = \tilde{z}_N \;\; ; \tilde{\lambda}_{N:0}\right)}, \tag{11}$$

where the tilde indicates negation of odd-parity variables, such as momentum and magnetic field.

The stochastic excess heat $\mathcal{Q}_{\mathrm{ex}}$ can be viewed as a piece of this path irreversibility—that associated with changes in the steady-state surprisal. What remains we term the *stochastic housekeeping heat* $\mathcal{Q}_{\mathrm{hk}}$:

$$\mathcal{Q}_{\mathrm{hk}} \doteq \mathcal{Q} - \mathcal{Q}_{\mathrm{ex}}. \tag{12}$$

In restricted cases it carries additional interpretation: if the stochastic dynamics are Markov (order 1) *and* the state and protocol variables are even (requiring no negation in the denominator), then $\mathcal{Q}_{\mathrm{hk}}$ measures *detailed balance violation* in the stochastic dynamics. If the state or control variables are odd, but we retain the Markov condition, then a *part* of $\mathcal{Q}_{\mathrm{hk}}$ measures detailed balance violation—see Refs. [13–15] for more on this breakdown. As stated, however, $\mathcal{Q}_{\mathrm{hk}}$ requires neither Markov dynamics nor even state and protocol variables, and we must take care not to over-interpret. In this general setting, it is simply that portion of a particular trajectory's irreversibility—conditioned on initial and final configurations—that is not attributable to changes in the system's steady-state surprisal along its forward path.

B. Nonequilibrium Dynamical States

In the special case where the system is Markovian (order 1) and subject to an indefinitely fixed drive—yielding a stationary Markov process—the rate of housekeeping heat takes the same form as that of asymptotic entropy production, familiar in stochastic thermodynamics [16]. The latter on average is sometimes taken to measure the system's fundamental time-reversal asymmetry [17, 18].

However, the following considers the more general case of systems that have not yet reached their steady-state distributions—processes that are not stationary. While Eqs. (8)–(12) leverage a suite of "would-be" stationary distributions, they are defined for arbitrary paths, including when the system is nowhere near such a steady state at any stage of the observed interval. We call such transient state distributions $\boldsymbol{\mu}_t \not\approx \boldsymbol{\pi}_{\lambda_t}$ *nonequilibrium dynamical states* (NEDSs). In treating system trajectories that begin and/or end in NEDSs, a final term appearing in our derivations and related results remains: the *nonsteady-state addition to stochastic free energy*:

$$\mathcal{F}^{\mathrm{nss}}_{\boldsymbol{\mu}\|\lambda}(z) \doteq \ln \frac{\mu(z)}{\pi_\lambda(z)}. \tag{13}$$

Like ϕ, $\mathcal{W}_{\mathrm{ex}}$, $\mathcal{Q}_{\mathrm{ex}}$, $\mathcal{Q}$, and $\mathcal{Q}_{\mathrm{hk}}$, this nonsteady-state addition to stochastic free energy's definition recalls connections to stochastic-thermodynamic energies and entropies of interest. We have been careful thus far to avoid over-interpretation in this vein, instead electing to treat these quantities as stochastic-dynamically meaningful in their own right. In the next section, however, we select a particular thermodynamic scheme and explicitly map the preceding functionals to their thermodynamically-meaningful counterparts.

C. Thermodynamic Scheme

To place physical constraints on our stochastically-evolving system of interest and to aid interpretation, we now embed it in an isothermal environment at inverse temperature $\beta \doteq k_{\mathrm{B}}T$, with k_{B} Boltzmann's constant, and connect it to two ideal [19] work reservoirs—one parameterized by λ that couples with the stochastic dynamics and one labeled an *auxiliary reservoir*, representing otherwise unaccounted-for degrees of freedom and providing for nonequilibrium steady states [8]. See Fig. 1 for an illustration. Additionally, we require that the underlying system dynamics are Markovian—or at least that there exists a Markov chain representation of the process for each λ, with which we can calculate the preceding functionals [20].

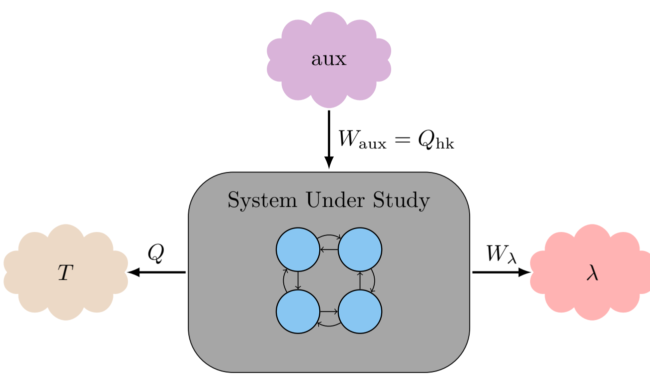

FIG. 1. The thermodynamic scheme considered: the Markovian stochastic system under study is coupled to an ideal heat bath, an ideal work reservoir parameterized by λ, and an auxiliary reservoir that accounts for maintaining nonequilibrium steady states. Here, we restrict the auxiliary reservoir's function solely to maintaining nonequilibrium steady states induced by the system's dynamics, so that $|W_{\mathrm{aux}}| = |Q_{\mathrm{hk}}|$.

If the system's dynamics are microscopic—or if the underlying coarse-graining scheme does not include hidden entropy-producing transitions—then the conditional path irreversibility maps directly to the total heat dissipated by the system to its thermal environment: $\mathcal{Q} \to \beta Q$ [21]. Otherwise, it provides a lower bound for total heat—there may be unaccounted-for dissipation in hidden degrees of freedom [22]. (Note that, following our unit convention, βQ—as well as the other energies per $k_{\mathrm{B}}T$ to appear shortly—is thus measured in nats.)

Similarly, the stochastic excess and housekeeping heats map to the excess and housekeeping heats of steady-state thermodynamics: $\mathcal{Q}_{\mathrm{ex}} \to \beta Q_{\mathrm{ex}}$, $\mathcal{Q}_{\mathrm{hk}} \to \beta Q_{\mathrm{hk}}$, and we have $Q = Q_{\mathrm{ex}} + Q_{\mathrm{hk}}$ [7, 8, 10, 11, 13, 15, 23, 24]. In this way, the total heat dissipation splits into one component due to the system's response to environmental stimuli—the excess heat Q_{ex}—and one due to maintaining NESSs—the housekeeping heat Q_{hk}.

The nonsteady-state addition to stochastic free energy, meanwhile, becomes the nonsteady-state (or nonequilibrium) addition to free energy, capturing state-dependent contributions to free energy resulting from initially- (and finally-) nonsteady-state configurations.

Absent the ESS limit, the stochastic excess work becomes just excess work via $\mathcal{W}_{\mathrm{ex}} \to \beta W_{\mathrm{ex}}$ and carries the interpretation as that work done atop the change in steady-state free energy that would be dissipated if the system relaxed to its stationary distribution, given that it started in one. Or, more specifically:

$$W_{\mathrm{diss}} \doteq W_{\mathrm{ex}} - k_{\mathrm{B}}T\Delta\mathcal{F}^{\mathrm{nss}}_{\boldsymbol{\mu}\|\lambda} \tag{14}$$

is the *dissipated work* [7, 8]—that done atop the change in nonequilibrium free energy as the system evolves between two NEDSs.

Finally, underlying each of these quantities, the steady-state surprisal ϕ_λ itself is interpreted as a *nonequilibrium potential* [25].

Unfortunately, in the NESS setting, defining a *steady-state free energy* analogous to the equilibrium free energy of ESS systems remains problematic [7, 8]. Furthermore, directly mapping surprisals to energies is in general impossible, as is operationally defining work and state energies [25]. As a result, we can give explicit construction of neither W_λ nor W_{aux}.

We can, however, make useful headway by placing an additional restriction on the auxiliary bath. Hereafter, we assume the bath provides *only* that energy required to maintain NESSs, so that $|W_\lambda| = |Q_{\mathrm{hk}}|$ at all times. This corresponds to assuming that the only unaccounted-for degrees of freedom are those strictly necessary for maintaining nonequilibrium steady states as implied by the observed stochastic dynamic. This means, in turn, the ESS limit of $\mathcal{Q}_{\mathrm{hk}} \to 0$ corresponds to eliminating the auxiliary bath, leaving a system coupled only to one ideal heat reservoir and one ideal work reservoir, mirroring common schemes in stochastic thermodynamics [5, 26, 27].

D. Excess and Thermodynamic First Laws

We call Eq. (8) the *excess First Law* due to its structural similarity to the First Law of thermodynamics. To see why, let us now take a detour to the $\mathcal{Q}_{\mathrm{hk}} \to 0$, ESS limit: We are left with the canonical ensemble of statistical mechanics. Our system is affixed only to an ideal heat bath and an ideal work reservoir and, without issue now, we assign to each state z an energy $E_\lambda(z)$ to obtain [26]:

$$\begin{aligned}\Delta E &= \underbrace{\Delta_\lambda E}_{} + \underbrace{\Delta_z E}_{} \\ &\doteq \; W \;\; - \;\; Q .\end{aligned} \tag{15}$$

Equation (15) defines work and heat in this restricted case as distinct contributions to the system's change in energy. Superficially, Eq. (8) is then a First Law for steady-state surprisal in exactly the same way that Eq. (15) is a First Law for energy.

The change of viewpoint from E to ϕ as the central object represents a subtle but useful generalization. Helpfully, it comes without additional risk, since the same restrictions that give Eq. (15) also imply:

- Boltzmann-distributed steady states [28], so that:

  $$k_\mathrm{B} T \phi_\lambda(z) = E_\lambda(z) - F^{\mathrm{eq}}, \tag{16}$$

  with F^{eq} the equilibrium free energy (the usual logarithm of the canonical partition function);

- For excess work [29]:

  $$\mathcal{W}_\mathrm{ex} \to \beta(W - \Delta F^{\mathrm{eq}}) \;, \tag{17}$$

  we now have a consistent notion of steady-state (equilibrium) free energy; and

- All dissipated heat is excess:

  $$\mathcal{Q}_\mathrm{ex} \to \beta Q . \tag{18}$$

And so, for ESSs:

$$\begin{aligned}k_\mathrm{B} T \Delta\phi &= \Delta E - \Delta F^{\mathrm{eq}} \\ &= (W - \Delta F^{eq}) - Q .\end{aligned} \tag{19}$$

Thus, mapping from energetic to surprisal-based First Laws involves only the switch in viewpoint from total work to *excess* work as the more direct quantity. Here, it is that work done atop the change in equilibrium free energy. Since fluctuation theorems are phrased quite naturally in terms of functionals of ϕ and realized path probabilities [7, 8, 10], taking the First Law of Eq. (8) as a starting point is particularly helpful when working with those theorems.

Treating ϕ as more fundamental than E carries utility beyond this convenience, however. There are many more-general settings than the canonical ensemble. These include, for example, biological, active matter, and other NESS systems not Boltzmann-distributed in the energies at stationarity [30–32]. In these cases, in defining $\mathcal{W}_\mathrm{ex}$ stochastic-dynamically, Eq. (8) circumvents issues with appropriately defining nonequilibrium steady-state free energies [7, 8]. Finally, in any situation where a relationship between E and ϕ *can* be derived, one can map Eq. (8) to Eq. (15) directly. Moreover, the former retains its meaning and, as we shall show, utility—even when the latter is far from familiar.

Such is often the case in highly coarse-grained, effective state-space models of mesoscopic complex phenomena where, at best, one estimates bounds on "true" entropy production [22, 33–36]. The coarse-grained dynamics themselves, however, may be directly observed. And in these cases, Eq. (8) holds exactly and remains interpretable *at the level of the observed phenomena*. This is reminiscent of several similarly-phrased fluctuation theorems; e.g., Ref. [8]'s NESS trajectory class fluctuation theorem.

E. Information Ratchets

We are especially interested in a particular decomposition of $\mathcal{Z}$ into distinct subspaces—a *ratchet* and a semi-infinite *information tape*. Figure 2 illustrates the setting.

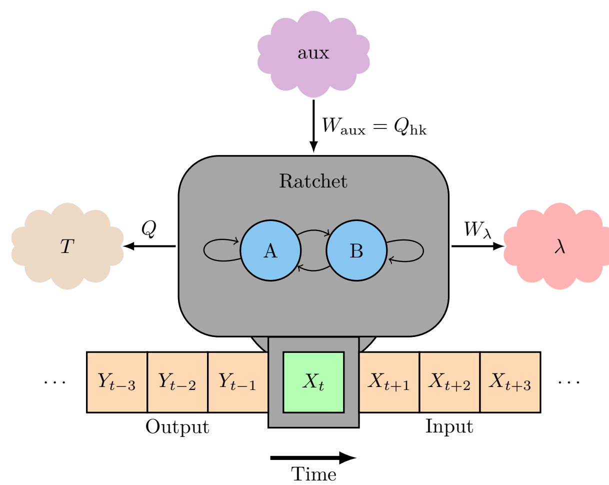

FIG. 2. Information ratchet system: At each time step, the ratchet moves along the tape, interacting with one symbol at a time and exchanging energy with the coupled reservoirs in the process. New here is the auxiliary reservoir that allows for nonequilibrium steady states and another mode of energy exchange with the ratchet-tape subsystem. (Illustration created in part by modifying Ref. [37]'s Fig. 1, with permission from the authors.)

The ratchet interacts directly with only a single information-tape cell at a time. (Hereafter, we refer to a *bit* since we consider a tape with a binary alphabet. Generalizing to other alphabet sizes is straightforward.) Furthermore, we assume that any violation of detailed balance is strictly due to the ratchet system interacting with a single bit. That is, there are no energetic fluxes through the extended tape except as facilitated by the ratchet-tape interaction. We assume that the joint dynamics of the ratchet, interacting bit, and reservoirs is Markovian. At each time step, the ratchet:

1. Moves one cell along the tape, putting it in contact with the next interaction bit; and

2. Thermalizes (perhaps incompletely) with the coupled reservoirs for a time τ.

We do not assign an energy to each state in the joint dynamics, since detailed balance is not required. (Previous studies imposed detailed balance during the thermalization step to fix relative state energies [4, 5, 38, 39].) Indeed, in the presence of nonequilibrium steady states, assignment of state energies—or even a formal, consistent definition of total work—is not generally possible [25]. This precludes direct comparison with previous ratchet studies [5, 6, 39], wherein total work extracted was upper bounded via a lower bound on total dissipated heat.

Rather, here we leverage the fact that in this isothermal setting $(\mathcal{Q}, \mathcal{Q}_{\mathrm{ex}}, \mathcal{Q}_{\mathrm{hk}}) \to (\beta Q, \beta Q_{\mathrm{ex}}, \beta Q_{\mathrm{hk}})$ and consider dissipated heats directly. In particular, when discussing thermodynamic functionality, we focus on exchanges with the heat bath. That is, instead of the "engine" regime previously defined by $\langle W \rangle > 0$ [5, 39], for example, we refer to the "heat engine" regime defined by $-\langle Q \rangle > 0$—where energy is on average extracted from the thermal reservoir.

To describe and decompose the information-bearing degrees of freedom, we split the random variable Z_n into three parts: the random variable R_n (with alphabet $\mathcal{R}_n$) corresponds to the ratchet subsystem's state at time n; the joint random variable $X_{n:\infty}$ to the *input tape* at time n—that portion of the information tape to which the ratchet has not yet written—and the joint random variable $Y_{0:n-1}$, for the *output tape*, to which the ratchet has written. Thus, at each n, $Z_n = (R_n, X_{n:\infty}, Y_{0:n-1})$. This mirrors the decompositions of Refs. [5, 39].

III. INFORMATION PROCESSING FIRST LAW

Let us return to the general stochastic-dynamical setting, with no assumption of any particular thermodynamic scheme. Let $\boldsymbol{\mu}_n$ be the NEDS at time step n, such that $Z_n \sim \boldsymbol{\mu}_n$. Suppose we have a system that begins in $\boldsymbol{\mu}_0$ and ends in $\boldsymbol{\mu}_N$, as driven by the protocol $\lambda_{0:N}$. We begin by equating the trajectory (over all possible state-space trajectories $z_{0:N}$) and state averages (justified in App. A) of the change in steady-state surprisal:

$$\begin{aligned} \langle \Delta \phi_\lambda \rangle &= \Delta \langle \mu | \phi_\lambda \rangle \\ &\doteq \langle \mu_N | \phi_{\lambda_N} \rangle - \langle \mu_0 | \phi_{\lambda_0} \rangle \, . \end{aligned} \tag{20}$$

The lefthand side is, by definition, $\langle \mathcal{W}_{\mathrm{ex}} \rangle - \langle \mathcal{Q}_{\mathrm{ex}} \rangle$. This is the averaged First Law for ϕ as in Eq. (8).

For the righthand side, notice that:

$$\begin{aligned} \langle \mu | \phi_\lambda \rangle &= \sum_{z \in \mathcal{Z}} \mu(z) \phi_\lambda(z) \\ &= -\sum_{z \in \mathcal{Z}} \mu(z) \ln \pi_\lambda(z) \\ &= H[Z] + D_{\mathrm{KL}}[Z \,\|\, \Lambda]. \end{aligned} \tag{21}$$

Hence, we have the *information processing first law* (IPFL):

$$\Delta H[Z] + \Delta D_{\mathrm{KL}}[Z \,\|\, \Lambda] = \langle \mathcal{W}_{\mathrm{ex}} \rangle - \langle \mathcal{Q}_{\mathrm{ex}} \rangle \, . \tag{22}$$

The lefthand side accounts for the "information processing": the Shannon entropy change of the system plus the change in its divergence from the local stationary distribution. (Alternatively, the change in cross entropy between the system's state distribution and the local steady-state distribution.)

Moving $\Delta D_{\mathrm{KL}}[Z \,\|\, \Lambda]$—the averaged change in nonsteady-state free energy from Eq. (13)—to the righthand side recovers Eq. (3). This is, quite directly, a First Law for information processing that expresses its changes in terms of averages of the stochastic-dynamical functionals $\mathcal{W}_{\mathrm{ex}}$, $\mathcal{Q}_{\mathrm{ex}}$, and $\mathcal{F}^{\mathrm{nss}}_{\boldsymbol{\mu}\|\lambda}$. These are the functionals that, under appropriate thermodynamic schemes as in Fig. 1, carry entropic and energetic meaning. This IPFL holds generally for transitions between NEDSs, implying validity for NESS and even nonthermal systems, since the generalized excess quantities are still well-defined by Eq. (8).

Stated in the form of Eq. (22), the IPFL makes no reference to the "conjugate" or "reversed" dynamics involved in the definitions of $\mathcal{Q}$ and by extension $\mathcal{Q}_{\mathrm{hk}}$. Rather, it is concerned strictly with averages weighted by forward trajectories. However, substituting Eq. (12) does involve these conjugated dynamics. This leads to expressing the IPFL equivalently as:

$$\Delta H[Z] + \langle \mathcal{Q} \rangle = \langle \mathcal{W}_{\mathrm{ex}} \rangle + \langle \mathcal{Q}_{\mathrm{hk}} \rangle - \Delta D_{\mathrm{KL}}[Z \,\|\, \Lambda]. \tag{23}$$

Here, the lefthand side is stochastic thermodynamics' familiar (average of) *total entropy production* ΔS_{tot}, broken into system ($\Delta H[Z]$) and environment ($\langle \mathcal{Q} \rangle$) pieces [7, 40, 41]. The righthand side thus represents an alternative decomposition of the total entropy production into excess environmental ($\langle \mathcal{W}_{\mathrm{ex}} \rangle$) and housekeeping ($\langle \mathcal{Q}_{\mathrm{hk}} \rangle$) components, as well as one due to initial (and final) state dependence ($\Delta D_{\mathrm{KL}}[Z \,\|\, \Lambda]$) [7, 8, 42, 43]. The IPFL, then, expresses a particular decomposition of the average total entropy production. In the appropriate settings, the decomposition directly links change in information content with thermodynamic processes *without* invoking conjugated dynamics.

A. Application to Information Ratchets

Arriving at Eq. (22) required minimal assumptions about the underlying SUS. Now, we wish to specialize it to the information ratchet system of Sec. II E and Fig. 2. In particular, the isothermal environment takes our stochastic-dynamical functionals to thermodynamic energies. And, the distinct ratchet and tape subspaces allow for meaningful decomposition of the information-bearing degrees of freedom.

First, we expand $\Delta H[Z] = H[Z_N] - H[Z_0]$ from Eq. (22). Splitting the joint Shannon entropies, making use of mutual informations—denoted $I[\cdot : \cdot]$ for the (symmetric) mutual information between two random variables [9]—and bearing in mind that changes in indices for X, Y, R are to be inferred from the breakdowns of Z_N and Z_0:

$$\begin{aligned}\Delta H[Z] &= \Delta H[R] + \Delta H[X,Y] - \Delta I[R : X, Y] \\ &= \Delta H[R] + \Delta H[X] + \Delta H[Y] \\ &\quad - \Delta I[R : X, Y] - \Delta I[X : Y]. \end{aligned} \tag{24}$$

This further decomposes the IPFL of Eq. (22):

$$\begin{aligned}&\langle \mathcal{W}_{\mathrm{ex}} \rangle - \langle \mathcal{Q}_{\mathrm{ex}} \rangle \\ &\quad = \Delta H[R] + \Delta H[X] + H[Y_{0:N-1}] \\ &\qquad - \Delta I[R : X, Y] - I[X_{N:\infty} : Y_{0:N-1}] \\ &\qquad + \Delta D_{\mathrm{KL}}[Z \,\|\, \Lambda]. \end{aligned} \tag{25}$$

Equivalently, the decomposition of average total entropy production in Eq. (23) becomes:

$$\begin{aligned}&\langle \mathcal{W}_{\mathrm{ex}} \rangle + \langle \mathcal{Q}_{\mathrm{hk}} \rangle - \Delta D_{\mathrm{KL}}[Z \,\|\, \Lambda] \\ &\quad = \langle \mathcal{Q} \rangle + \Delta H[R] + \Delta H[X] + H[Y_{0:N-1}] \\ &\qquad - \Delta I[R : X, Y] - I[X_{N:\infty} : Y_{0:N-1}]. \end{aligned} \tag{26}$$

Equation (24)'s decomposition took Eqs. (22) and (23) to Eqs. (25) and (26), respectively. The decomposition is identical to that in Ref. [5]. However, there the goal was to take asymptotic rates. That, together with the finite-state ratchet requirement, removed several terms. Here, we pause to interpret each term in its finite-time context and comment on its contribution to the averaged total entropy production.

The first term $\langle \mathcal{Q} \rangle$ is the environment's contribution to the total entropy production. All that remains is contributed by the joint ratchet-tape system.

The second term $\Delta H[R]$ monitors the change in information content of the ratchet's states—a change in the ratchet's internal memory. If, as the ratchet interacts with the tape, it gains memory in this sense, this specific part of the joint system must become more randomized. And, equivalently, this term contributes an increase to the total entropy production.

The third term $\Delta H[X] = H[X_{N:\infty}] - H[X_{0:\infty}]$ quantifies a change in the information content of the input tape. Or, more specifically for finite alphabets, this is strictly nonpositive—the opposite of the information contributed by the random variables $X_{0:N-1}$. And so, the more random the input tape, the more negative this term can be. We expect memoryless inputs to reduce the potential to extract heat compared to memoryful ones. That is, colloquially there is less pattern to scramble [39]. We shall see later that this is indeed the case for IPSLs. For the IPFL, in the meantime, this term's negativity reduces averaged total entropy production. Intuitively, removing randomness in the input tape reduces overall entropy production.

The fourth term $\Delta H[Y]$ is the output tape's information content. Its impact on countable spaces—as assumed—is straightforward. Due to Shannon information's nonnegativity, the more random the ratchet makes the output tape, the greater the positive contribution to average total entropy production.

The fifth term $\Delta I[R : X, Y]$ tracks the change in *shared* information between the ratchet and tape. As the ratchet interacts with bits from the input tape and writes to the output tape, it induces correlation between it and the tape. While at first glance this recalls Ref. [39]'s (de)randomizer axes, it is altogether different. Those axes tracked induced correlations *internal* to the information tape, whereas this term tracks induced correlation *between the ratchet and the (entire input-output) tape.* Mutual information's positivity makes the contribution to Eq. (24) strictly nonpositive, lowering the average total entropy production. In other words, inducing correlation between the ratchet and tape reduces the joint system's entropy production and vice versa.

This mutual information is especially important to consider in tandem with $\Delta H[R]$. While an increase in ratchet memory alone—unrelated to tape correlation—acts to increase entropy production, generically an increase in ratchet memory also enables greater correlation with the information tape via $\Delta I[R : X, Y]$. Thus, the ratchet memory's effect on entropy production involves both terms: correlation between the ratchet and tape, enabled by ratchet memory to capture temporal patterns in the tape, counteracts the entropy produced by an increase in ratchet memory alone.

Finally, the sixth term $I[X_{N:\infty} : Y_{0:N-1}]$ is the mutual information between the input and output tapes. And so, again due to mutual information's nonnegativity, it has the effect of decreasing the averaged total entropy production.

Taken altogether, Eqs. (25) and (26) delineate exact links between finite-time ratchet-tape information processing and the joint system's thermodynamic behavior. It is a specialization of the IPFL to the case of a system constructed as in Fig. 2. Shortly, we use it as a starting point to derive and generalize the previously-reported IPSLs for ratchet-tape systems [5, 6]. However, the inequalities in IPSLs are replaced by equalities of the IPFL in the same way that fluctuation theorems of stochastic thermodynamics replace inequalities of thermodynamic Second Laws with strict equalities. In point of fact, as we now show, fluctuation theorems directly take the IPFL to IPSLs.

IV. RATCHET FIRST TO SECOND LAWS

The following derives several specializations to the information ratchet system class of Sec. II E and Fig. 2, starting from Eqs. (22) and (23). To do this, it first leverages an integral fluctuation relation to take the equality to an inequality. It then splits the effective state space as in Sec. II E, along the way generalizing part of a recently-reported finite-tape IPSL [6]. Finally, it takes asymptotic limits to similarly generalize the previous asymptotic IPSL [5, 38, 39] for these regimes. Since we adopt the same assumptions as Sec. II E, hereafter the underlying dynamics of the joint ratchet-tape space are Markovian. However, the statistical process that produces input and output tape symbol sequences need not be Markovian. In the infinite tape case, they may even possess infinite-range temporal correlations.

A. Fluctuations and Second Laws

A crowning achievement of stochastic thermodynamics over the last several decades was the development fluctuation relations and fluctuation theorems that capture fluctuations arbitrarily far from equilibrium. (See Ref. [40] for a recent review.) These come in three main types: (i) *integral*, relating to exponential averages over all possible trajectories [10, 26, 27, 44]; (ii) *detailed*, exposing a time reversal (a)symmetry between forward and reverse paths [7, 21, 45–47]; and *trajectory class*, interpolating between the two [8, 41, 48].

A comprehensive review would go too far afield here; rather see Refs. [24, 49]. Nonetheless, the following uses, in particular, two integral fluctuation theorems:

$$1 = \left\langle \mathrm{e}^{-(\Delta S_{\mathrm{tot}})} \right\rangle \quad \text{and} \tag{27}$$

$$1 = \left\langle \mathrm{e}^{-\left(\mathcal{W}_{\mathrm{ex}} + \mathcal{F}^{\mathrm{nss}}_{\mu_0 \| \lambda_0}\right)} \right\rangle . \tag{28}$$

Invoking the convexity of the exponential, we apply Jensen's inequality to derive the *generalized Second Laws*:

$$\Delta H[Z] + \langle \mathcal{Q} \rangle \geq 0 \quad \text{and} \tag{29}$$

$$\langle \mathcal{W}_{\mathrm{ex}} \rangle + D_{\mathrm{KL}}[Z_0 \,\|\, \Lambda_0] \geq 0. \tag{30}$$

Equation (29) is thus a consequence of the total entropy production integral fluctuation theorem. Equation (28), first introduced in Ref. [8], generalizes the integral fluctuation theorem of Ref. [10] to include initial-state dependence. The resulting inequality in Eq. (30) shows that initially-nonsteady states *lower* the bound on $\mathcal{W}_{\mathrm{ex}}$.

The following now demonstrates a suite of IPSLs that result directly from applying these integral fluctuation theorems and Jensen's inequality to the IPFL.

First, Eq. (29) gives, directly:

$$-\langle \mathcal{Q} \rangle \leq \Delta H[Z]. \tag{31}$$

While there was no need to substitute into an IPFL expression, note that Eq. (29)'s lefthand side is identical to Eq. (23)'s righthand side, and so Eq. (31) is a specialization of that equality.

Notice that $\langle \mathcal{Q} \rangle$ can be negative so long as $\Delta H[Z]$ is positive. In an appropriate thermal environment, such as that of Fig. 2, this upper bounds the finite-time extracted heat and it is (the negation of) Ref. [5]'s Eq. (A7). This is at the core of the latter's subsequent derivation, as it shows how energy may be extracted from a heat bath at the cost of an increase in the system's information-bearing entropy.

In the ESS ratchet setting, such as that considered by Refs. [5, 6, 39], we may also rephrase this bound in terms of the averaged work $\beta \langle W \rangle$ done *by* the system [50]:

$$\beta \langle W \rangle \leq \Delta H[Z]. \tag{32}$$

This is exactly Ref. [6]'s finite-tape, single-pass IPSL, where Z denotes the joint random variable of their ratchet and tape subspaces.

However, in the NESS setting one cannot directly equate negative dissipated heat with positive work production, as detailed previously in Sec. II E. To avoid confusion, then, we focus on Eq. (31)'s heat bound: Negative averaged total heat indicates the system's function as a *heat engine*, a net extraction of energy from the thermal environment.

The NESS setting, however, affords us additional IFTs. We can, for example, substitute Eq. (28) into Eq. (22), yielding:

$$-\langle \mathcal{Q}_{\mathrm{ex}} \rangle \leq \Delta H[Z] + D_{\mathrm{KL}}[Z_N \,\|\, \Lambda_N] \tag{33}$$

or, equivalently, via Eq. (12):

$$-\langle \mathcal{Q} \rangle \leq \Delta H[Z] + D_{\mathrm{KL}}[Z_N \,\|\, \Lambda_N] - \langle \mathcal{Q}_{\mathrm{hk}} \rangle . \tag{34}$$

This adjusts the finite-time extracted heat bound to account for NEDSs. First, for finite spaces, nonsteady final states raise the bound on extracted heat. That is, we need not dissipate to full relaxation. The presence of NESSs instead *lowers* the bound by the amount of the total housekeeping heat. Thus, as long as $\mathcal{Q}_{\mathrm{hk}} \geq D_{\mathrm{KL}}[Z_N \,\|\, \Lambda_N]$—the result of an established IFT in even state spaces [8]—Eq. (34) is always a tighter bound than Eq. (31).

In this way, Eq. (34) reveals the effect of NEDSs on the heat-bath equivalent of Ref. [6]'s Eq. (19) and establishes it under very general conditions. Additionally, two new effects appear. The ensemble-averaged housekeeping heat $\mathcal{Q}_{\mathrm{hk}}$ *lowers* the bound on heat extraction—as an additional source of dissipation—while the final-state dependence $D_{\mathrm{KL}}[Z_N \parallel \Lambda_N]$ raises it. That is, we need not account for what *would be* dissipation if the system fully relaxed to its steady states [42]. As before, this bound is always tighter than Eq. (32) in the case of even state spaces.

Finally, applying the preceding decomposition of $\Delta H[Z]$ to Eq. (34) gives the analogue to Ref. [6]'s finite-tape IPSL, but further decomposed to both account for NEDSs and delineate ratchet-tape information dynamics:

$$\begin{aligned} -\beta \left\langle Q \right\rangle \leq & \, D_{\mathrm{KL}}[Z_N \parallel \Lambda_N] - \left\langle \mathcal{Q}_{\mathrm{hk}} \right\rangle \\ & + \Delta H[R] + \Delta H[X] + H[Y_{0:n-1}] \\ & - \Delta I[R : X, Y] - I[X_{N:\infty} : Y_{0:N-1}]. \end{aligned} \quad (35)$$

With this, we can translate how each term of $\Delta H[Z]$ affected the averaged total entropy production to its effect on the maximum extracted work. In short, information processing that reduces the averaged total entropy production identically reduces the upper bound on ensemble-averaged heat extraction. That is, even for NEDS, extracting heat requires producing entropy.

B. General Asymptotics

The preceding results apply for all finite times or, equivalently, for finite tapes. Now, we address asymptotics. Our procedure is to take $N \to \infty$ and divide the quantities of interest by N, giving an asymptotic rate per time step. For notational simplicity, we use the dot notation for the thermodynamic quantities:

$$\langle \dot{\mathcal{Q}} \rangle \doteq \lim_{N \to \infty} \frac{1}{N} \left\langle \mathcal{Q} \right\rangle \quad (36)$$

and so on, for $\langle \dot{\mathcal{W}_{\mathrm{ex}}} \rangle$, $\langle \dot{\mathcal{Q}_{\mathrm{ex}}} \rangle$, $\langle \dot{\mathcal{Q}_{\mathrm{hk}}} \rangle$, and $\langle \dot{W} \rangle$.

Let's take the asymptotic limit of Eq. (25). In particular, as in Ref. [5] we have (i) $\lim_{N\to\infty} \Delta H[X]/N = -h_\mu$, (minus) the Shannon entropy rate of the process generating the input tape; (ii) $\lim_{N\to\infty} H[Y_{0:N-1}]/N = h'_\mu$, the Shannon entropy rate of the process generating the output tape; and (iii) $\lim_{N\to\infty} I[X_{N:\infty} : Y_{0:N-1}]/N = 0$.

The remaining two pieces of $\Delta H[Z]$, however, vanish only under restricting to finite-state ratchets. Without that assumption we are left with an asymptotic IPFL:

$$\begin{aligned} \langle \dot{\mathcal{W}_{\mathrm{ex}}} \rangle - \langle \dot{\mathcal{Q}_{\mathrm{ex}}} \rangle = & \, \Delta h_\mu \\ & + \lim_{N \to \infty} \frac{1}{N} (\Delta H[R] - \Delta I[R : X, Y]) \\ & + \lim_{N \to \infty} \frac{1}{N} \Delta D_{\mathrm{KL}}[Z_N \parallel \Lambda_N]. \end{aligned} \quad (37)$$

And, similarly, write the heat-extraction asymptotic IPSL:

$$\begin{aligned} -\beta \langle \dot{Q} \rangle \leq & \, \Delta h_\mu - \langle \dot{\mathcal{Q}_{\mathrm{hk}}} \rangle \\ & + \lim_{N \to \infty} \frac{1}{N} (\Delta H[R] - \Delta I[R : X, Y]) \\ & + \lim_{N \to \infty} \frac{1}{N} D_{\mathrm{KL}}[Z_N \parallel \Lambda_N]. \end{aligned} \quad (38)$$

This generalizes the previous bound by accounting explicitly for final-state dependence, nonequilibrium steady states, and potentially infinite-state ratchets.

We leave detailed analytical consideration of the remaining limits for infinite-state ratchets and their con/divergence to a sequel. However, we will interpret the contextual meaning of the remaining limits for countably infinite ratchets.

First, $\lim_{N\to\infty} \Delta H[R]/N$ is the rate of change of the ratchet's *statistical complexity* $C_\mu[R]$ per time step, lower bounded by the statistical complexity of its ϵ-machine representation from computational mechanics [51, 52]. In essence, this limit measures the rate of increase in ratchet memory as it reads an infinite stream of incoming bits. It is only nonzero for a ratchet with an infinite memory capacity. The resulting device is able to violate the finite-state asymptotic IPSL by leveraging its infinite internal memory to produce work in excess of that bound [38]. For any finite-state ratchet $\lim_{N\to\infty} \Delta H[R]/N$ vanishes since in that case $H[R_N]$ is bounded from above. It vanishes also for any infinite-state ratchet that does not asymptotically *gain* memory from an infinite stream of inputs. More precisely, this holds for a ratchet whose internal state distribution approaches a fixed value unaffected by the incoming bit stream.

Second, $\lim_{N\to\infty} \Delta I[R : X, Y]/N$ is the rate of change of correlation between the ratchet and the total information tape. This limit is nonzero only if (i) the ratchet continually gains memory as above and (ii) the ratchet continually induces correlation between itself and the total input-output tape.

Finally, $\lim_{N\to\infty} \Delta D_{\mathrm{KL}}[Z \parallel \Lambda]/N$ monitors (asymptotic) movement away from steady-state conditions. Specifically, it is nonzero only when $D_{\mathrm{KL}}[Z_N \parallel \Lambda_N]$ diverges with N—the system approaches a distribution infinitely far from stationarity—perhaps by moving progressively farther away from the underlying stationary distribution at each step. Colloquially, the interaction timescale is so short that the ratchet-tape system moves further away from thermalization by constantly changing the interaction bit. The presence of this term at all implies the existence of a stationary distribution for the ratchet-tape system. This is a fact not guaranteed for infinite ratchets [38, 53], but assumed by our stochastic (thermo)dynamical formalism.

C. Finite Ratchet Asymptotics

Assuming a finite-state ratchet—in line with potential physical implementation—simplifies the asymptotic analysis. (As it did in Refs. [5, 38].) This results in an asymptotic IPFL for finite-state systems:

$$\langle \dot{\mathcal{W}}_{\rm ex} \rangle - \langle \dot{\mathcal{Q}}_{\rm ex} \rangle = \Delta h_\mu . \tag{39}$$

And, finally, there is the correction to the previously-reported IPSL—rewritten as a bound on extracted heat—for finite ratchets interacting with an infinite tape:

$$-\beta \langle \dot{Q} \rangle \leq \Delta h_\mu - \langle \dot{\mathcal{Q}}_{\rm hk} \rangle . \tag{40}$$

The correction is simply $\beta \langle \dot{Q}_{\rm hk} \rangle$ in the isothermal setting. For even state spaces this is nonnegative and so tightens the previous bound. Said simply, housekeeping dissipation reduces the maximum extracted heat—one cannot harness what must go toward maintaining NESSs.

V. ASYMMETRIC STOCHASTIC 4-CYCLE

The presence of housekeeping dissipation in Eq. (40) suggests meaningful change in ratchet functional thermodynamics [54], depending on the degree to which the joint ratchet-bit system violates detailed balance. To demonstrate this dependence we introduce the *asymmetric stochastic* 4-*cycle* (AS4C)—a two-state ratchet coupled to an information tape. The states are labeled A and B. With joint ratchet-bit Markov chain states ordered by $(\mathrm{A} \otimes 0, \mathrm{A} \otimes 1, \mathrm{B} \otimes 1, \mathrm{B} \otimes 0)$, the dynamics obey the row-stochastic transition matrix:

$$\mathbf{T}(p,q) \doteq \begin{bmatrix} 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \\ 0 & 1-qp & 0 & qp \\ p & 0 & 1-p & 0 \end{bmatrix} . \tag{41}$$

This two-parameter ratchet family, pictured in Fig. 3, generically violates detailed balance and allows $0 \to 1$ and $1 \to 0$ transitions to be unequally favored in terms of transition probabilities. The latter fact manifests as a rotational asymmetry in the cycle—given by the scaling parameter $q \in (0,1]$. When $q = 1$, the cycle is symmetric: $0 \to 1$ and $1 \to 0$ transitions are equally favored, but the system exhibits stationary directionality in its joint state space. In the symmetric case, detailed balance is achieved only when $p = \frac{1}{2}$.

The extent to which a discrete- and even-state Markov chain system violates detailed balance on average is given by $\langle \dot{\mathcal{Q}}_{\rm hk} \rangle$. Via a single-step average of Eq. (12) we thus obtain for discrete time:

$$\langle \dot{\mathcal{Q}}_{\rm hk} \rangle = \sum_{i \neq j} \pi(i) [\mathbf{T}]_{ij} \log \frac{\pi(i)[\mathbf{T}]_{ij}}{\pi(j)[\mathbf{T}]_{ji}} , \tag{42}$$

where i and j index the states.

This is also the exact amount by which the previous asymptotic ESS IPSL Eq. (1) was tightened by our NESS IPSL in Eq. (40). To visualize this difference—the degree of tightening—Fig. 4 plots $\langle \dot{\mathcal{Q}}_{\rm hk} \rangle$ while sweeping parameters p and q.

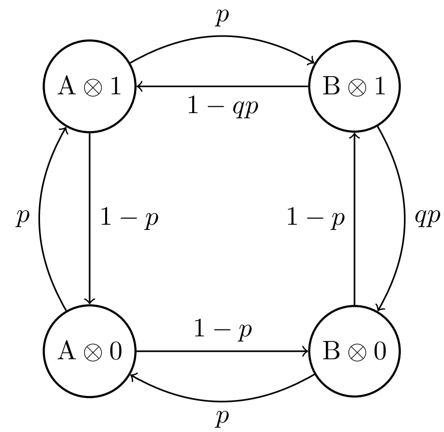

FIG. 3. Markov chain describing the joint ratchet-bit dynamics of the asymmetric stochastic 4-cycle (AS4C) ratchet family. The behavior is parameterized by $p \in (0,1)$ and $q \in (0,1]$. The ratchet generically violates detailed balance. When $q \neq 1$ it controls rotational asymmetry and can probabilistically favor either $0 \to 1$ or $1 \to 0$ transitions. The symmetric value $q = 1$ equally favors these transitions, but the dynamics of the joint space still exhibits directionality. Detailed balance is satisfied in this case only for $p = 1/2$.

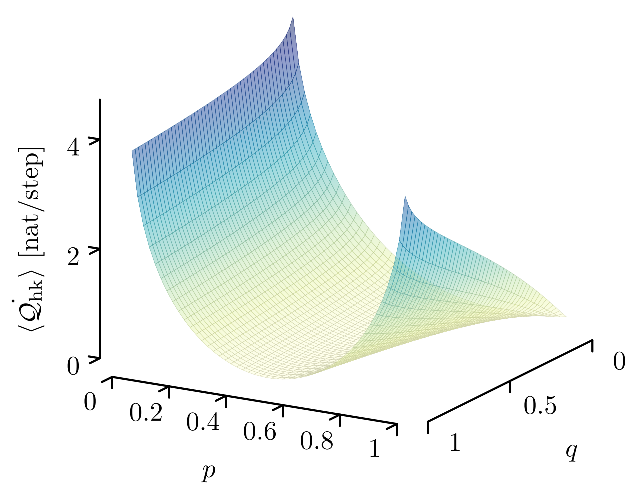

FIG. 4. Averaged rate of housekeeping entropy production $\langle \dot{\mathcal{Q}}_{\rm hk} \rangle$—measured in nats per time step, in keeping with our convention that surprisal takes units of nat—for the asymmetric stochastic 4-cycle, as a function of parameters p and q. This is the exact amount by which Eq. (40) tightens Eq. (1).

A. Input-Output Transducer

As one sees, $\langle \dot{\mathcal{Q}}_{\text{hk}} \rangle$ is far from zero over a wide range of the parameter space. These are entropies that must be produced—equivalently in the isothermal setting, heat that must be dissipated—to maintain the system's NESS character. And so, one expects, they significantly impact the system's ability to leverage an information reservoir to extract heat from a thermal environment. This is to say, with our correction to the ESS IPSL in hand, we can analyze bounds on the functional thermodynamics of this ratchet family.

To do so, we must calculate the remaining terms in Eq. (40), namely the Shannon entropy rates h_μ and h'_μ of the processes generating the input and output state sequences. Following Refs. [5, 39], we achieve this by first translating our 4-state joint ratchet-bit Markov chain into a 2-state ratchet *transducer* that accepts as input the process generating the input symbol statistics—in the form of a hidden Markov chain (HMC)—and produces as output the HMC generating the output symbol statistics.

A transducer is specified by its input-output-labeled matrices $\mathbf{M}^{(\text{out}|\text{in})}$:

$$\mathbf{M}^{(\text{out}|\text{in})} \doteq \mathbb{P}_{\text{in}}^{\mathsf{T}} \, \mathbf{T} \, \mathbb{P}_{\text{out}} . \tag{43}$$

The AS4C ratchet has two projection matrices $\mathbb{P}_0$ and $\mathbb{P}_1$ given by:

$$\mathbb{P}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbb{P}_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} . \tag{44}$$

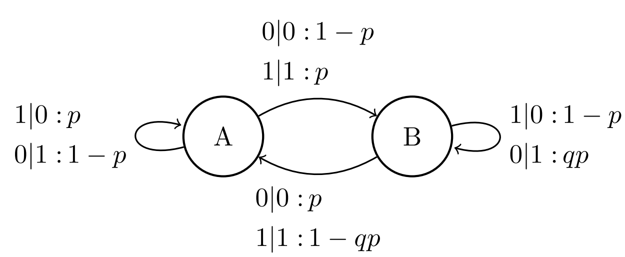

FIG. 5. The transducer corresponding to the 4-state joint ratchet-bit Markov chain of the asymmetric stochastic 4-cycle (AS4C). The transducer has two internal states, A and B, and accepts 0 or 1 as an input bit symbol. The transitions (graph edges) are labeled output bit|input bit : probability.

This defines the AS4C's transducer, whose state-transition diagram is visualized in Fig. 5. Now, we compose it with any input HMC—specified by its symbol-labeled transition matrices $\mathbf{U}^{(x)}$—to give the output HMC producing the symbol statistics on the output tape, specified by $\mathbf{V}^{(y)}$ [39, 55]. The output HMC state space is the Cartesian product of the state spaces of the input HMC and the transducer. Let i and j index the states of the input HMC and i' and j' index those of the transducer. Then:

$$V^{(y)}_{i \times i' \to j \times j'} = \sum_x M^{(y|x)}_{i'j'} U^{(x)}_{ij} . \tag{45}$$

B. All-1s Driving

To simplify determining h'_μ, we drive the AS4C transducer with the *all-ones process*: an input tape of all 1s, exhibiting no randomness whatsoever. Note that generically the output HMC of a memoryless ratchet driven by a memoryless input process results in a highly nonunifilar output HMC [56], for which determining the entropy rate is very challenging [39]. However, for all-1s driving, the AS4C produces the unifilar output HMC shown in Fig. 6.

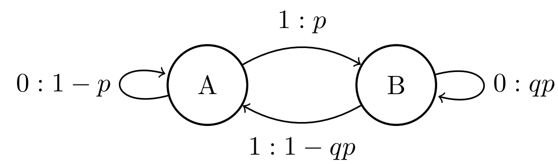

FIG. 6. Output HMC given by the AS4C transducer acting on the all-1s input tape. The ratchet in this case scrambles an informationless input, thereby introducing the capacity to do work.

Since this HMC is unifilar—an internal state and an output symbol completely determine the next internal state—and since its two states make probabilistically distinct future predictions, it is a finite-state ϵ-machine of computational mechanics [52]. That the output tape's process can be described this way enables direct calculation of the output entropy rate [57]. Letting i and j index the output HMC's internal states, $\boldsymbol{\pi}$ be its stationary distribution, and y an output symbol, one has:

$$h'_\mu = - \sum_{y,i,j} \pi(i) V^{(y)}_{ij} \log V^{(y)}_{ij} . \tag{46}$$

Figure 7 plots this over the parameter space.

Setting the input process to have zero randomness also sets $h_\mu = 0$ for it: $\Delta h_\mu = h'_\mu$, all intrinsic randomness in the output tape is induced by the ratchet and, therefore, is available as a thermodynamic resource for heat extraction. For the case of totally ordered input, Eq. (40) reads:

$$-\beta \langle \dot{Q} \rangle \leq h'_\mu - \langle \dot{\mathcal{Q}}_{\text{hk}} \rangle . \tag{47}$$

Summarizing the requirements for net heat extraction: the ratchet must, at minimum, induce randomness in the output tape faster than it dissipates entropy to maintain its NESS.

Since with this particular driving $\Delta h_\mu > 0$ for all parameter combinations, the information eraser functionality of

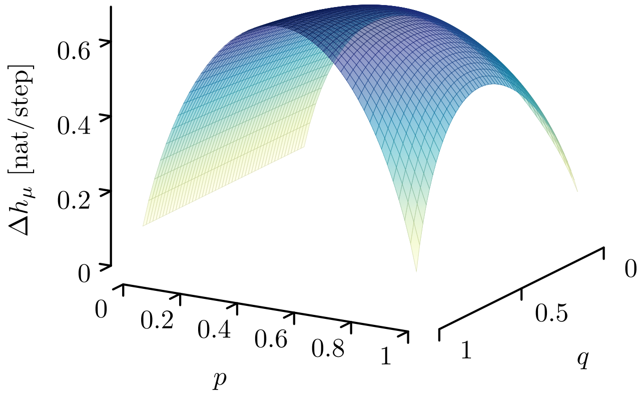

FIG. 7. Change in Shannon entropy rate Δh_μ of the information tape, as generated by the AS4C ratchet driven by the all-1s process. Since the input process is entirely ordered with $h_\mu = 0$, this represents the ESS IPSL's maximum upper bound $-\beta\langle\dot{Q}\rangle_{\min}$. And this, in turn, precludes eraser functionality. That is, one cannot erase information that was never there.

Refs. [5, 39] is precluded. Instead, we have either a heat engine ($\langle Q\rangle < 0$) or a dud ($\langle Q\rangle \geq 0$). Most importantly, the presence of $\langle\dot{\mathcal{Q}}_{\mathrm{hk}}\rangle$ here restricts the regions of parameter space where the ratchet *can* function as a heat engine. Or, alternatively, for some p and q the housekeeping costs are higher than the ratchet's ability to compensate by scrambling the information tape. This forces the previous "potential engine" regions into dud regions.

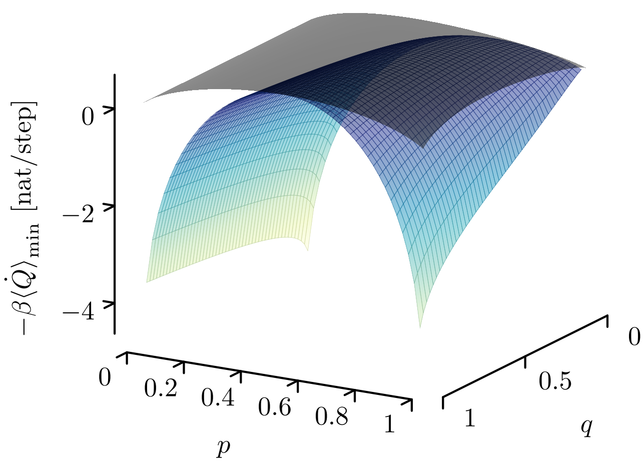

FIG. 8. NESS-tightened upper bound on heat extraction $-\beta\langle\dot{Q}\rangle_{\min} = \Delta h_\mu - \langle\dot{\mathcal{Q}}_{\mathrm{hk}}\rangle$ for the AS4C driven by the all-1s process. To the extent that this differs from Fig. 7, superimposed here in gray, it represents a change of maximum possible net heat extraction.

This is indeed the case, as Fig. 8 shows. In fact, only a small part of engine functionality remains within bounds. Figure 9 shows this directly, where only "potential engine" regions of parameter space are colored. Since an entirely ordered input drives the ratchet, without accounting for the NESS correction one would expect all parameter space to allow potential heat extraction. In this way, explicitly accounting for a system's NESS nature enables qualitative (and quantitative) correction to its allowed behaviors.

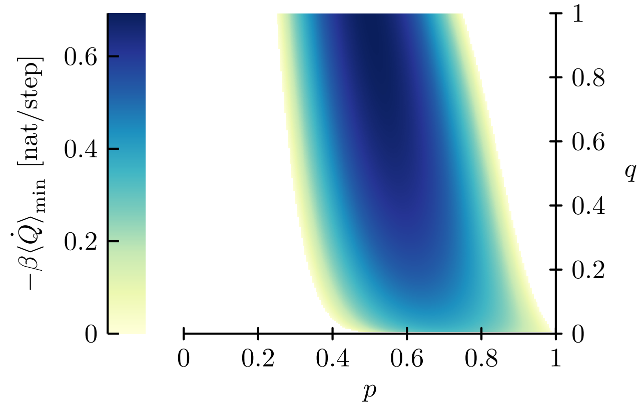

FIG. 9. Parameter space regions that permit the all-1s-driven AS4C ratchet to function as a heat engine. Notably, this includes only a band centered around detailed-balanced dynamics. The remainder of the uncolored parameter space forces $-\beta\langle\dot{Q}\rangle_{\min} < 0$, where the ratchet dissipates heat on average.

VI. CONCLUSION

We began by deriving, under very general circumstances, an IPFL that connects ensemble-averaged thermodynamic behavior to a system's information processing via a strict equality. We showed that this equality is, equivalently, a decomposition of stochastic thermodynamics' average total entropy production. To get there, we placed very few restrictions on the underlying system's dynamics, considering transitions between nonequilibrium dynamical states.

From this First Law, we then applied integral fluctuation theorems to take the equalities to inequalities, reproducing and then tightening established bounds on average heat extraction. By splitting the system into ratchet and tape subspaces and considering both finite and infinite-time cases, we similarly reproduced and then tightened previous IPSLs of autonomous Maxwellian ratchets [5, 6, 38, 39] to explicate the effects of nonequilibrium dynamical states. Finally, we illustrated these results with an example ratchet-tape system—the AS4C, driven by the ordered all-1s process. This demonstrated that, even under extreme simplification, the presence of NESSs introduced qualitative corrections to a ratchet's allowed behavior. In short, the presence of housekeeping entropy costs, induced by NESSs, directly counteracts a ratchet's ability to leverage information creation to extract energy from a heat bath.

Much room for further development remains, particularly in light of the role of fluctuation theorems in deriving these

IPSLs. While our derivation concerned full ensemble averages, recent development of trajectory-class fluctuation theorems [8, 41] highlight opportunities to derive trajectory class IPSLs that are more amenable to experimental verification via their freedom from rare-event statistical errors [58].

That odd-parity variables allow for meaningful decomposition of the housekeeping heat suggests further explication of their effects on the derived IPFLs and IPSLs, including bounds on asymptotic work extraction. Indeed, recent results in stochastic thermodynamics show that where a known, constrained splitting of the joint state space is available, it may be used to tighten the corresponding Second Laws [59]. Additionally, considering that infinite-state ratchets revealed new contributions to the underlying IPSL, their convergence or divergence in general cases warrants detailed analytical investigation.

Finally, the fact that the NESS setting does not straightforwardly account for work extraction itself warrants further study. Indeed, even without the formal challenges surrounding work calculation, the presence of three thermodynamic reservoirs implies additional net fluxes. While the result remains that information-bearing degrees of freedom can—in principle—provide a thermodynamic resource, the presence of additional dissipation from NESSs suggests that care must be taken when determining information engine efficiency: two ratchets that produce the same asymptotic work may nevertheless produce (potentially very) different heats.

Taken together, these results demonstrate that combining familiar tools—average change in steady-state surprisal and a single integral fluctuation theorem—simplifies and generalizes deriving IPSLs. In turn, these bound the extent to which systems can leverage information-bearing degrees of freedom to support thermodynamic functionality. Furthermore, we showed explicitly how such inequalities arise from underlying equalities. This appeared in much the same way as stochastic thermodynamics' fluctuation theorems simplify to the original statement of the Second Law [26, 27].

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Appendix A: Trajectory versus State Averaging

The main result relies on the equivalence between $\langle \Delta\phi_\lambda \rangle$ and $\Delta \langle \mu | \phi_\lambda \rangle = \langle \mu_N | \phi_{\lambda_N} \rangle - \langle \mu_0 | \phi_{\lambda_0} \rangle$. The former refers to $\Delta\phi_\lambda$'s average over an ensemble $\{z_{0:N}\}$ of repeated trajectories and, thus, means $\langle \Delta\phi_\lambda \rangle = \langle \mathcal{W}_{\mathrm{ex}} \rangle - \langle \mathcal{Q}_{\mathrm{ex}} \rangle$. The latter refers to two specific state averages of ϕ_λ—namely, those at the trajectory's endpoints. And, it is equal to $\Delta H[Z] + \Delta D_{\mathrm{KL}}[Z \,\|\, \Lambda]$, via the arguments in Eq. (21). We establish the equivalence between the path and state averages of $\Delta\phi_\lambda$ here.

The *trajectory average* of a path-dependent functional $g : \mathcal{Z}^N \to \mathbb{R}$, denoted $\langle g \rangle$, is:

$$\langle g \rangle \doteq \int g(z_{0:N}) \Pr(z_{0:N}) \left(\prod_{i=0}^{N} \mathrm{d}z_i \right). \tag{A1}$$

We will extend this definition to functionals from $\mathcal{Z}^n \to \mathbb{R}$ for integer n, where $1 \leq n < N$, by simply placing the functional in the integral above while keeping the average over the full trajectory space $\mathcal{Z}^N$.

The *state average* of a function $f : \mathcal{Z} \to \mathbb{R}$, denoted $\langle \mu_i | f \rangle$, is:

$$\langle \mu_i | f \rangle \doteq \int f(z_i) \Pr(z_i) \, \mathrm{d}z_i \,. \tag{A2}$$

Claim. For any $f(z_n)$ that depends only on one point z_n, $0 \leq n \leq N$ in the path, the path and state averages are equal: $\langle f \rangle = \langle \mu_n | f \rangle$.

Proof. We explicitly evaluate the trajectory average. Consider two cases: (i) $n = N$, and (ii) $0 \leq n < N$.

(i) First, split the path probability into two pieces: $\Pr(z_{0:N}) = \Pr(z_{0:N-1}) \Pr(z_N \mid z_{0:N-1})$. Now, evaluate the integrals for $\mathrm{d}z_0$ through $\mathrm{d}z_{N-1}$:

$$\int \Pr(z_{0:N-1}) \Pr(z_N \mid z_{0:N-1}) \left(\prod_{i=0}^{N-1} \mathrm{d}z_i \right) = \Pr(z_N) \,,$$

by the law of total probability. The remainder is the $\mathrm{d}z_N$ integral:

$$\int \Pr(z_N) \, f(z_N) \, \mathrm{d}z_N = \langle \mu_N | f \rangle \,,$$

by definition.

(ii) Again split the probability, but now as $\Pr(z_{0:N}) = \Pr(z_{0:n}) \Pr(z_{n+1:N} \mid z_{0:n})$. Evaluate the integrals for $\mathrm{d}z_{n+1}$ through $\mathrm{d}z_N$:

$$\int \Pr(z_{n+1:N} \mid z_{0:n}) \left(\prod_{i=n+1}^{N} \mathrm{d}z_i \right) = 1$$

by probability conservation. What remains is exactly case (i).

□

This assumes a truly finite stochastic process, such that no conditioning before z_0 or after z_N is possible or relevant. However, the result is robust in the limit of a bi-infinite stochastic process. Evaluating the future integral in (ii) still yields 1 in the $N \to \infty$ limit. And, then, the past integral in (i) still gives $\Pr(z_n)$, even as the lower bound extends to $-\infty$.

Furthermore, we did not require Markovity, ergodicity, or stationarity for the underlying stochastic process. The result, then, appears quite general. This is not too surprising: a point function's average over paths should not depend on the path. Yet the link between path-independent ($\Delta\phi_\lambda$) and path-dependent ($\mathcal{W}_{\mathrm{ex}}$ and $\mathcal{Q}_{\mathrm{ex}}$) quantities provided by the nonaveraged First Law renders it particularly useful.

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