

State Diagnostics of Complexity in Open Quantum Systems

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We study the emergence of complexity in finite-size quantum systems as their dynamics transition from closed and coherent evolution to interacting and effectively open behavior. Using a state-based geometric framework, we represent mixed quantum states as probability measures on complex projective Hilbert space. This representation allows us to track how interactions reshape the underlying pure-state geometry. We introduce two complementary diagnostics: a distinguishability measure, based on the Wasserstein distance between probability-measure representations of mixed states, that quantifies sensitivity to initial states, and a state-space coverage index that measures long-time exploration of the subsystem state space. These diagnostics provide a geometric perspective on the emergence and evolution of quantum dynamical complexity. When applied to the quantum kicked top, both diagnostics generally increase with interaction strength. Their dependence on environment size is structured by parity symmetry, with integer-spin systems often exhibiting greater sensitivity and state-space coverage than half-integer-spin systems. These results highlight finite-size quantum effects and provide a geometric approach to quantifying dynamical complexity deep in the quantum regime.

Keywords: dynamical complexity, coherence, transition, geometric quantum states, probability measures, complex projective Hilbert space, Wasserstein distance, mixed states, finite-size effects, quantum kicked top

I. INTRODUCTION

What does it mean for a dynamical system to be complex? In classical mechanics, complexity has clear geometric interpretations: small perturbations grow, periodic motion breaks down, competing recurrent or periodic behaviors emerge, and phase-space structures reorganize [1–4]. In integrable systems, motion is periodic or quasiperiodic, with trajectories confined to invariant tori. When nonlinear interactions are introduced, these structures deform and eventually break down, giving rise to chaotic regions characterized by sensitive dependence on initial conditions and fractal geometry of invariant sets [5, 6]. Quantities such as the maximal Lyapunov exponent [7–9] and information dimension [10, 11] quantify this transition by measuring trajectory instability and geometric reorganization of invariant measures [12]. Classical complexity can thus be understood as the breakdown and reorganization of periodic or near-integrable phase-space structure under interactions.

Defining complexity in quantum systems is much more subtle [13, 14]. Global quantum evolution is linear and

unitary, preserving inner products and therefore the distinguishability between pure states. As a result, the classical notion of state-based trajectory instability does not directly carry over. Quantum complexity is often quantified using operator-based methods such as out-of-time-ordered correlators (OTOCs) [15–18], which probe operator growth, and the Loschmidt echo [19–21], which measures reversibility. In addition, Krylov complexity [22] and entanglement entropy [23] quantify information spreading and correlation generation.

These measures are powerful and capture key aspects of information scrambling and correlation generation in quantum systems. However, they do not directly describe how geometric structure of the state space reorganizes under interactions. In a closed system, unitary evolution preserves inner products and confines motion to invariant manifolds in complex projective space CP^{d-1} , leading to periodic or quasiperiodic dynamics analogous to integrable classical motion (Fig. 1(a),(c)). When interactions are introduced or the system becomes open, this periodic structure is reshaped at the level of reduced states, as decoherence spreads probability across the state space (Fig. 1(d)). Existing diagnostics lack a direct geometric analogue of classical torus-breaking or a description of how periodic structure becomes unstable.

To more directly address the emergence of these aspects of quantum complexity, the central idea of the following is that a geometric perspective emerges naturally

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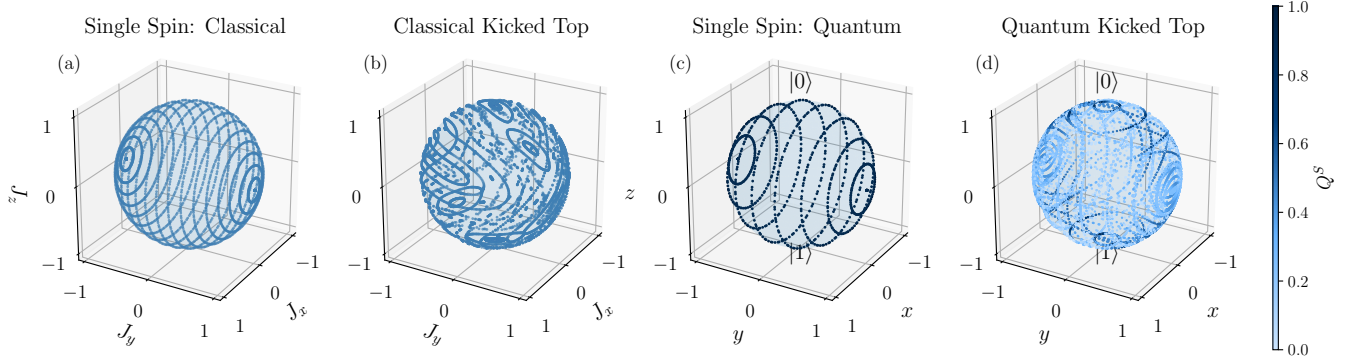


Figure 1: From periodic motion to interaction-induced complexity. (a) Classical single-spin precession ($\kappa = 0$). (b) Classical kicked top ($\kappa > 0$) showing trajectory deformation. (c) Closed single-spin quantum evolution with periodic motion on $\mathbb{CP}^1$. (d) Quantum kicked top, where interactions deform this periodic motion into an ensemble spreading over $\mathbb{CP}^1$.

when an interacting quantum system is partitioned into a system S and an environment E . In the absence of measurement, the global quantum state evolves linearly and unitarily. However, interactions between S and E generate entanglement, rendering the reduced state of S mixed. Beyond its representation as a density matrix, this reduced state can be expressed in a more geometrically resolved form as a probability distribution over pure states on complex projective Hilbert space. This environment-conditioned ensemble representation is what we refer to as a geometric quantum state (GQS) [24–27]. The reduced dynamics of S is then described as the evolution of probability measures on projective Hilbert space. Within this framework, the signature of quantum complexity is (i) interaction-induced spreading and (ii) reorganization of geometric quantum states on projective Hilbert space.

Given this, we introduce two complementary diagnostics of complexity within the geometric framework. First, we define a distinguishability measure based on the Wasserstein distance between nearby geometric quantum states. This serves as an open-system analogue of the classical maximal Lyapunov exponent. Rather than tracking the divergence of trajectories, it measures the separation of probability measures on projective Hilbert space. Second, to quantify long-time exploration, we introduce a State-Space Coverage Index (SSCI), defined by the Wasserstein proximity of the time-aggregated measure relative to the uniform distribution. These diagnostics capture finite-time instability as well as long-time, state-space-scale geometric reorganization induced by interactions.

We illustrate this framework using the quantum kicked top, taking a single qubit as the system S and the re-

maining qubits as its environment (Fig. 1(c),(d)). In the absence of interactions, this single-qubit state undergoes periodic precession on the Bloch sphere, mirroring integrable classical motion (Fig. 1(c)). When interactions are introduced, entanglement deforms this periodic evolution into a spreading ensemble on $\mathbb{CP}^1$ (Fig. 1(d)). The loss of periodic structure is reflected in a positive distinguishability measure and increased state-space coverage. In this way, complexity in open quantum systems parallels classical structural instability at the level of evolving probability measures.

Our approach restores a geometric interpretation of complexity in the open quantum setting. By treating reduced states as evolving ensembles and quantifying their instability using optimal transport, we provide a bridge between classical dynamical intuition and open quantum dynamics. This perspective complements existing operator-based diagnostics and provides a natural language for understanding how interactions reorganize quantum state space.

The following develops this framework systematically. Section II contrasts global unitary invariance with reduced-state dynamics, highlighting why instability arises at the level of reduced states. It introduces the geometric quantum state (GQS) representation, expressing reduced states as probability measures on complex projective space. Section III then develops the optimal transport geometry used to compare geometric quantum states, introducing Wasserstein distances on $\mathbb{CP}^{d-1}$. Building on this structure, Section IV defines two complementary diagnostics: a distinguishability measure capturing finite-time instability and a State-Space Coverage Index characterizing long-time exploration. Section V ap-

plies the diagnostics to the quantum kicked top, examining their dependence on interaction strength and environment size. We conclude in Section VI.

II. CLOSED VERSUS OPEN: WHY GEOMETRY MATTERS

We consider a finite-dimensional many-body quantum system whose total Hilbert space factorizes as

$$\mathcal{H}_{SE} = \mathcal{H}_S \otimes \mathcal{H}_E, \quad (1)$$

where $\dim \mathcal{H}_S = d_S$ and $\dim \mathcal{H}_E = d_E$. The subsystem S is the object of interest, while E denotes its environment.

The global system evolves under the Hamiltonian

$$H_{SE} = H_0 + H_I, \quad (2)$$

where H_0 generates local dynamics and H_I couples S and E . The total system is closed and evolves unitarily,

$$|\Psi_{SE}(t)\rangle = U(t)|\Psi_{SE}(0)\rangle, \text{ with } U(t) = e^{-iH_{SE}t/\hbar}. \quad (3)$$

A. Global Unitary Invariance

The global state is a pure vector in $\mathcal{H}_{SE}$, whose projective space is $\mathbb{C}P^{d_S d_E - 1}$. Consider two nearby pure states $|\Psi_{SE}(0)\rangle$ and $|\Psi'_{SE}(0)\rangle$. Their separation is quantified by the Fubini–Study distance

$$d_{FS}(|\Psi\rangle, |\Phi\rangle) = \arccos(|\langle\Psi|\Phi\rangle|). \quad (4)$$

Under unitary evolution,

$$\begin{aligned} \langle\Psi_{SE}(t)|\Psi'_{SE}(t)\rangle &= \langle\Psi_{SE}(0)|U^\dagger U|\Psi'_{SE}(0)\rangle \\ &= \langle\Psi_{SE}(0)|\Psi'_{SE}(0)\rangle, \end{aligned} \quad (5)$$

so that

$$d_{FS}(|\Psi_{SE}(t)\rangle, |\Psi'_{SE}(t)\rangle) = d_{FS}(|\Psi_{SE}(0)\rangle, |\Psi'_{SE}(0)\rangle). \quad (6)$$

Thus, on the full projective space $\mathbb{C}P^{d_S d_E - 1}$, global distinguishability is conserved for all times. Unlike classical trajectories, which may separate exponentially under nonlinear dynamics, the global quantum state exhibits no intrinsic trajectory instability.

B. Loss of Invariance and the Emergence of Mixed Subsystems

The situation changes when we restrict attention to the subsystem S . Studying a subsystem that interacts with

an environment (other subsystems) helps us understand how interactions shape dynamics of an open quantum system. Suppose the initial state is unentangled,

$$|\Psi_{SE}(0)\rangle = |\psi_S\rangle \otimes |\psi_E\rangle, \quad (7)$$

with $|\psi_S\rangle \in \mathcal{H}_S$ and $|\psi_E\rangle \in \mathcal{H}_E$.

Interactions generated by H_I generally produce entanglement, so that at later times

$$|\Psi_{SE}(t)\rangle = \sum_{k=1}^{d_S} \sum_{j=1}^{d_E} \psi_{kj}(t) |s_k\rangle \otimes |e_j\rangle, \quad (8)$$

where $\{|s_k\rangle\}$ and $\{|e_j\rangle\}$ form orthonormal bases of $\mathcal{H}_S$ and $\mathcal{H}_E$, respectively. In general, this state is no longer separable.

The subsystem state is obtained by tracing out the environment,

$$\rho_S(t) = \text{Tr}_E(|\Psi_{SE}(t)\rangle\langle\Psi_{SE}(t)|). \quad (9)$$

Even though the global state remains pure in $\mathbb{C}P^{d_S d_E - 1}$, the reduced state $\rho_S(t)$ is, in general, mixed and can no longer be described by a single vector or pure state $|\psi_S\rangle \in \mathcal{H}_S$.

C. From Pure States to Density Matrices

The geometric object describing the dynamics changes fundamentally when we pass from a closed quantum system to an open one.

For the closed composite system $S + E$ with Hilbert space $\mathcal{H}_{SE} \cong \mathbb{C}^{d_S d_E}$, pure states evolve on the projective manifold $\mathbb{C}P^{d_S d_E - 1}$.

Unitary evolution preserves the Fubini–Study distance on this manifold. When we restrict attention to the subsystem S , however, the situation changes. Pure states of S live on $\mathbb{C}P^{d_S - 1}$, whereas mixed states are elements of the convex set

$$\mathcal{D}_{d_S} = \{\rho \in \text{Herm}(d_S) \mid \rho \geq 0, \text{Tr}(\rho) = 1\}, \quad (10)$$

where $\text{Herm}(d_S)$ denotes the space of $d_S \times d_S$ Hermitian operators. Points in $\mathcal{D}_{d_S}$ represent statistical mixtures rather than individual pure-state configurations. Thus, open-system dynamics no longer corresponds to motion on a projective state-space manifold, but to evolution within the density-operator space $\mathcal{D}_{d_S}$.

For initially pure subsystem states, distinguishability is measured by the Fubini–Study distance

$$d_{FS}(|\psi_S\rangle, |\psi'_S\rangle) = \arccos(|\langle\psi_S|\psi'_S\rangle|). \quad (11)$$

If the subsystem evolved unitarily, this distance would be conserved. However, once entanglement with the environment develops, the reduced dynamics becomes non-unitary. Distinguishability between mixed states can be quantified by operator-level metrics such as the Bures distance,

$$d_B(\rho_S, \rho'_S) = \sqrt{2 - 2\sqrt{F(\rho_S, \rho'_S)}}, \quad (12)$$

where $F(\rho_S, \rho'_S)$ denotes the quantum fidelity, defined as $F(\rho_S, \rho'_S) = [\text{Tr} \sqrt{\sqrt{\rho_S} \rho'_S \sqrt{\rho_S}}]^2$.

In general,

$$d_B(\rho_S(t), \rho'_S(t)) \neq d_B(\rho_S(0), \rho'_S(0)), \quad (13)$$

which indicates that the reduced dynamics does not, in general, preserve distances.

D. Limitations of the Density Matrix Description

The transition from the smooth projective Hilbert space of pure states to the operator space of density matrices makes the direct translation of classical, state-based complexity measures difficult.

Every point on $\mathbb{CP}^{d_S-1}$ corresponds uniquely to a single pure quantum state, up to an overall phase. In contrast, points in the operator space $\mathcal{D}_{d_S}$ do not admit such a one-to-one interpretation. A mixed density matrix generally admits infinitely many convex decompositions,

$$\rho_S = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|, \quad \lambda_j \geq 0, \quad \sum_j \lambda_j = 1, \quad (14)$$

and these distinct ensembles generally correspond to different probability measures on $\mathbb{CP}^{d_S-1}$.

However, all such decompositions map to the same operator ρ_S . Thus the space $\mathcal{D}_{d_S}$ identifies many geometrically distinct probability distributions on the projective manifold $\mathbb{CP}^{d_S-1}$ as a single point. Consequently, operator-level distances such as the Bures metric compare density matrices only at the level of their averaged statistical effect. They do not uniquely capture how probability mass is arranged across $\mathbb{CP}^{d_S-1}$. In particular, two density matrices may therefore be arbitrarily close in Bures distance while their underlying pure-state components occupy widely separated regions of projective space. This degeneracy becomes particularly significant when formulating a quantum analogue of classical trajectory instability. If one wishes to track how quantum ensembles spread, reorganize, or develop geometric structure, then working solely within $\mathcal{D}_{d_S}$ is insufficient.

E. From Operator Geometry to Geometric Quantum States

A natural ensemble structure emerges directly from the global system-environment state. Expanding the global wavefunction introduced in Eq. (8) in the environment basis $\{|e_j\rangle\}$ yields the environment-conditioned decomposition

$$|\Psi_{SE}(t)\rangle = \sum_{j=1}^{d_E} \sqrt{\lambda_j^E(t)} |\chi_j^S(t)\rangle |e_j\rangle. \quad (15)$$

Throughout this work, the environment basis $\{|e_j\rangle\}$ is taken to be the computational basis. For an environment of L_E qubits, where $d_E = 2^{L_E}$, this is the product basis

$$\{|e_j\rangle\}_{j=1}^{2^{L_E}} = \{|b_1 b_2 \cdots b_{L_E}\rangle : b_\ell \in \{0, 1\}\}.$$

Here, j simply enumerates the 2^{L_E} binary strings. Although the resulting geometric quantum states depend on this choice of conditioning, it provides a natural reference for qubit systems. For example, when $d_E = 4$,

$$\{|e_j\rangle\}_{j=1}^4 = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}.$$

The conditional subsystem states are:

$$|\chi_j^S(t)\rangle = \frac{1}{\sqrt{\lambda_j^E(t)}} \sum_{k=1}^{d_S} \psi_{kj}(t) |s_k\rangle, \quad (16)$$

with associated probabilities

$$\lambda_j^E(t) = \sum_{k=1}^{d_S} |\psi_{kj}(t)|^2. \quad (17)$$

The set of conditional pure states $\{|\chi_j^S(t)\rangle\}$ together with probabilities $\{\lambda_j^E(t)\}$ therefore defines a probability distribution over the projective Hilbert space $\mathbb{CP}^{d_S-1}$. Terms with $\lambda_j^E(t) = 0$ are omitted, since the corresponding conditional states in Eq. (16) are undefined. Interactions with the environment redistribute probability weight among these components, reshaping the geometric organization of the subsystem ensemble on projective space, as illustrated in Fig. 2. Tracing over the environment yields the reduced density matrix

$$\rho_S(t) = \sum_{j=1}^{d_E} \lambda_j^E(t) |\chi_j^S(t)\rangle\langle\chi_j^S(t)|. \quad (18)$$

While $\rho_S(t)$ reproduces all observable statistics of the subsystem, it encodes this ensemble structure only implicitly. Distinct probability distributions on $\mathbb{CP}^{d_S-1}$ may

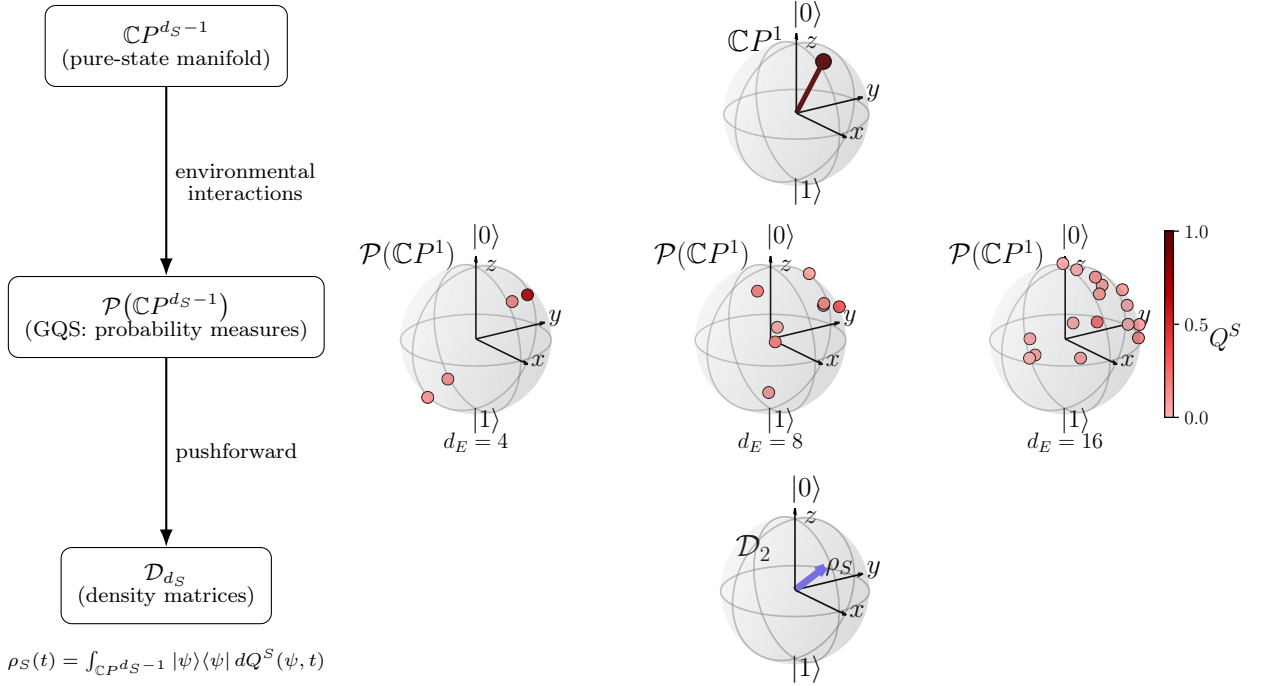


Figure 2: Left: Relation between pure states, GQS, and density matrices. Right: For a single-qubit subsystem, interactions with different environment sizes ($L_E = 2, 3, 4$, $d_E = 2^{L_E}$) produce distinct ensembles on $\mathbb{CP}^1$ with the same reduced density matrix ρ_S , showing that ρ_S does not capture the geometric structure induced by environment conditioning.

correspond to the same density matrix, so the operator description does not retain information about how environment conditioning arranges probability mass on the manifold itself.

For this reason we lift the reduced dynamics to the space of probability measures on projective Hilbert space, $\mathcal{P}(\mathbb{CP}^{d_S-1})$, and represent the subsystem state as a geometric quantum state (GQS)

$$Q^S(Z, t) = \sum_{j=1}^{d_E} \lambda_j^E(t) \delta_{\mathbf{Z}_j^S(t)}^Z \in \mathcal{P}(\mathbb{CP}^{d_S-1}). \quad (19)$$

Each conditional pure state

$$|\chi_j^S(t)\rangle = \sum_{k=1}^{d_S} c_k^{(j)}(t) |s_k\rangle \quad (20)$$

defines a point in projective Hilbert space

$$\mathbf{Z}_j^S(t) = [c_1^{(j)}(t) : \dots : c_{d_S}^{(j)}(t)] \in \mathbb{CP}^{d_S-1}. \quad (21)$$

If the subsystem and environment remain unentangled, the distribution collapses to a single point

$$Q^S(Z, t) = \delta_{\mathbf{Z}^S(t)}^Z, \quad (22)$$

corresponding to a pure state in $\mathbb{CP}^{d_S-1}$.

Fig. 2 illustrates this distinction for a single qubit. In the density matrix picture, a mixed state corresponds to a single point inside the Bloch ball. In the GQS representation, the same state is described as a probability distribution over pure states on the Bloch sphere ($\mathbb{CP}^1$), thereby retaining information about how probability mass is arranged on the manifold itself upon conditioning on the environment. The density matrix can be recovered as the pushforward:

$$\rho_S(t) = \int_{\mathbb{CP}^{d_S-1}} |\psi\rangle\langle\psi| dQ^S(\psi, t), \quad (23)$$

but the measure Q^S retains additional environment-conditioned geometric information not specified by ρ_S alone.

In this framework, instability is formulated not in terms of operator differences in $\mathcal{D}_{d_S}$ but as the evolution of probability mass on $\mathbb{CP}^{d_S-1}$. Crucially, $\mathbb{CP}^{d_S-1}$ is a smooth Riemannian manifold equipped with the Fubini-Study metric, allowing transport-based distances to be defined directly between subsystem ensembles. Thus, the GQS

representation restores a geometric description of subsystem dynamics in terms of probability measures evolving on a smooth manifold, enabling the application of transport-based metrics analogous to those used in classical dynamical systems.

III. OPTIMAL TRANSPORT ON PROJECTIVE HILBERT SPACE

Subsystem states are represented as probability measures on the projective Hilbert space $\mathbb{C}P^{d_S-1}$ within the geometric quantum state framework (Sec. II). To quantify distances between such measures, we employ the Wasserstein metric from optimal transport theory [28].

Quantum extensions of optimal transport and related distance-based approaches have been developed in operator settings, including formulations on density matrices and quantum channels [29–32]. In contrast, the geometric quantum formulation adopted here works at the level of probability measures over pure states. This allows us to use classical optimal transport definitions and solvers, with the Fubini–Study metric providing the natural quantum cost on projective Hilbert space.

Consider two geometric quantum states

$$Q^S(Z) = \sum_i \lambda_i^S \delta_{\mathbf{z}_i^S}^Z, \quad (24)$$

$$Q^{S'}(Z) = \sum_j \lambda_j^{S'} \delta_{\mathbf{z}_j^{S'}}^Z. \quad (25)$$

The Wasserstein distance of order p between these measures is defined as

$$W_p(Q^S, Q^{S'}) = \left(\inf_{\pi \in \Pi(Q^S, Q^{S'})} \sum_{i,j} \pi_{ij} d_{FS}(\mathbf{z}_i^S, \mathbf{z}_j^{S'})^p \right)^{1/p}, \quad (26)$$

where d_{FS} denotes the Fubini–Study distance on $\mathbb{C}P^{d_S-1}$.

The matrix π_{ij} represents a transport plan satisfying

$$\pi_{ij} \geq 0, \quad (27)$$

$$\sum_j \pi_{ij} = \lambda_i^S, \quad (28)$$

$$\sum_i \pi_{ij} = \lambda_j^{S'}. \quad (29)$$

The set $\Pi(Q^S, Q^{S'})$ denotes the collection of all such admissible couplings between the two distributions. The Wasserstein distance therefore measures the minimal cost required to transport probability mass from one geometric quantum state to another, where the transport cost is

determined by the Fubini–Study distance between pure states on projective Hilbert space. Whereas the Bures metric compares density operators directly, the Wasserstein distance compares probability measures over pure states and therefore captures geometric rearrangements of subsystem ensembles on $\mathbb{C}P^{d_S-1}$ upon interactions with the environment.

Throughout this following we focus on the case $p = 1$, referred to as the Wasserstein distance $W_1(Q^S, Q^{S'})$. For pure states the Wasserstein distance reduces to the Fubini–Study distance,

$$W_1(\delta_\psi, \delta_\phi) = d_{FS}(|\psi\rangle, |\phi\rangle), \quad (30)$$

ensuring consistency with the underlying geometry of projective Hilbert space. The Wasserstein distance provides a unified way to track the evolution of subsystem states as they transition from pure to mixed under interactions with the environment. Details of the discrete formulation and numerical implementation used in this work are provided in Appendix C. The following section uses this metric to define a dynamical diagnostic of complexity based on how nearby geometric quantum states evolve under interactions.

IV. STATE-BASED DIAGNOSTICS OF COMPLEXITY

A. Distinguishability Measure

Sensitivity to initial conditions is a defining feature of complexity in classical dynamical systems, typically quantified by the maximal Lyapunov exponent, which measures the exponential divergence of nearby trajectories in phase space.

In open quantum systems, subsystem states do not follow single trajectories. Interactions generate mixed states, redistributing probability from an initially localized state across $\mathbb{C}P^{d_S-1}$. In the geometric quantum framework, this evolution is described by geometric quantum states (GQS), i.e., probability measures on $\mathbb{C}P^{d_S-1}$. This motivates replacing trajectory-based diagnostics with an ensemble-based notion of sensitivity. Using the Wasserstein distance (Sec. III), we quantify how nearby subsystem ensembles evolve under identical dynamics.

Let Q^S denote a reference GQS, and let $\{Q_m^{S'}\}_{m=1}^M$ be a collection of nearby perturbed states characterized by the perturbation scale ϵ , such that $Q_m^{S'} \rightarrow Q^S$ as $\epsilon \rightarrow 0$. For each perturbation, we define the time-dependent separation between the evolved perturbed and reference

GQSS as

$$W_{1,m}(t) = W_1(Q_m^{S'}(Z, t), Q^S(Z, t)). \quad (31)$$

Here, $Q^S(Z, t)$ is the evolved reference state and $Q_m^{S'}(Z, t)$ denotes the evolution of the m -th perturbation. The distinguishability measure is then

$$\Gamma = \frac{1}{T} \sum_{t=0}^{T-1} \left\langle \ln \left[\frac{W_{1,m}(t)}{W_{1,m}(0)} \right] \right\rangle_m, \quad (32)$$

where $\langle \cdot \rangle_m$ denotes an average over the M perturbations. This average reduces dependence on a particular perturbation and captures the typical distinguishability growth near the reference GQS.

The parameter T specifies the duration over which the distinguishability measure is averaged. In the presence of recurrences, T is chosen according to the numerically estimated recurrence timescale; otherwise, it is taken sufficiently large to capture the long-time behavior of the distinguishability measure.

The quantity Γ measures the average logarithmic growth of the Wasserstein distance between initially nearby subsystem ensembles. In this sense, it acts as a Lyapunov-type diagnostic on the space of probability measures $\mathcal{P}(\mathbb{C}P^{d_S-1})$. In an isolated system undergoing unitary evolution, states remain pure and the Wasserstein distance reduces to the Fubini-Study distance (Eq. 30), so separations are preserved and $\Gamma = 0$. With interactions, the reduced dynamics may either amplify or contract ensemble separations.

A positive Γ signals sensitivity and complex dynamics; $\Gamma = 0$ corresponds to preserved distinguishability, as in the noninteracting periodic case; and $\Gamma < 0$ indicates average contraction over the observation window. Ensemble distinguishability thus provides a natural extension of classical sensitivity measures to open quantum systems.

While Γ captures finite-time sensitivity, complexity also appears in long-time organization. To quantify this complementary aspect, we next introduce the state-space coverage index.

B. State-Space Coverage Index

While the distinguishability measure Γ captures local sensitivity, it does not quantify how extensively the underlying state space is explored. In classical systems, sensitivity and global organization are distinct because trajectories may remain confined to invariant structures such as tori, or they may spread across larger regions of phase space as these structures break down. The latter is often

quantified by geometric measures such as the information dimension [10, 11], which characterizes the long-time spread of dynamics (see Appendix D).

In the present setting, where subsystem dynamics are described by probability distributions on projective Hilbert space, an analogous question arises: to what extent do interactions redistribute probability mass across the state space? To capture this, we introduce a geometric diagnostic that measures the long-time spread of subsystem states on $\mathbb{C}P^{d_S-1}$.

Let $M = \mathbb{C}P^{d_S-1}$ denote projective Hilbert space equipped with the Fubini-Study distance, and let $\sigma \in \mathcal{P}(M)$ denote the uniform distribution over M . For each time t , the geometric quantum state defines a probability measure

$$Q^S(\cdot, t) \in \mathcal{P}(M). \quad (33)$$

We define the time-aggregated geometric quantum state as

$$\nu^S(A) = \frac{1}{T_S} \sum_{t=0}^{T_S-1} Q^S(A, t), \quad A \subset M, \quad (34)$$

where T_S is the averaging time. This measure represents the aggregate distribution of subsystem states across projective Hilbert space over T_S time steps. Numerically, T_S is taken sufficiently large so that ν^S approximates the long-time aggregate distribution. Formally, this corresponds to the limit $T_S \rightarrow \infty$, when the limit exists.

The State-Space Coverage Index (SSCI) is then defined as:

$$\mathcal{S}_p = 1 - \frac{W_p(\nu^S, \sigma)}{W_p(\delta_{Z_0}^Z, \sigma)}, \quad (35)$$

where W_p denotes the p -Wasserstein distance. Throughout the numerical analysis, we take $p = 1$ and therefore report $\mathcal{S}_1$, which measures coverage using the same Wasserstein geometry used for the distinguishability measure. Here, σ is the uniform probability measure on $\mathbb{C}P^{d_S-1}$ induced by the Fubini-Study geometry. It represents the reference case of uniform coverage, in which no pure state in state space is preferred. The measure $\delta_{Z_0}^Z$ is a Dirac measure concentrated at a single point Z_0 and represents maximal localization, corresponding to a distribution concentrated on a single pure state. Thus, $W_p(\delta_{Z_0}^Z, \sigma)$ measures the distance between the maximally localized distribution and the uniformly spread distribution. Because all pure states are geometrically equivalent under the Fubini-Study geometry, the choice of Z_0 does not affect this normalization, so $W_p(\delta_{Z_0}^Z, \sigma)$ provides a unique normalization scale for measuring state-space coverage.

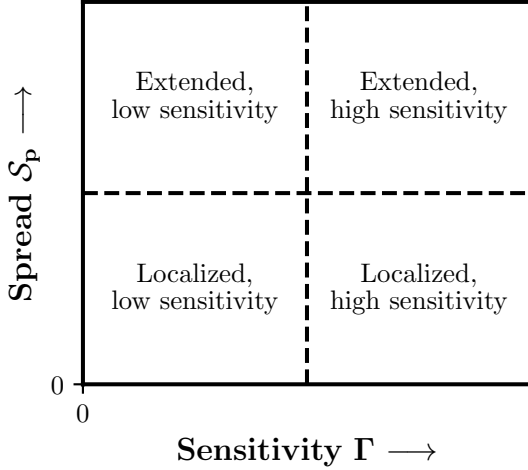


Figure 3: Schematic classification of subsystem dynamics in the nonnegative- Γ regime using the pair $(\Gamma, \mathcal{S}_p)$. Increasing Γ indicates greater sensitivity to initial states, while increasing $\mathcal{S}_p$ indicates broader long-time exploration of state space. The four quadrants distinguish localized low-sensitivity, extended low-sensitivity, localized high-sensitivity, and extended high-sensitivity dynamics.

By construction, $\mathcal{S}_p = 1$ if and only if $\nu^S = \sigma$, corresponding to uniform exploration of the state space. In this case, probability is uniformly distributed over $\mathbb{CP}^{d_S-1}$, so no pure state in state space is preferred. In contrast, $\mathcal{S}_p = 0$ corresponds to maximal localization, where the distribution remains concentrated on a single pure state throughout the evolution. Intermediate values indicate increasing long-time spread across $\mathbb{CP}^{d_S-1}$. Thus, $\mathcal{S}_p$ captures the global geometric footprint of subsystem dynamics by quantifying how broadly probability mass is distributed over projective Hilbert space, independently of local separation rates.

The pair $(\Gamma, \mathcal{S}_p)$ provides a two-dimensional characterization of subsystem dynamics by separating sensitivity to initial states from long-time state-space exploration, as summarized schematically in Fig. 3 for the nonnegative- Γ regime. Small Γ and small $\mathcal{S}_p$ indicate localized dynamics with weak sensitivity, as in periodic or quasiperiodic motion confined to a small region of $\mathbb{CP}^{d_S-1}$. Small Γ but large $\mathcal{S}_p$ indicates weakly sensitive dynamics that explore an extended region of state space. Larger Γ corresponds to stronger distinguishability growth and hence greater sensitivity. When $\mathcal{S}_p$ remains small, this sensitivity is spatially confined; when $\mathcal{S}_p$ is large, it is accompanied by broad exploration of the subsystem state space.

The State-Space Coverage Index thus complements the distinguishability measure by quantifying the extent of exploration, enabling a geometric interpretation of subsystem dynamics in terms of both sensitivity and spread. All values reported below are computed using conditioning in the computational environment basis and are therefore basis dependent.

V. SENSITIVITY AND STATE-SPACE EXPLORATION IN THE QUANTUM KICKED TOP

We apply the framework developed in Sec. IV to the quantum kicked top. The quantum kicked top is a many-body system that serves as a foundational model for studying the transition from periodic to complex behavior [33], with accessible experimental realizations [23, 34–36], as well as a classically chaotic counterpart, making it an ideal example.

The quantum kicked top is described by the time-dependent Hamiltonian:

$$H(t) = \frac{\pi}{2\tau} J_y + \frac{\kappa}{2j} J_z^2 \sum_{n \in \mathbb{Z}} \delta(t - n\tau), \quad (36)$$

where the first term generates a rotation about the y -axis and the second term introduces nonlinear interactions between the qubits. Throughout this work, we set $\hbar = 1$ and $\tau = 1$. Time is therefore measured in units of the kick period, with $t_n = n$ denoting the number of Floquet kicks. We have $J_y = \frac{\hbar}{2} \sum_{i=1}^L \sigma_y^{(i)}$ and $J_z = \frac{\hbar}{2} \sum_{i=1}^L \sigma_z^{(i)}$, which makes the nonlinear term

$$J_z^2 = \frac{\hbar^2}{4} \left(LI + 2 \sum_{i < k} \sigma_z^{(i)} \sigma_z^{(k)} \right). \quad (37)$$

The identity term contributes only an overall phase, while the remaining terms describe homogeneous pairwise interactions between all qubits. Thus, the interaction strength κ controls the collective coupling, with $L = 2j$ denoting the total number of qubits and j the total spin.

For $L = 3$, the all-to-all interaction graph is equivalent to a three-site ring with periodic nearest-neighbor couplings. For $L > 3$, the collective J_z^2 term also couples non-nearest-neighbor qubits, introducing additional interaction pathways beyond those present in the three-qubit system.

In this setting, the subsystem and environment dimensions introduced earlier correspond to $d_S = 2^{L_S}$ and $d_E = 2^{L_E}$, with $L = L_S + L_E$. The corresponding Floquet operator

$$U = e^{-i\kappa J_z^2 / 2j\hbar} e^{-i\pi J_y / 2\hbar} \quad (38)$$

governs the discrete-time unitary evolution and drives the system away from product structure through the generation of multipartite entanglement.

We initialize the system in a globally pure spin-coherent product state,

$$|\psi(\theta, \phi)\rangle = \cos(\frac{\theta}{2}) |0\rangle + e^{i\phi} \sin(\frac{\theta}{2}) |1\rangle, \quad (39)$$

$$|\Psi_{SE}(0)\rangle = \bigotimes_{\ell=1}^L |\psi(\theta, \phi)\rangle. \quad (40)$$

We then focus on the reduced dynamics of a single-qubit subsystem ($L_S = 1$, $d_S = 2$). Subsystem states are represented as geometric quantum states (GQS) obtained by conditioning on an environment basis, taken throughout to be the computational basis.

At the initial time, this subsystem is described by a localized geometric quantum state,

$$Q^S(Z, 0) = \delta_{\mathbf{Z}^S}^Z, \quad (41)$$

where $\mathbf{Z}^S \in \mathbb{CP}^1$ is the point corresponding to $|\psi(\theta, \phi)\rangle$ on the Bloch sphere.

To probe sensitivity to initial conditions, we perturb the initial spin-coherent state using small isotropic random rotations with Gaussian-distributed rotation angles of standard deviation ϵ . This generates an ensemble of nearby product states,

$$|\Psi'_{SE_m}(0)\rangle = \bigotimes_{\ell=1}^L |\psi(\theta'_m, \phi'_m)\rangle, \quad (42)$$

where (θ'_m, ϕ'_m) denotes the perturbed Bloch-sphere coordinates of the m th state.

The corresponding subsystem states are then

$$Q_m^{S'}(Z, 0) = \delta_{\mathbf{Z}_m^{S'}}^Z, \quad (43)$$

with $\mathbf{Z}_m^{S'} \in \mathbb{CP}^1$.

Both the reference and perturbed states are evolved under the same Floquet operator. Their subsequent dynamics are compared using the distinguishability measure Γ and the state-space coverage index $\mathcal{S}_1$, which quantify, respectively, the local instability and long-time exploration of subsystem states. The average $\langle \cdot \rangle$ is taken over the ensemble of perturbations to ensure that the measured sensitivity reflects intrinsic dynamical behavior rather than dependence on a particular choice of initial perturbation.

In the following subsections, we examine how varying the interaction strength κ and system size L drives a transition from periodic, confined dynamics to unstable and globally spreading behavior.

A. From Periodic Motion to Interaction-Induced Deformation

We first examine how interactions reshape subsystem dynamics in a three-qubit quantum kicked top, where interactions are limited to nearest-neighbor couplings, allowing us to isolate the onset of interaction-induced complexity. Figure 4 shows the evolution of two nearby geometric quantum states of a qubit for increasing interaction strength κ , starting from a localized spin-coherent state.

For $\kappa = 0$, the subsystem state remains localized and follows a periodic trajectory on the Bloch sphere, confined to a narrow region of $\mathbb{CP}^1$, analogous to classical motion on invariant tori. Distinguishability between two nearby states is preserved at all times.

For $\kappa > 0$, this structure is progressively deformed. The distribution is sheared and stretched, and probability mass redistributes across projective Hilbert space.

The motion loses periodicity, and the geometric quantum state develops increasingly complex structure. At larger κ , this deformation leads to broad coverage of $\mathbb{CP}^1$, indicating substantial reorganization of the subsystem state. This transition mirrors the classical breakdown of invariant phase-space structures, now realized through the redistribution of probability mass on projective Hilbert space.

These observations suggest interaction-induced sensitivity at the level of subsystem ensembles, which we quantify next.

1. Local Instability: Distinguishability Measure

To quantify the sensitivity suggested by the deformation in Fig. 4, we compute the distinguishability measure Γ (Sec. IV), which captures the average separation of nearby geometric quantum states under time evolution.

We first consider the spin-coherent initial state $(\theta, \phi) = (\pi/2 + 0.5, \pi/2)$ and track the evolution of the Wasserstein distance between two nearby subsystem states for varying interaction strength κ . The resulting distance evolution is shown in Fig. 5(a). In the absence of interactions ($\kappa = 0$), the distance remains constant, reflecting stable, coherent dynamics. As κ increases, the distance shows stronger growth over time, indicating increasing sensitivity to perturbations.

This behavior is summarized in Fig. 5(b), which shows the distinguishability measure Γ as a function of κ . For small κ , the sensitivity Γ is small, consistent with periodic or quasiperiodic dynamics. As κ increases, Γ becomes positive, signaling the onset of instability in subsystem dynamics.

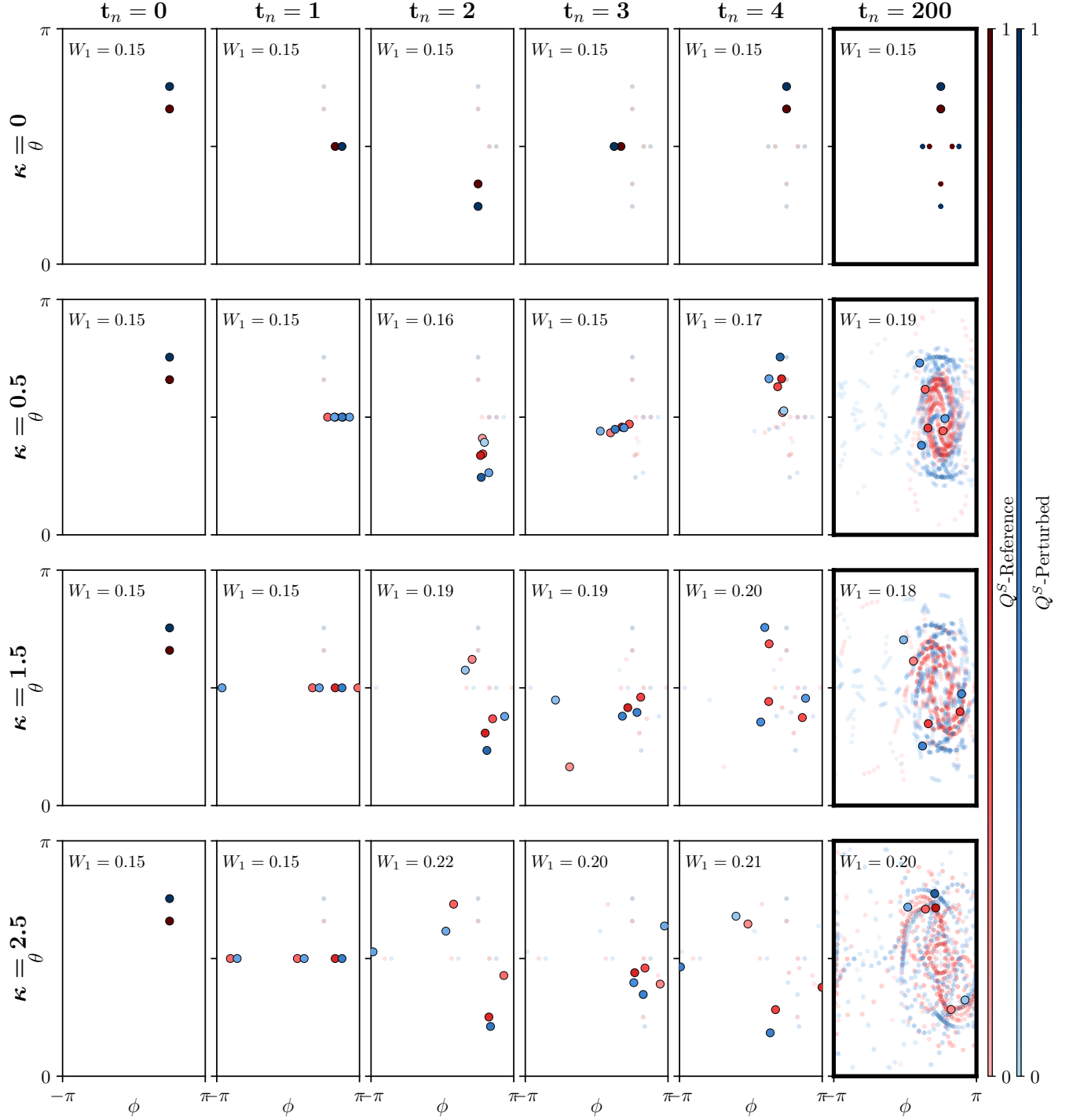


Figure 4: Evolution of two neighboring geometric quantum states in the three-qubit quantum kicked top. Columns show environment-conditioned ensembles (GQSs) on $\mathbb{CP}^1$ after $t_n = 0, 1, 2, 3, 4$, and 200 Floquet kicks, while rows correspond to interaction strengths κ . The separation $W_1(t_n) = d_{FS}$ is preserved for noninteracting dynamics ($\kappa = 0$). For $\kappa > 0$, $W_1(t_n)$ varies in time, indicating loss of invariance as the ensembles deform and spread across state space, with stronger effects at larger κ .

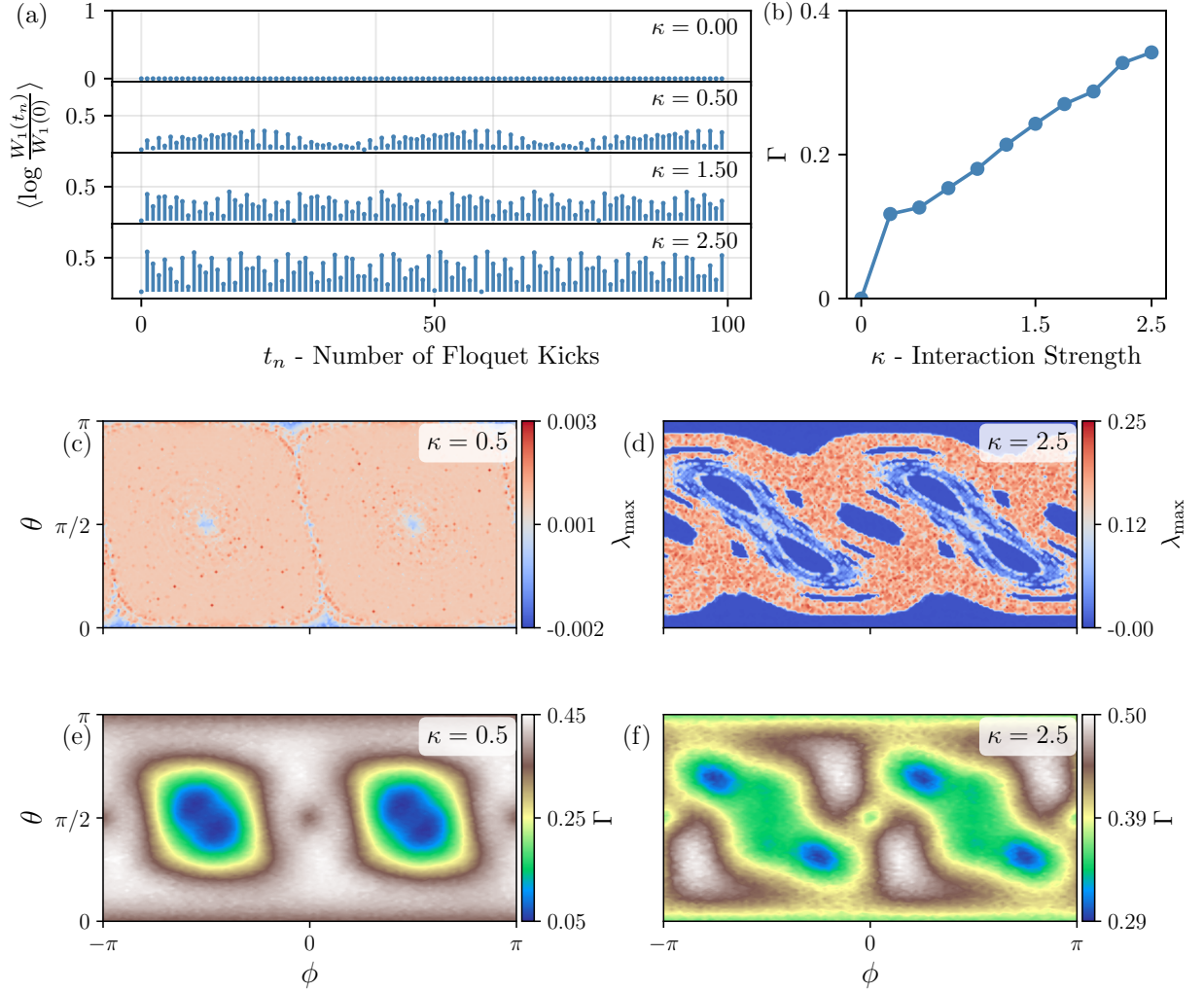


Figure 5: Distinguishability growth and phase-space structure in the three-qubit kicked top. (a) Time evolution of the average relative distance between perturbed and unperturbed GQs for increasing kick strength κ , starting from a spin-coherent state at $(\theta, \phi) = (\pi/2 + 0.5, \pi/2)$. (b) Corresponding distinguishability measure Γ as a function of κ . (c,d) Classical maximal Lyapunov exponent $\lambda_{\max}$ over initial conditions (θ, ϕ) for $\kappa = 0.5$ and 2.5 . (e,f) Quantum distinguishability measure Γ evaluated over spin-coherent product states for the same interaction strengths. Increasing κ enhances both $\lambda_{\max}$ in the classical case and the distinguishability growth Γ in the quantum case. The classical and quantum phase-space maps also exhibit qualitative similarities, indicating that the underlying kicked-top geometry continues to organize the reduced quantum dynamics.

To probe initial-state dependence, we compute Γ across all spin-coherent states on the Bloch sphere. Figures 5(e,f) show that Γ exhibits a strongly structured dependence on the initial condition. For moderate κ , regions of low and high sensitivity coexist, while for larger κ , regions of enhanced instability expand across the state space.

This structure closely parallels the phase-space behavior of the classical kicked top (Figs. 5(c,d)), where periodic and chaotic regions coexist, but here it emerges through evolution of probability measures on projective Hilbert space.

These results show that interactions induce a transition from stable to unstable subsystem dynamics, with sensi-

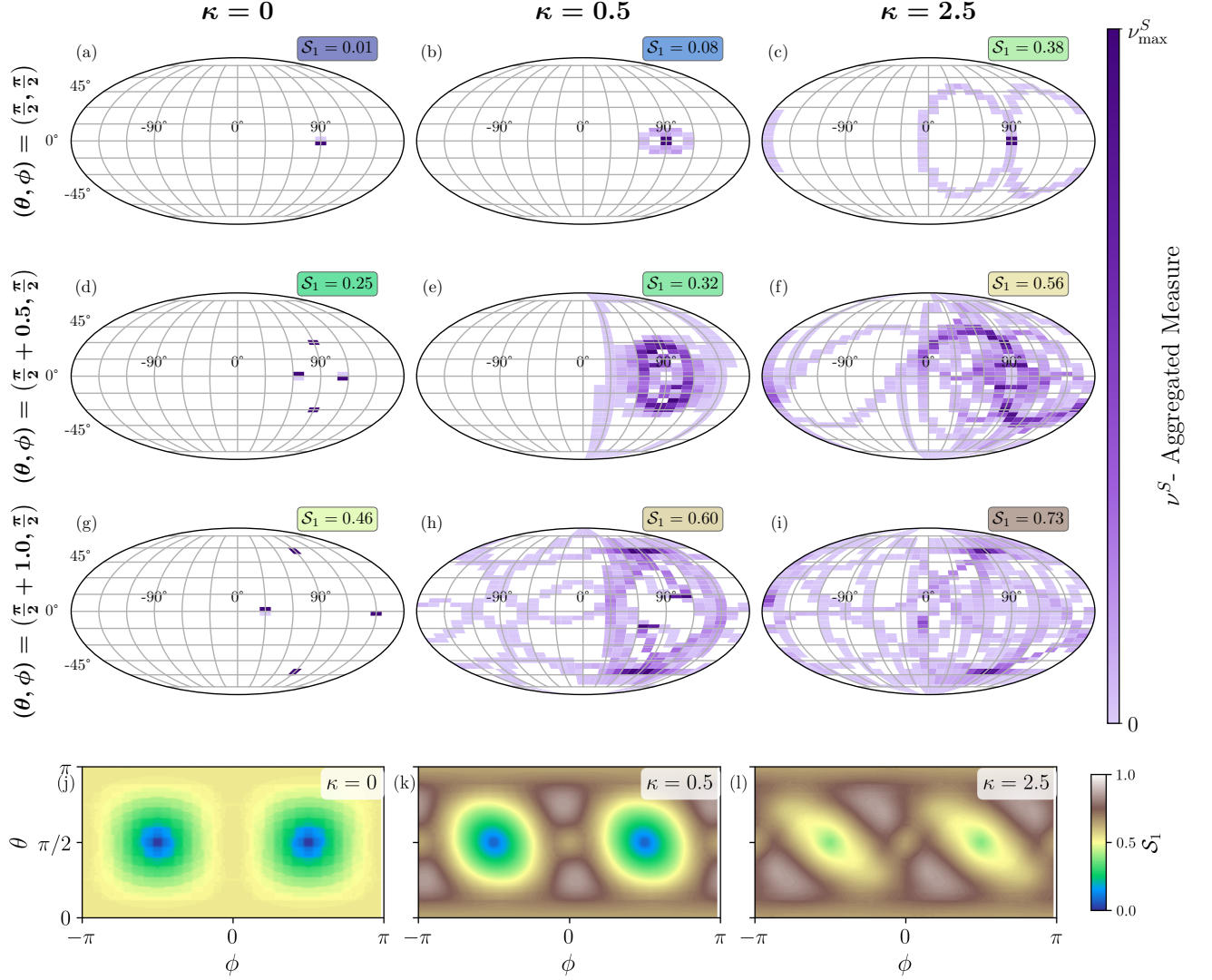


Figure 6: Interaction-induced spreading in the three-qubit kicked top. (a-i) Time-averaged measures ν^S for three initial states (rows) and $\kappa = 0, 0.5, 2.5$ (columns). For $\kappa = 0$, dynamics remain periodic on $\mathbb{CP}^1$, yielding nonzero $\mathcal{S}_1$ set by orbit geometry: states near $(\pi/2, \pi/2)$ remain confined, while others explore larger regions. With increasing κ , probability mass is redistributed across the sphere, transitioning from structured spreading to broad coverage. (j-l) Corresponding $\mathcal{S}_1$ values over all initial states, showing increased exploration and persistent dependence on initial condition.

tivity that is both interaction-dependent and geometrically organized across state space.

2. Sensitivity and Spread: A Two-Dimensional Picture

While the distinguishability measure Γ quantifies local instability of subsystem dynamics, it does not capture how extensively the dynamics explore the underlying state space. To characterize this complementary aspect, we use the State-Space Coverage Index $\mathcal{S}_1$

(Sec. IV), which measures the long-time spread of the time-aggregated geometric quantum state on $\mathbb{CP}^1$.

Figure 6 shows the time-averaged geometric quantum state for three representative initial conditions and increasing interaction strength κ . For $\kappa = 0$, the dynamics is periodic and $\Gamma = 0$ for all initial states. Nevertheless, $\mathcal{S}_1$ remains nonzero, reflecting periodic motion on $\mathbb{CP}^1$. Its value is set by orbit geometry. The spin-coherent state at $(\pi/2, \pi/2)$, which is an eigenstate of $(\pi/2\tau)J_y$, remains confined and yields small $\mathcal{S}_1$, while states farther away trace larger orbits and produce greater coverage.

As interactions are introduced ($\kappa > 0$), probability mass is progressively redistributed across the Bloch sphere and $\mathcal{S}_1$ increases. For intermediate κ , the distribution spreads along structured regions, while for larger κ it covers a substantial portion of $\mathbb{CP}^1$, indicating enhanced global exploration. This spreading remains strongly dependent on the initial state, with states farther from $(\pi/2, \pi/2)$ exhibiting larger coverage across all κ .

This dependence is summarized in Fig. 6(j-l), which shows $\mathcal{S}_1$ over all spin-coherent states. As κ increases, regions of low coverage shrink while regions of high coverage expand, reflecting a transition from confined motion to broad exploration. Importantly, the spatial structure of $\mathcal{S}_1$ mirrors that of the distinguishability growth Γ , with regions of low sensitivity corresponding to low coverage and regions of high sensitivity corresponding to high coverage. Thus, the organization of instability is reflected in the organization of long-time spreading. Despite this overall increase, the imprint of the initial state persists, indicating that interactions enhance exploration without fully erasing underlying structure.

Together, Γ and $\mathcal{S}_1$ provide a two-dimensional geometric description of subsystem dynamics by distinguishing the separation of nearby states from the long-time redistribution of probability over projective Hilbert space. Their combined behavior shows that quantum dynamical complexity is governed not only by interaction strength but also by the location of the initial state on the Bloch sphere.

B. From Interaction-Induced Deformation to Environment-Driven Complexity

We now examine how environment size influences subsystem dynamics within the geometric framework. We consider systems of varying total size L , with a fixed subsystem size $L_S = 1$ and environment size $L_E = L - L_S$.

For $L > 3$, the collective J_z^2 interaction introduces additional non-nearest-neighbor couplings beyond those present in the three-qubit system. The additional inter-

action pathways increase the redistribution of information across the system. This leads to increased entanglement generation and provides more channels through which perturbations can propagate, thereby influencing both local instability (Γ) and global spreading ($\mathcal{S}_1$).

System size also determines symmetry properties of the Hamiltonian. Even- and odd-qubit systems exhibit distinct parity symmetries, inherited by the Floquet operator, leading to qualitative differences in the dynamics [37–40].

In this framework, distinguishability growth generally increases with L within each parity sector, whereas state-space coverage varies nonmonotonically. Both diagnostics are structured by parity symmetry, with even- L and odd- L systems displaying systematically different behavior.

In the following, we quantify these effects using Γ and $\mathcal{S}_1$, showing how environment size drives increasingly complex subsystem dynamics.

1. Weak-Interaction Regime

We first examine the effect of increasing environment size in the weak-interaction regime, shown in Fig. 7 for $\kappa = 0.5$. In this regime, the dynamics retain significant structure, allowing us to isolate how environment size modifies subsystem behavior before the onset of strong mixing.

Figure 7 shows the evolution of two nearby subsystem states as L increases. For small L , the motion is structured. As L increases, trajectories become more diffuse and less overtly organized, but retain visible organization, indicating that interactions are not yet strong enough to fully disrupt the underlying structure.

The corresponding time-averaged states and coverage index $\mathcal{S}_1$, shown in Fig. 8, remain localized and nonuniform, indicating limited state-space exploration. As L changes, the distributions broaden and deform modestly, but the coverage remains constrained and does not grow monotonically. A weak parity dependence is already visible, with even- L systems exhibiting slightly larger coverage than odd- L systems. This regime provides a baseline in which environment effects are present but not yet dominant.

2. Strong-Interaction Regime

We now consider the strong-interaction regime, shown in Fig. 9 for $\kappa = 2.5$. In contrast to the weak interaction case, the dynamics exhibit substantial mixing, and the



Figure 7: Evolution of two neighboring GQSs in the weakly interacting L -qubit quantum kicked top, $\kappa = 0.5$. Rows correspond to increasing total qubit number L , and columns show the states after Floquet kicks $t_n = 0, 1, 2, 3, 4, 200$. As L increases, the ensembles exhibit greater sensitivity to initial perturbations and more pronounced deformation, while remaining confined to a limited region of $\mathbb{CP}^1$.

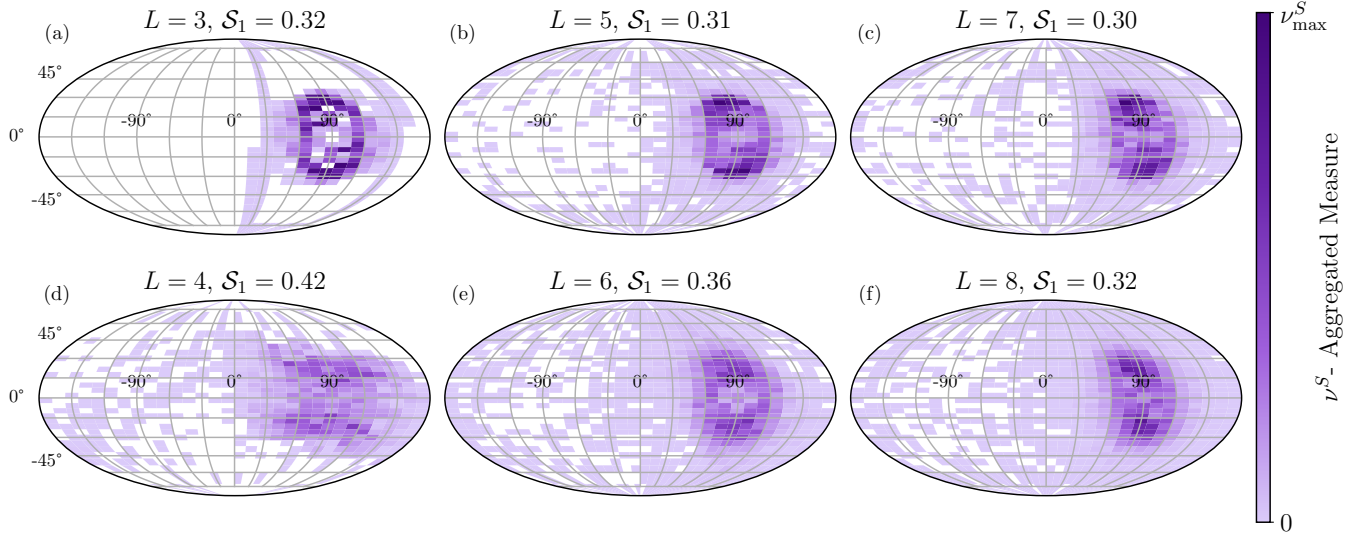


Figure 8: Time-aggregated GQSs ν^S and coverage index $\mathcal{S}_1$ in the weakly interacting quantum kicked top, $\kappa = 0.5$. Top row (a–c) shows odd- L systems, and bottom row (d–f) shows even- L systems, with L increasing across each row. The color scale gives the time-aggregated probability density on $\mathbb{CP}^1$. Even- L systems exhibit slightly larger coverage than odd- L systems, indicating weak finite-size parity-symmetry effects.

influence of environment size becomes significantly more pronounced.

Figure 9 shows the evolution of two nearby subsystem states as L increases. For small L , the dynamics already show substantial deformation. As L increases, trajectories rapidly lose structure and spread across the Bloch sphere, showing strong sensitivity and efficient propagation of perturbations.

The corresponding time-aggregated geometric quantum states and spread $\mathcal{S}_1$ are shown in Fig. 10. Unlike the weak interaction regime, the distributions now cover a large fraction of $\mathbb{CP}^1$, approaching near-uniform coverage for larger system sizes. This reflects a transition to global exploration of state space.

Parity effects become more pronounced in this regime. Even- L systems exhibit more uniform coverage and larger $\mathcal{S}_1$, while odd- L systems retain residual structure. This indicates that symmetry continues to organize the dynamics even under strong interactions. Overall, increasing environment size strongly enhances both sensitivity and global exploration. Unlike the weak-interaction regime, the dynamics are no longer constrained by underlying structure, and environment size plays a dominant role in driving complexity.

3. Joint Dependence on Interaction Strength and Environment Size

We now examine the joint dependence of sensitivity and spread on interaction strength κ and environment size L . Figures 11(a,b) show the evolution of separation between nearby subsystem states as L increases. For weak interactions, growth is modest, while for strong interactions it becomes rapid and strongly fluctuating. Increasing L enhances both the magnitude and fluctuations of this growth, indicating more effective propagation of perturbations.

The joint behavior of Γ and $\mathcal{S}_1$ (Figs. 11(c,d)) reveals a clear organization of the dynamics. Both generally increase with κ , while their dependence on L is structured by parity symmetry. Distinguishability growth tends to increase with L within each parity sector, whereas state-space coverage varies nonmonotonically.

This trend is further illustrated in Figs. 11(e,f), which display Γ and $\mathcal{S}_1$ as functions of κ for different values of L . Both quantities generally increase with κ across system sizes, indicating that stronger coupling promotes greater sensitivity to perturbations and more extensive exploration of the accessible state space. A particularly notable change occurs between the $L = 3$ system, for which the complete interaction graph coincides with a three-site

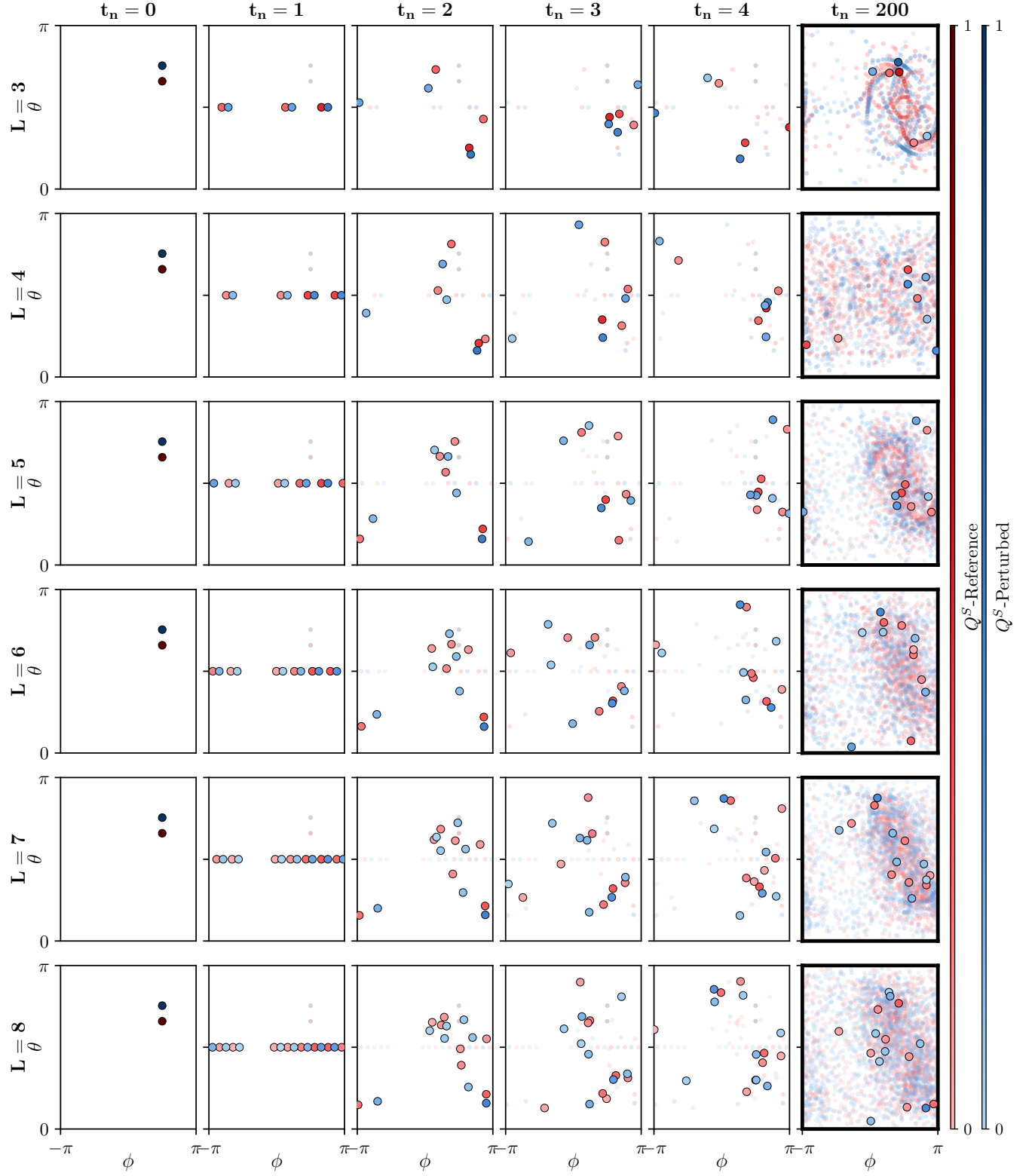


Figure 9: Evolution of two neighboring GQSs in the strongly interacting L -qubit quantum kicked top with $\kappa = 2.5$. Rows correspond to increasing values of L , while columns represent kicks at $t_n = 0, 1, 2, 3, 4, 200$. As L increases, the ensembles spread across $\mathbb{CP}^1$, rapidly lose visible structure, and display greater sensitivity compared to the weakly interacting case. Systems with even L tend to exhibit stronger sensitivity and more uniform spreading than those with odd L , highlighting finite-size parity symmetry effects.

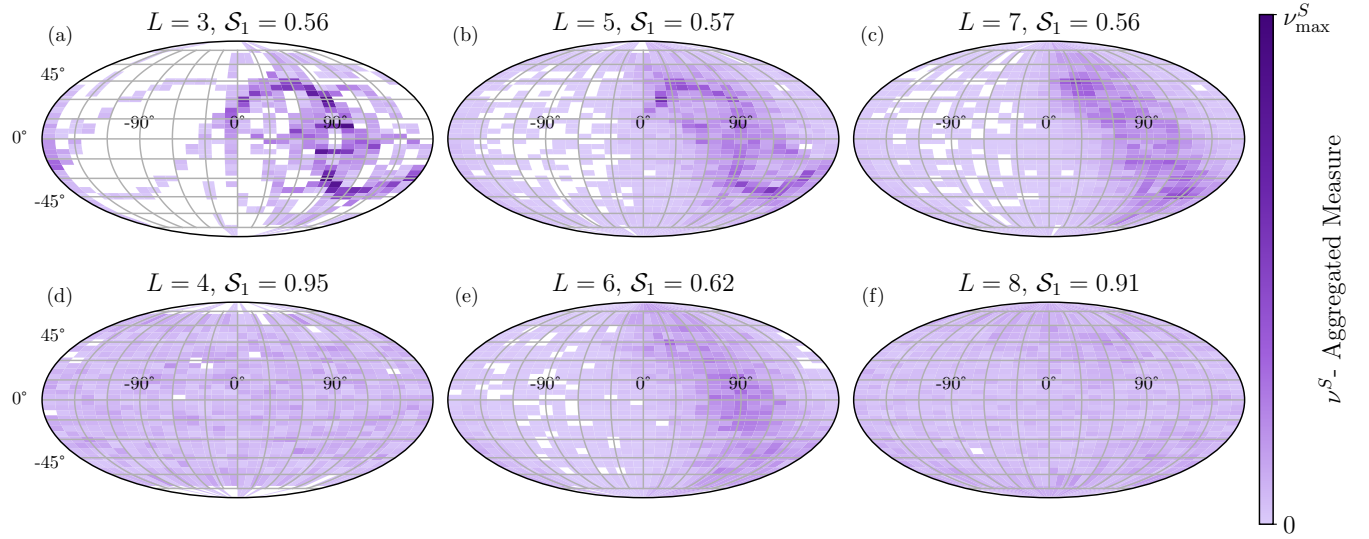


Figure 10: Time-aggregated GQSs ν^S and coverage index S_1 in the strongly interacting quantum kicked top, $\kappa = 2.5$. Top row (a–c) shows odd- L systems, and bottom row (d–f) shows even- L systems, with L increasing across each row. The color scale gives the time-aggregated probability density on $\mathbb{CP}^1$. Even- L systems exhibit larger coverage and more uniform spreading than odd- L systems, revealing pronounced finite-size parity-symmetry effects.

nearest-neighbor ring, and systems with $L > 3$, where the collective J_z^2 interaction introduces additional non-nearest-neighbor couplings. The introduction of these additional interaction pathways leads to a pronounced increase in complexity, suggesting that longer-range correlations substantially enrich the dynamics. Overall, larger values of L tend to yield higher values of both Γ and S_1 , reflecting an enhanced capacity for global exploration and a progressively richer dynamical structure.

The enhancement is not uniform across system sizes. Even- L systems exhibit systematically higher Γ and S_1 than odd- L systems, reflecting the underlying differences in parity symmetry. This separation becomes more pronounced at larger κ and L , where mixing is strongest. This parity-dependent organization is a finite-size quantum effect and does not persist in the classical limit $j \rightarrow \infty$ [40, 41].

Overall, quantum dynamical complexity is jointly controlled by the interaction strength and system size. While the interaction strength κ drives the transition from periodic to complex dynamics, changing the system size L , and hence the environment size $L - 1$, modifies the dynamics in a parity-dependent and generally nonmonotonic manner.

VI. CONCLUSION AND OUTLOOK

Understanding the emergence of complexity in finite interacting quantum systems is central to a wide range of contemporary problems, including digital and analog quantum simulation [17, 42–44], information scrambling [45, 46], and the control and stability of quantum devices [47–49]. In these settings, the interplay between sensitivity to perturbations and global exploration of state space directly impacts both computational utility and physical behavior.

In this work, we introduced a geometric, state-based framework for quantifying dynamical complexity in quantum systems. By representing subsystem dynamics as probability measures on projective Hilbert space via environment-conditioned states, we provide a direct and physically transparent way to track how interactions reshape quantum states.

Within this framework, noninteracting dynamics remain confined to periodic or quasiperiodic motion on projective space, analogous to trajectories on invariant tori. Interactions induce a qualitative transition: probability mass initially localized on a single pure state deforms and spreads across the state space, reflecting the buildup of correlations and mixedness. This perspective makes explicit that complexity arises through the redistribution of

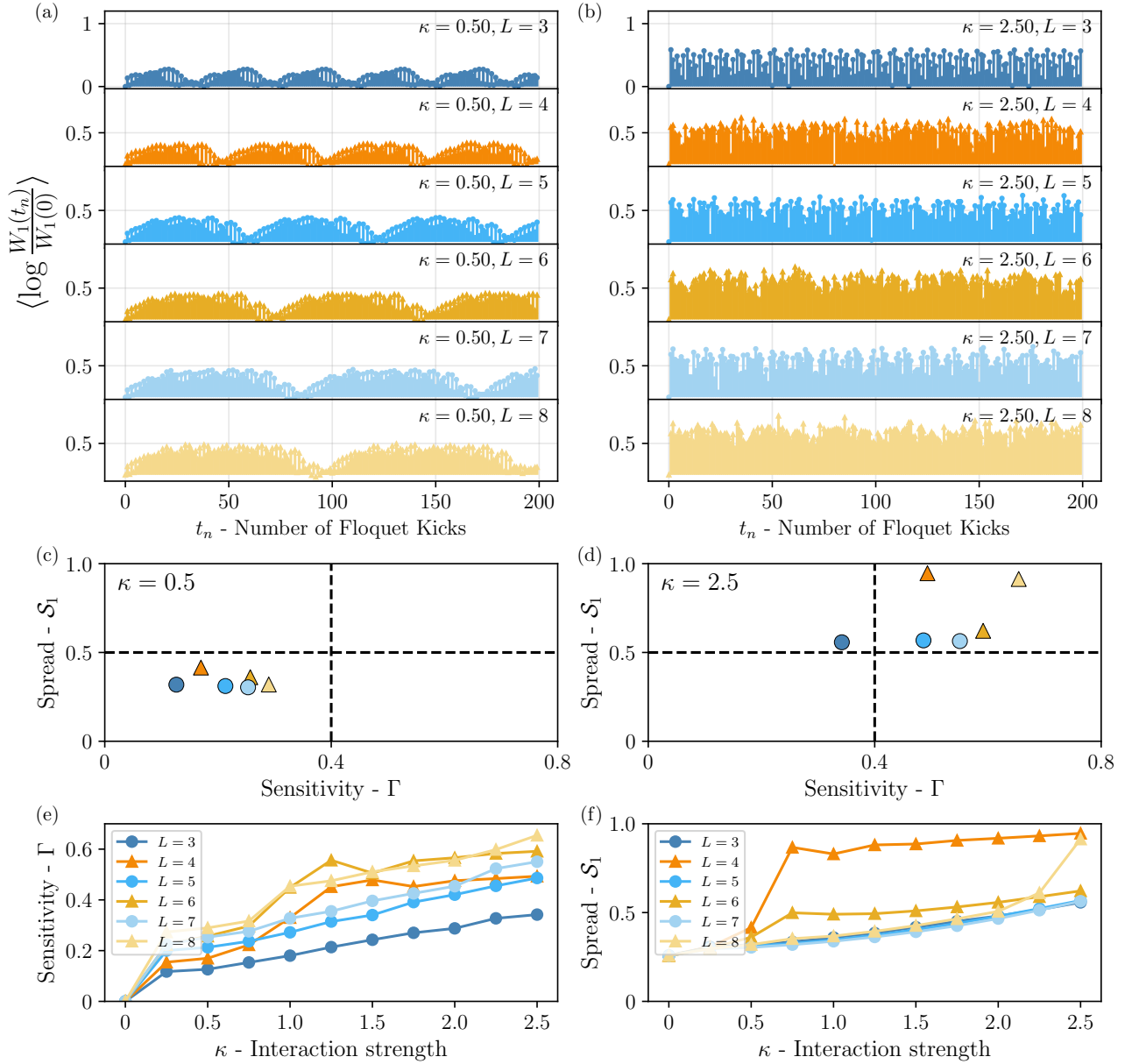


Figure 11: Sensitivity and spreading in the L -qubit kicked top for the spin-coherent initial state $(\theta, \phi) = (\pi/2 + 0.5, \pi/2)$. (a,b) Separation dynamics of nearby subsystem states for different L at $\kappa = 0.5$ and 2.5 , showing weaker and stronger distinguishability growth, respectively. (c,d) Joint behavior of the distinguishability measure Γ and the state-space coverage index $\mathcal{S}_1$. Dashed lines serve as visual guides to the qualitative dynamical regimes. (e,f) Dependence of Γ and $\mathcal{S}_1$ on κ for different L . Both generally increase with κ , while their dependence on L is structured by parity symmetry, with even- L systems often exhibiting larger values than neighboring odd- L systems.

probability mass on projective Hilbert space.

To quantify these effects, we introduced two complementary diagnostics. The distinguishability measure Γ captures sensitivity to initial states, while the state-space coverage index $\mathcal{S}_1$ characterizes long-time spreading. Applied to the quantum kicked top, both diagnostics generally increase with the interaction strength κ . Their dependence on the environment size L_E , however, is non-monotonic and structured by finite-size parity symmetry, with even- L and odd- L systems exhibiting systematically different behavior.

These results show that complexity in quantum systems is not solely an asymptotic phenomenon but can be meaningfully characterized at finite, experimentally relevant sizes. The growth of distinguishability and the redistribution of probability mass provide direct indicators of transitions from stable to highly sensitive, broadly spreading reduced-state dynamics, with implications for simulation accuracy, error propagation, and control.

Unlike operator-based diagnostics such as out-of-time-ordered correlators or Loschmidt echoes, the present approach is explicitly state-based and geometric, tracking the evolution of quantum states themselves. It also complements Husimi-based phase-space complexity measures for discrete-variable states and channels [50] by focusing on dynamical deformation and spreading rather than static state or channel structure. This framework is particularly useful in regimes where observables remain stable while the underlying state structure undergoes significant change.

The framework is readily extendable to larger subsystems ($L_S > 1$) and remains computationally tractable for

the system sizes considered here. Future work includes exploring the role of larger environments, incorporating scalable methods such as tensor networks, and clarifying the relationship between these geometric diagnostics and established measures of quantum chaos. The dependence on the choice of environment basis and extensions to driven, dissipative, or noisy systems also remain important open directions.

These results establish a geometric, state-resolved perspective on quantum dynamical complexity in finite interacting systems, where interaction-induced deformation and spreading of probability distributions provide a natural extension of classical notions of instability and exploration to the quantum setting.

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APPENDICES

The appendices provide additional conceptual, computational, and numerical details supporting the main text. Appendix A compares the environment-resolved geometric quantum state (GQS) representation with the Schmidt decomposition, while Appendix B distinguishes GQSs from Husimi representations. Appendix C describes the computation of the Wasserstein distance on $\mathbb{CP}^{d_S-1}$, and Appendix D relates the State-Space Coverage Index to the classical information dimension. Appendix E summarizes the numerical implementation and simulation parameters. Appendix F examines quantum recurrence and its relation to distinguishability growth and state-space coverage. Finally, Appendix G explores how these diagnostics vary across the spin-coherent-state phase space of the L -qubit kicked top.

Appendix A: Geometric Quantum States and Schmidt Decomposition

For a pure bipartite state $|\Psi_{SE}(t)\rangle \in \mathcal{H}_S \otimes \mathcal{H}_E$, the Schmidt decomposition is

$$|\Psi_{SE}(t)\rangle = \sum_{\alpha=1}^r \sqrt{\lambda_{\alpha}(t)} |\tilde{s}_{\alpha}(t)\rangle \otimes |\tilde{e}_{\alpha}(t)\rangle, \quad (\text{A1})$$

where $\lambda_{\alpha}(t) \geq 0$, $\sum_{\alpha} \lambda_{\alpha}(t) = 1$, and $r \leq \min(\dim \mathcal{H}_S, \dim \mathcal{H}_E)$. Tracing over the environment we get:

$$\rho_S(t) = \sum_{\alpha=1}^r \lambda_{\alpha}(t) |\tilde{s}_{\alpha}(t)\rangle \langle \tilde{s}_{\alpha}(t)|. \quad (\text{A2})$$

The quantities λ_{α} are the eigenvalues of $\rho_S(t)$, while $\sqrt{\lambda_{\alpha}}$ are the Schmidt coefficients.

In the geometric quantum framework, instead of diagonalizing $\rho_S(t)$, we consider an environment-conditioned decomposition with respect to a fixed environment basis $\{|e_j\rangle\}$:

$$|\Psi_{SE}(t)\rangle = \sum_{j=1}^{d_E} \sqrt{\lambda_j^E(t)} |\chi_j^S(t)\rangle \otimes |e_j\rangle, \quad (\text{A3})$$

with

$$\lambda_j^E(t) = \sum_{k=1}^{d_S} |\psi_{kj}(t)|^2, \quad |\chi_j^S(t)\rangle = \frac{1}{\sqrt{\lambda_j^E(t)}} \sum_{k=1}^{d_S} \psi_{kj}(t) |s_k\rangle. \quad (\text{A4})$$

Tracing over the environment gives

$$\rho_S(t) = \sum_{j=1}^{d_E} \lambda_j^E(t) |\chi_j^S(t)\rangle \langle \chi_j^S(t)|. \quad (\text{A5})$$

Unlike the Schmidt decomposition, the states $\{|\chi_j^S(t)\rangle\}$ are not generally orthogonal, and the weights $\{\lambda_j^E(t)\}$ are not eigenvalues of $\rho_S(t)$. Within the GQS framework, the pairs $\{\lambda_j^E(t), |\chi_j^S(t)\rangle\}$ define a probability distribution over subsystem pure states. This representation preserves the explicit correspondence between environment configurations and subsystem states. This association with the chosen environment basis is lost after rotation to the Schmidt basis.

Thus, while the Schmidt decomposition characterizes the magnitude of entanglement through its spectrum, the environment-resolved decomposition provides the structural information needed to study the geometry and transport of subsystem state distributions.

Appendix B: Geometric Quantum States and Husimi Representations

In quantum mechanics, mixed states are often visualized through phase-space representations such as the Husimi Q -function. In this work, we instead employ the geometric quantum state (GQS), which arises naturally from the system–environment decomposition and admits a direct geometric interpretation. The relationship between these representations is summarized schematically in Fig. 12.

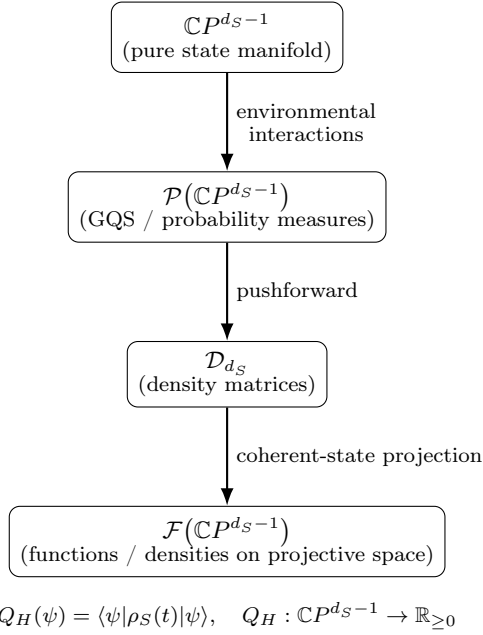


Figure 12: Geometric quantum states (GQS) describe subsystem states as probability measures on $\mathbb{C}P^{d_S-1}$.

The reduced density matrix ρ_S is obtained via the pushforward map Φ , and can be used to construct a Husimi representation through projection onto coherent states. Unlike the GQS, the Husimi representation is a nonnegative function rather than a probability measure on $\mathbb{C}P^{d_S-1}$.

The GQS representation introduced in Sec. II.E expresses subsystem states as probability measures on projective Hilbert space. For completeness, we recall that

$$Q^S(Z, t) = \sum_{j=1}^{d_E} \lambda_j^E(t) \delta_{Z_j^S(t)}^Z \in \mathcal{P}(\mathbb{C}P^{d_S-1}), \quad (\text{B1})$$

where the weights and conditional states are determined by the system–environment decomposition of the global wavefunction (see Sec. II.E).

While the reduced density matrix $\rho_S(t)$ captures all observable statistics, it does not uniquely specify how probability is distributed across pure states. The GQS representation retains this ensemble structure explicitly, enabling one to track how probability mass is organized and evolves on $\mathbb{C}P^{d_S-1}$ under interactions.

Given a reduced density matrix $\rho_S(t)$, the Husimi function is defined as

$$Q_H(\psi) = \langle \psi | \rho_S(t) | \psi \rangle, \quad (\text{B2})$$

where $|\psi\rangle \in \mathbb{C}P^{d_S-1}$ denotes a chosen family of coherent or reference states. This defines a nonnegative function on projective Hilbert space and provides a smooth visualization of observable statistics.

The key distinction between the GQS and Husimi representations lies in how they encode ensemble structure. Distinct probability distributions on $\mathbb{C}P^{d_S-1}$ may correspond to the same density matrix $\rho_S(t)$ and are therefore indistinguishable by any measurement on the subsystem. Since the Husimi function is constructed from $\rho_S(t)$, it inherits this indistinguishability: geometrically distinct GQS that share the same density matrix yield identical Husimi representations.

Consequently, while the Husimi representation is well-suited for visualizing observable statistics, the GQS framework preserves the underlying ensemble structure, enabling the study of dynamical complexity through the evolution and transport of probability mass on projective Hilbert space.

Appendix C: Computation of Wasserstein Distance on $\mathbb{C}P^{d_S-1}$

The definition of the Wasserstein distance on projective Hilbert space is given in Sec. III. Here, we summarize its formulation and numerical implementation for the geometric quantum states considered in this work.

For geometric quantum states of the form

$$Q = \sum_i \lambda_i \delta_{Z_i}^Z, \quad Q' = \sum_j \lambda'_j \delta_{Z'_j}^Z, \quad (\text{C1})$$

the computation of the Wasserstein distance reduces to a finite-dimensional optimal transport problem. The cost of transporting probability mass between points Z_i and Z'_j is given by the Fubini–Study distance,

$$C_{ij} = d_{FS}(Z_i, Z'_j). \quad (\text{C2})$$

The Wasserstein distance $W_1(Q, Q')$ is then obtained by solving for the optimal transport plan π_{ij} that minimizes the total transport cost,

$$W_1(Q, Q') = \min_{\pi} \sum_{i,j} \pi_{ij} C_{ij}, \quad (\text{C3})$$

subject to the constraints

$$\sum_j \pi_{ij} = \lambda_i, \quad \sum_i \pi_{ij} = \lambda'_j. \quad (\text{C4})$$

This formulation corresponds to a linear programming problem over the transport plan π_{ij} .

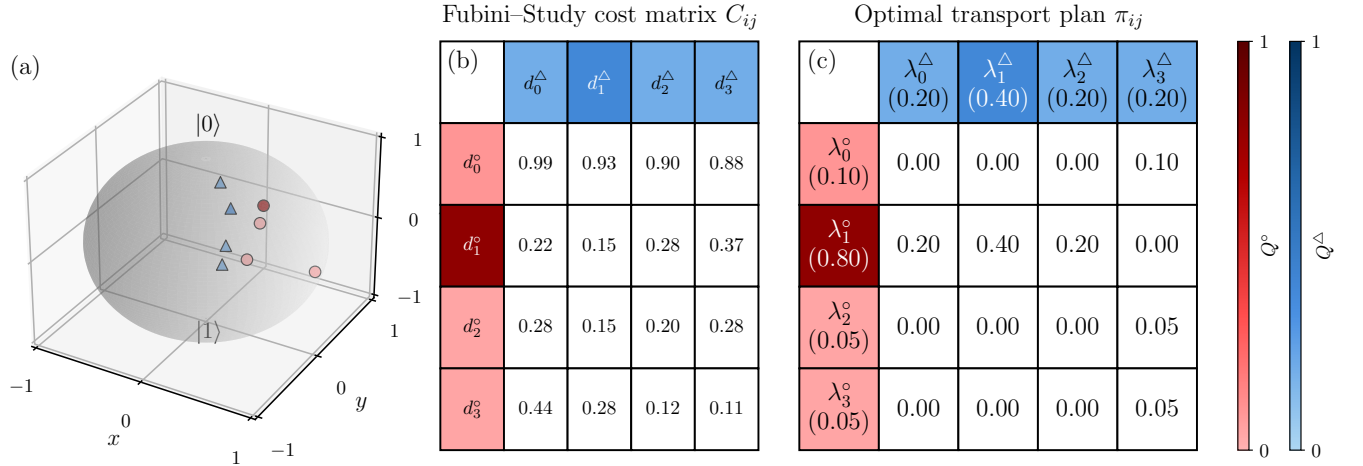


Figure 13: Geometric computation of the Wasserstein distance between two geometric quantum states (GQSs). (a) Discrete probability measures Q^Δ and Q° on the Bloch sphere CP^1 . (b) Fubini-Study distance cost matrix C_{ij} between support points. (c) Optimal transport plan π_{ij} , illustrating how probability mass is redistributed to obtain $W_1(Q^\Delta, Q^\circ) = 0.27$.

In practice, the computation proceeds by constructing the cost matrix C_{ij} using pairwise Fubini-Study distances between the support points of the two geometric quantum states. For geometric quantum states with at most d_E support points, the cost matrix has size $d_E \times d_E$, so the dominant computational cost comes from solving the discrete optimal transport problem over these environment-conditioned components rather than from the subsystem dimension d_S itself. The optimal transport problem is then solved numerically to obtain the transport plan π_{ij} and the corresponding Wasserstein distance [51].

In this work, we employ Earth Mover's Distance (EMD) solver. For the system sizes considered here, the discrete formulation remains tractable and allows for direct evaluation of the Wasserstein distance without additional approximations.

This discrete optimal transport formulation enables a direct comparison of subsystem states by quantifying how probability mass is redistributed across projective Hilbert space.

Appendix D: State-Space Coverage Index and Information Dimension

In Sec. IV, we introduced the State-Space Coverage Index (SSCI) as a measure of long-time dynamical behavior, inspired by the classical notion of information dimension. Here, we briefly review the definition of information di-

mension and explain the challenges in extending it to the present setting.

In classical dynamical systems, long-time geometric complexity is often characterized through the information dimension of an invariant measure. Let $T : M \rightarrow M$ be a dynamical map on a compact metric space (M, d) , and let $\mu \in \mathcal{P}(M)$ be an invariant probability measure. μ can be approximated by the empirical (time-averaged) measure generated by a trajectory $\{x_n\}_{n=0}^{N-1}$,

$$\mu_N(A) = \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_A(x_n), \quad A \subset M, \quad (D1)$$

which converges to μ under appropriate ergodicity assumptions.

To probe the geometric structure of μ , one partitions M into sets $\{\Delta_i(\epsilon)\}$ of diameter ϵ , defining probabilities

$$p_i(\epsilon) = \mu(\Delta_i(\epsilon)). \quad (D2)$$

The associated Shannon entropy at scale ϵ is

$$H(\epsilon) = - \sum_i p_i(\epsilon) \log p_i(\epsilon), \quad (D3)$$

and the information dimension is defined as

$$D_1 = \lim_{\epsilon \rightarrow 0} \frac{H(\epsilon)}{\log(1/\epsilon)}, \quad (D4)$$

whenever the limit exists.

Table I: Numerical parameters used to compute the distinguishability measure Γ and the state-space coverage index $\mathcal{S}_1$.

Category	Parameter	Value or implementation detail
Geometric quantum states	Environment basis	Computational basis
	Zero-weight cutoff	Conditional states with $\lambda_j^E < 1e-12$ are omitted
	State representation	Normalized state vectors / Bloch-sphere coordinates
Optimal transport	Wasserstein order	$p = 1$
	Ground metric	Fubini–Study distance
	Optimal-transport solver	EMD
Sensitivity- Γ calculations	Number of Floquet kicks	$T = 200$
	Time step	$t_n = n$, with $\tau = 1$
	Number of perturbations	$M = 200$
	Perturbation scale	Rotation angles are drawn from $\alpha \sim \mathcal{N}(0, \epsilon^2)$, with $\epsilon = 0.2$ radians
	Perturbation distribution	Isotropic random rotations with uniformly distributed rotation axes and Gaussian-distributed rotation angles
Coverage index	Number of bins over Bloch sphere	30×30
	Binning method	Equal area bins on the Bloch sphere
	Averaging Time	$T_S = 5000$

Geometrically, D_1 characterizes how probability mass is distributed across phase space at fine scales. For smooth measures, it coincides with the dimension of the underlying space, while for singular measures (e.g., strange attractors), it can take non-integer values, reflecting fractal structure.

In the geometric quantum framework considered here, the dynamical object is not a trajectory of points in $M = \mathbb{C}P^{ds-1}$, but a trajectory of probability measures

$$Q^S(\cdot, t) \in \mathcal{P}(M).$$

A direct analogue of the empirical construction in Eq. (D1) would therefore yield a measure over measures,

$$\mathbb{M}_T = \frac{1}{T} \sum_{t=0}^{T-1} \delta_{Q^S(\cdot, t)} \in \mathcal{P}(\mathcal{P}(M)), \quad (\text{D5})$$

which resides on an infinite-dimensional space. Constructing partitions and defining entropy scaling in this space is not straightforward and becomes computationally intractable upon discretization.

For this reason, we adopt the State-Space Coverage Index introduced in the main text as a tractable alternative, providing a finite-time measure of how subsystem states spread across projective Hilbert space.

Appendix E: Numerical Implementation Details

Numerical simulations are performed as described in the main text, including the construction of geometric quantum states, computation of Wasserstein distances via discrete optimal transport, and evaluation of dynamical diagnostics such as Γ and the State-Space Coverage Index. Table I lists the numerical parameters used in these calculations. A complete implementation of the methods used in this work is available at <https://github.com/Kommmi/Qaos>.

Appendix F: Complexity and Quantum Recurrence

In the main text, we examined complexity for $\kappa \in [0, 2.5]$. Here, we focus on interaction strengths at which quantum recurrences are known to occur [37, 39, 40]. We consider the quantum kicked top for both half-integer spins $j \in \{3/2, 5/2, 7/2\}$ ($L = 3, 5, 7$) and integer spins $j \in \{2, 3, 4\}$ ($L = 4, 6, 8$), allowing us to probe behavior across different parity symmetries.

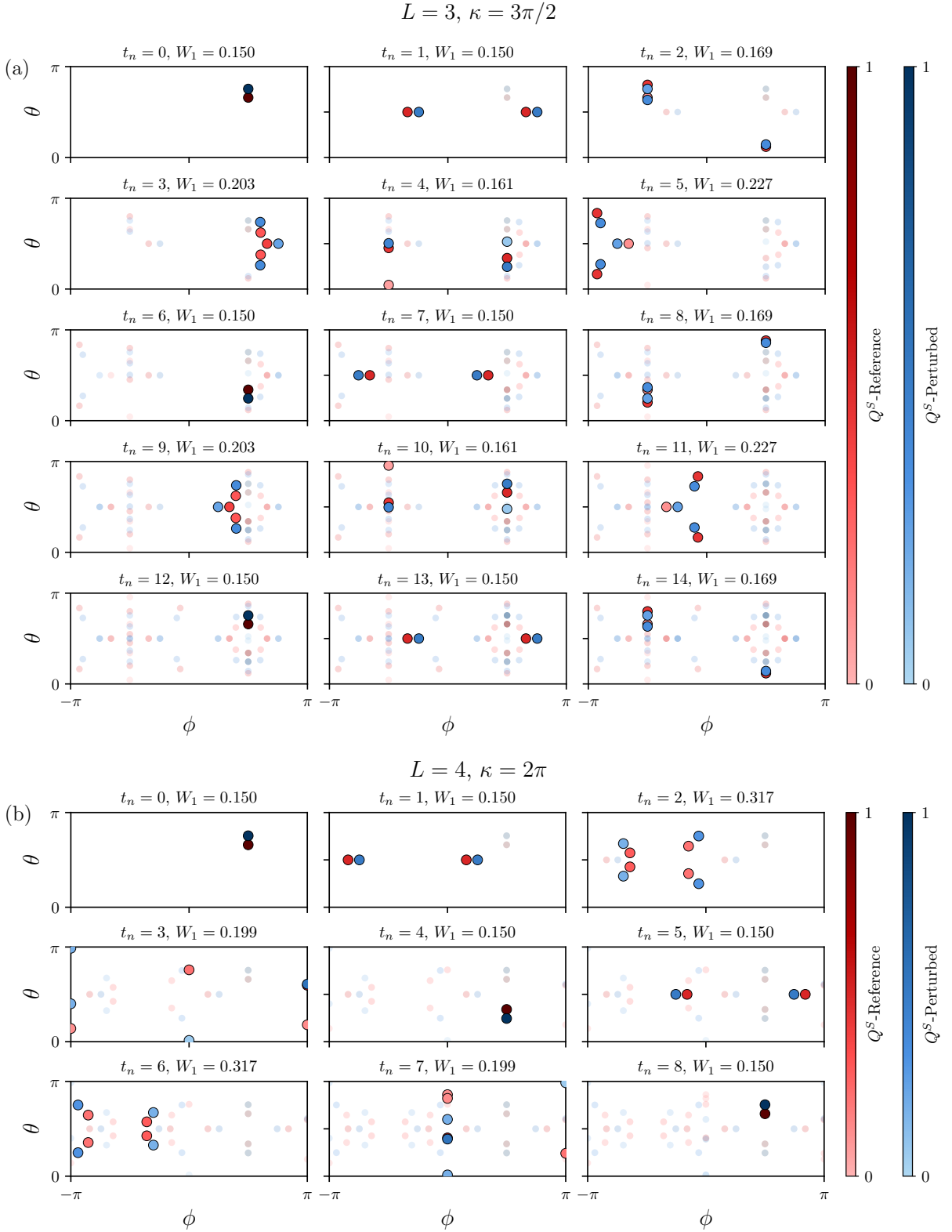


Figure 14: Subsystem dynamics of two nearby QGSs and their Wasserstein separation in the recurrent regime $\kappa = \pi j$. (a) $L = 3, \kappa = 3\pi/2$. (b) $L = 4, \kappa = 2\pi$. Recurrence occurs after 12 and 8 kicks, respectively. Although the dynamics are recurrent, distinguishability is not preserved for either the integer- or half-integer-spin kicked top at $\kappa = \pi j$.

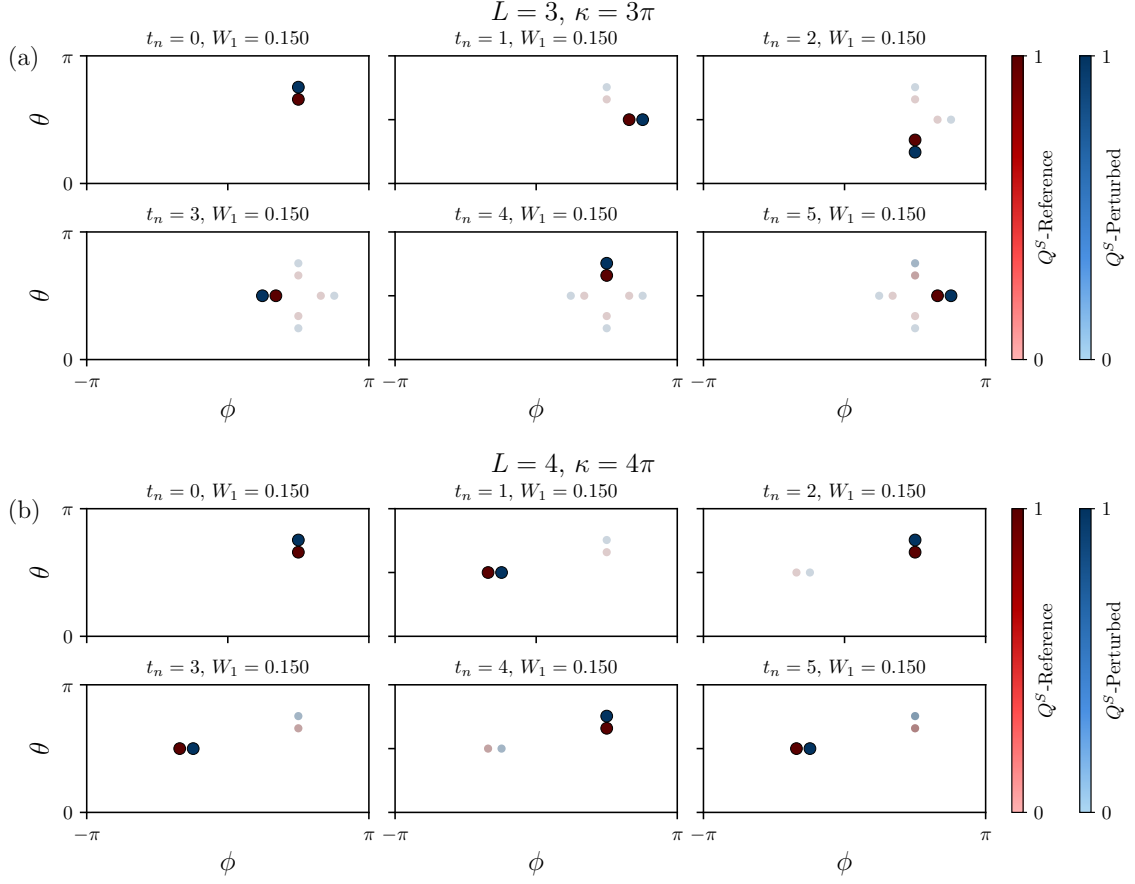


Figure 15: Subsystem dynamics of two nearby GQs and their separation. (a) $(L, \kappa) = (3, 3\pi)$ and (b) $(L, \kappa) = (4, 4\pi)$. Recurrence occurs after 4 and 2 kicks, respectively, with separation preserved for $\kappa = 2\pi j$.

1. Interaction strength $\kappa = \pi j$

We examine the evolution of two nearby GQs initialized from spin-coherent states $(\theta, \phi) = (\pi/2 + 0.5, \pi/2)$ (circles) and $(\pi/2 + 0.8, \pi/2)$ (triangles). As shown in Fig. 14, the subsystem exhibits recurrence after 12 kicks for $L = 3$ (half-integer spin) and 8 kicks for $L = 4$ (integer spin), consistent with theoretical predictions [40].

However, recurrence of the trajectory does not imply preservation of nearby-state separation. As shown in Fig. 16(a), the distinguishability between the two GQs undergoes oscillatory growth rather than remaining fixed. Thus, unlike the noninteracting case $\kappa = 0$, the subsystem dynamics are recurrent but distinguishability is not preserved. This yields a positive distinguishability measure, $\Gamma > 0$, indicating sensitivity to perturbations.

Figure 16(c) summarizes this behavior in the

sensitivity-spread plane, where Γ quantifies average distinguishability growth and $\mathcal{S}_1$ quantifies long-time state-space coverage. As in Sec. V, integer-spin systems, corresponding to even L , exhibit larger sensitivity and spread than half-integer-spin systems, corresponding to odd L . In the two-dimensional classification of Fig. 3, this places the $\kappa = \pi j$ recurrent dynamics in an extended but low-sensitivity regime.

2. Interaction strength $\kappa = 2\pi j$

We now consider the same initial states for interaction strength $\kappa = 2\pi j$. As shown in Fig. 15, recurrence occurs after 4 kicks for $L = 3$ and 2 kicks for $L = 4$, again consistent with theoretical predictions [40].

In contrast to the previous case, the separation between

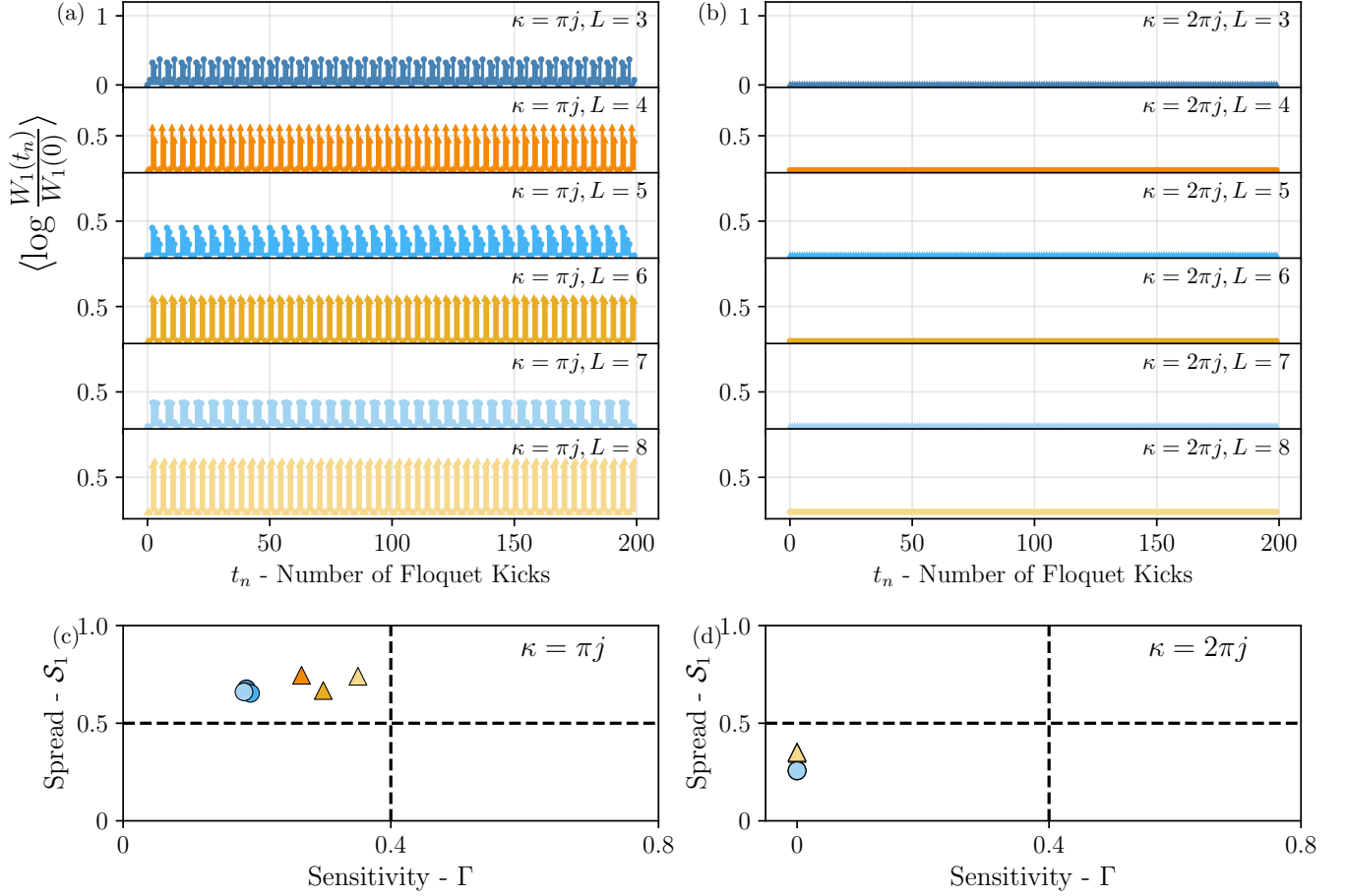


Figure 16: Separation dynamics and state-space exploration in recurrent kicked-top regimes for increasing system size L . (a) For $\kappa = \pi j$, the average logarithmic relative separation $\langle \ln[W_1(t_n)/W_1(0)] \rangle$ oscillates persistently, yielding $\Gamma > 0$ and indicating sensitivity to perturbations. Integer-spin systems exhibit larger changes in distinguishability than half-integer-spin systems. (b) For $\kappa = 2\pi j$, the distinguishability is preserved, giving $\Gamma \simeq 0$. (c) The sensitivity-spread diagram for $\kappa = \pi j$ shows low sensitivity but relatively large coverage $\mathcal{S}_1$. (d) For $\kappa = 2\pi j$, both Γ and $\mathcal{S}_1$ remain small, indicating localized recurrent dynamics with limited state-space exploration. The dashed lines in panels (c) and (d) serve as visual guides separating the qualitative dynamical regimes and do not represent sharp thresholds.

the two states is preserved throughout the evolution. The subsystem dynamics remain confined, similar to the non-interacting case $\kappa = 0$, and the distinguishability measure vanishes, $\Gamma = 0$, indicating the absence of dynamical complexity.

Fig. 16(d) shows that both Γ and $\mathcal{S}_1$ remain small for $\kappa = 2\pi j$. Within the sensitivity-spread classification of Fig. 3, this corresponds to localized periodic dynamics. The comparison between $\kappa = \pi j$ and $\kappa = 2\pi j$ shows that recurrence alone is not sufficient to characterize dynamical

simplicity. What matters is whether recurrence preserves nearby-state distinguishability or allows sustained growth and broader state-space exploration.

Appendix G: Phase-space structures for the L -qubit kicked top

To understand how open-system complexity depends on the initial condition, we compute the distinguishability

measure Γ and the state-space coverage index $\mathcal{S}_1$ over the spin-coherent state phase space of the L -qubit kicked top. For each initial condition (θ, ϕ) , the total system is initialized as a product spin-coherent state,

$$|\Psi_{SE}(0)\rangle = \bigotimes_{\ell=1}^L |\psi(\theta, \phi)\rangle. \quad (\text{G1})$$

We then evolve the full L -qubit state unitarily, treat one qubit as the subsystem, and construct the corresponding geometric quantum state on the subsystem Bloch sphere. The quantities Γ and $\mathcal{S}_1$ are evaluated on a 100×100 grid of initial conditions, producing phase-space maps that reveal how sensitivity and state-space exploration depend on the interaction strength κ and the effective environment size $L - 1$.

Fig. 17 shows the phase-space structure of the distinguishability measure Γ for $L = 3, \dots, 8$ and $\kappa = 0, 0.5, 2.5$. In the noninteracting limit, $\kappa = 0$, Γ is uniformly zero for all L . Although the initial condition determines the subsystem trajectory, distance between two nearby states does not change with time, providing a baseline for periodic dynamics.

At $\kappa = 0.5$, regions of enhanced Γ emerge, showing that sensitivity depends strongly on the initial spin-coherent state. Some initial conditions remain weakly sensitive, while others generate larger distinguishability growth through interaction-induced entanglement with the remaining qubits.

For stronger interactions, $\kappa = 2.5$, the high- Γ regions broaden and become more intricate. Sensitivity increases over a larger portion of phase space, indicating that stronger subsystem-environment interactions promote more widespread distinguishability growth. The maps also reveal finite-size parity-symmetry effects, with even- L systems (integer total spin) generally exhibiting larger Γ values and stronger phase-space deformation than odd- L systems (half-integer total spin).

Fig. 18 shows the corresponding phase-space structure of the state-space coverage index $\mathcal{S}_1$. While Γ measures distinguishability growth between nearby GQSSs, $\mathcal{S}_1$ quan-

tifies the long-time extent of subsystem exploration over the Bloch sphere. The two diagnostics therefore capture complementary aspects of the dynamics, namely local sensitivity and global state-space spreading.

For $\kappa = 0$, the coverage maps retain patterns reflecting periodic dynamics and are strongly organized by the initial spin-coherent state. Unlike Γ , which is uniformly zero in this limit, $\mathcal{S}_1$ varies across phase space because it is set by the geometry of the trajectory. States near $(\theta, \phi) = (\pi/2, \pi/2)$, an eigenstate of the non-interacting portion of the Hamiltonian $H_0 = (\pi/2\tau)J_y$, remain confined and yield small $\mathcal{S}_1$, whereas states farther away trace larger orbits and produce greater long-time coverage.

As κ increases to 0.5, the coverage patterns broaden and deform. At $\kappa = 2.5$, the coverage becomes larger and more extended, with many regions reaching $\mathcal{S}_1 > 0.5$. Stronger interactions therefore promote global exploration while preserving the underlying phase-space organization.

Parity-symmetry effects are particularly visible in the coverage maps compared to sensitivity maps. Even- L systems generally show broader and more uniform coverage than odd- L systems, indicating differences between integer-spin and half-integer-spin dynamics that are sometimes more evident in $\mathcal{S}_1$ than in Γ .

Figs. 17 and 18 show that interaction strength and environment size reshape the open-system phase-space structure in complementary ways. Increasing κ enhances both distinguishability growth and long-time coverage, while increasing L reveals systematic parity effects. The resulting patterns also resemble the classical kicked-top structures in the maximal Lyapunov exponent maps of Fig. 5(c,d), where regions of enhanced quantum distinguishability and coverage appear in organized bands and islands rather than uniformly. This suggests that the geometry of the underlying kicked-top phase space continues to organize the open-system complexity, even though the subsystem dynamics are described by probability measures on quantum state space rather than classical trajectories.

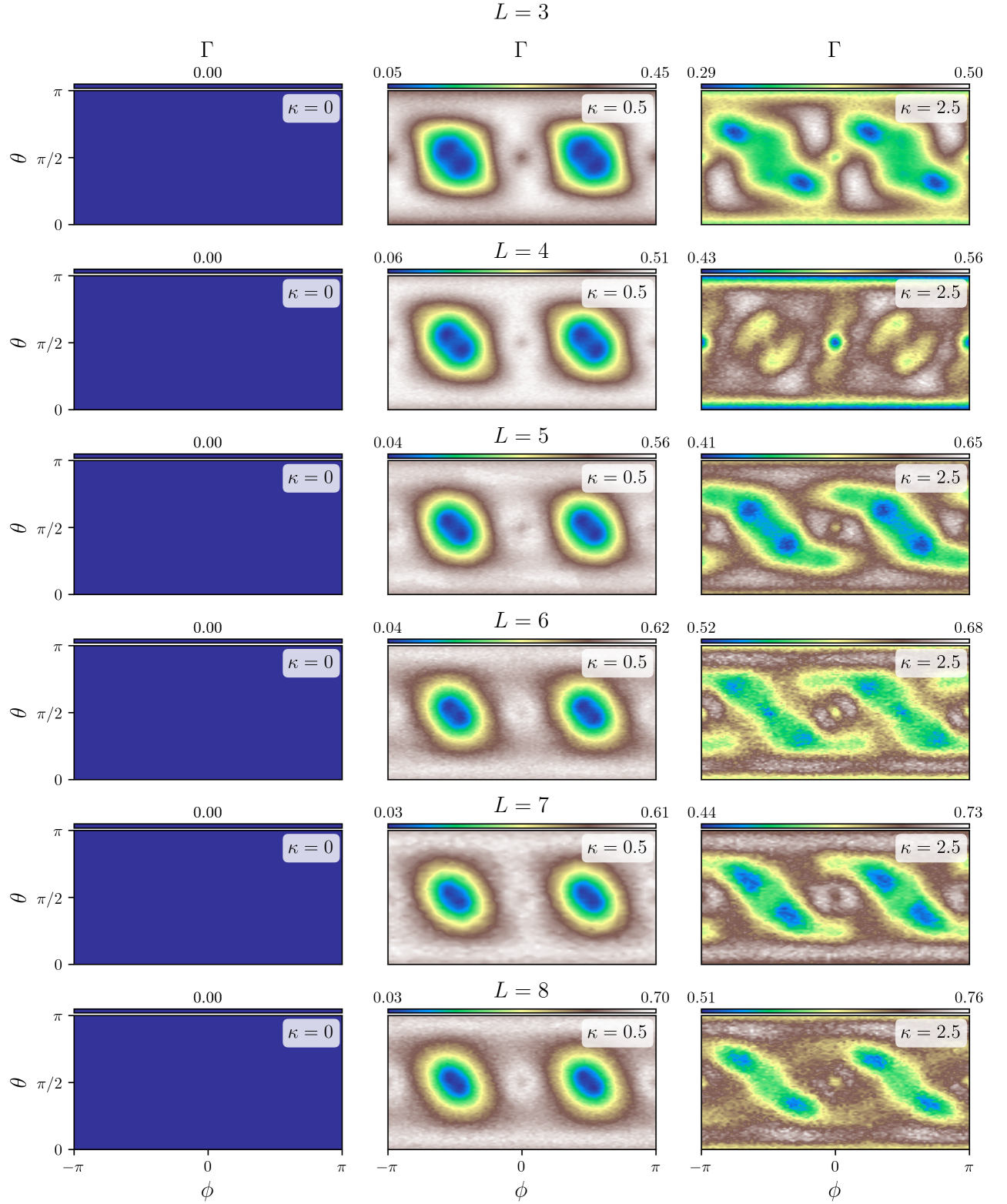


Figure 17: Phase-space structure of Γ for the L -qubit kicked top. Columns show $\kappa = 0, 0.5, 2.5$, and rows show $L = 3, \dots, 8$. For $\kappa = 0$, $\Gamma = 0$ uniformly. Increasing κ produces increasingly extended and intricate regions of distinguishability growth, with even- L systems generally showing slightly stronger sensitivity than odd- L systems.

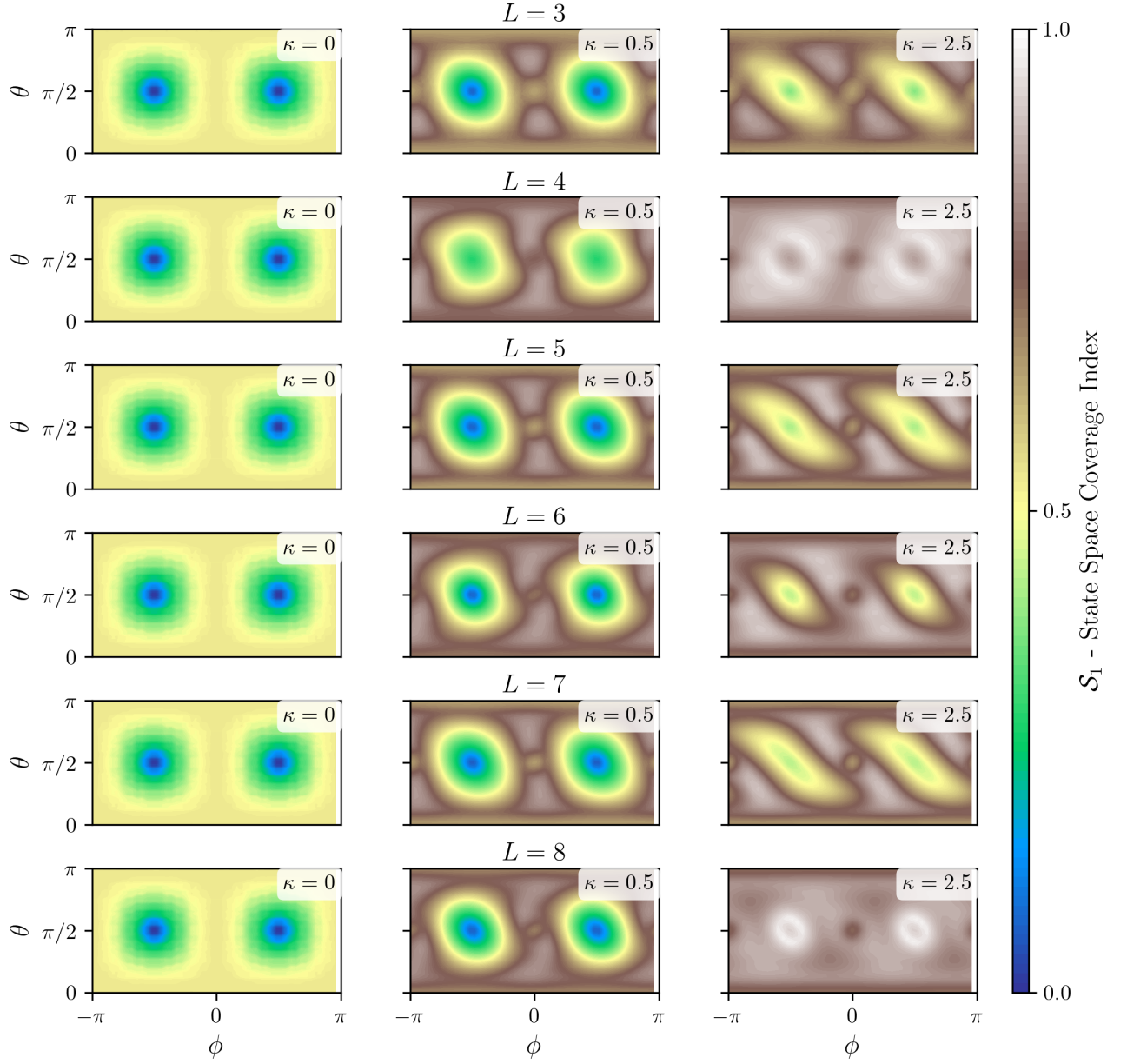


Figure 18: Phase-space structure of the coverage index S_1 for the L -qubit kicked top. Columns show $\kappa = 0, 0.5, 2.5$ and rows show $L = 3, 4, 5, 6, 7, 8$. Increasing κ deforms the periodic coverage patterns and promotes broader exploration of the subsystem state space. Even- L systems generally exhibit larger coverage than odd- L systems, indicating parity-dependent differences in interaction-induced spreading.