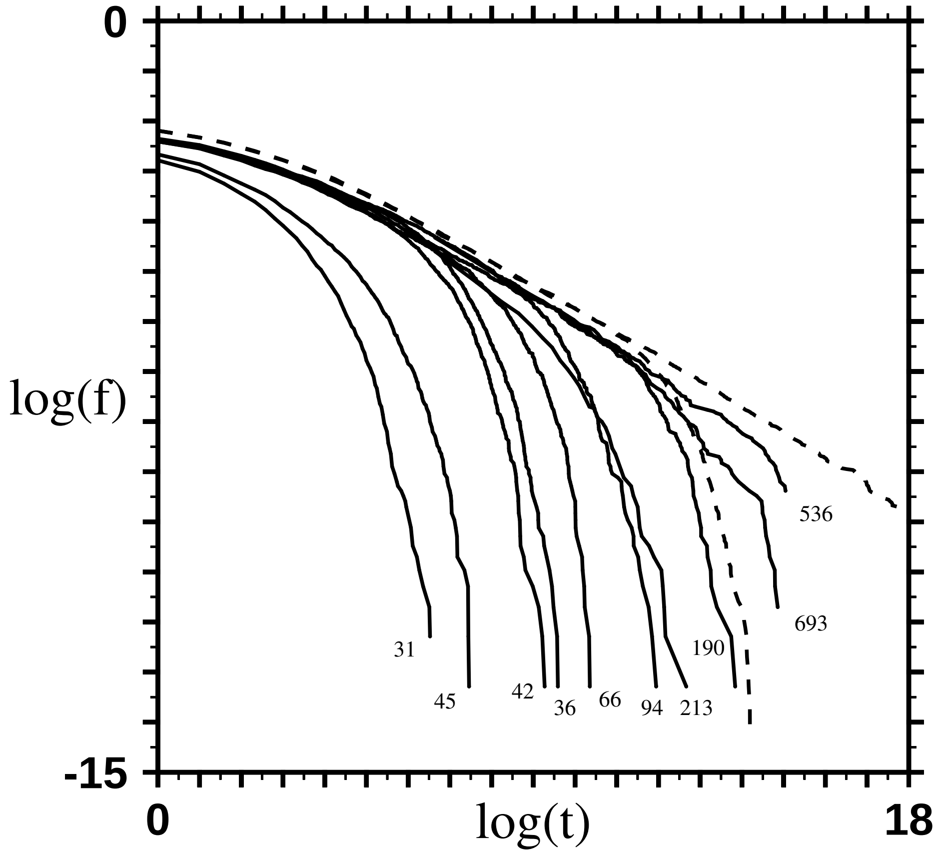

Fig. 13.5 The effect of noise: $f_P(t)$ vs. t for representative lattices $N \in \{31, 36, 42, 45, 66, 94, 190, 213, 536, 693\}$ from each class, with minimal stochastic perturbations in the domains. All curves are labeled by lattice size. Lattices $N \in \{536, 693\}$ were halted at $t = 30,000$ before they reached their asymptotic values; all other lattices decayed to $f_P = 0$ and are plotted to $t = T_c - 1$. As before, $f_{RCW}(t)$ is shown as a dashed line. The RMW decay curve for lattice $N = 190$, which is in the decelerating-decay class for noise-free evolution, is also shown as the dashed line curving down off the bottom of the page.

the time-1 fraction $f_P(1)$. The time-1 fraction increased monotonically with lattice size to a limiting value. The profiles of the lattices fell into the same classification scheme as for V_H-decay, but the functional forms were different.

The stochastic model corresponding to the b_2-subbasin vicinity on a lattice of size N is a random mesa walker with a uniform distribution of interdislocation distances $d(0)$ on the interval $0 < d(0) \leq N/2$. The probability of a given site being occupied by a random walker is proportional to $1/N$, so that as N grows, relatively fewer random walkers fall off the cliffs in the first time-step. Numerical experiments showed that the time-1 fraction $f_{RMW}(1)$ varies with lattice size as $f_{RMW}(1) = c_1(1 - 1/N)$, with $c_1 = 0.25$. The observed decay curves quickly became exponential, $f_{RMW}(t) \sim 2^{-mt}$, and the decay rate m depended on lattice size as $m(N) = kN^{-\alpha}$, with $k = 16.0$ and $\alpha = 2.0$.

Finally, when a small amount of noise was added to the evolution of V_b, all lattices decayed to $f_P = 0$. All of the decay profiles for even-size lattices were approximately exponential, with a decay rate that decreased with increasing N. As for the RMW model, the decay rate on even-size lattices varied with lattice size roughly as $m(N) = kN^{-\alpha},\ \alpha \approx 2,\ k \approx 16$; the agreement with the RMW was better for larger N. Odd lattices decayed slightly faster than even lattices of corresponding size.

All of these results indicate that, while the functional forms of the V_b decay profiles differ in details from those of V_H, the convergence times, instability to perturbations, and overall classification scheme are preserved.

13.6 Finite-Size Scaling

We conclude this chapter with a brief discussion of the dependence of the asymptotic nonattracted fraction f_P on the lattice size N. At various points in the above discussions of temporal vicinity decay, we have mentioned values of f_P for certain lattice sizes, and have hinted at the general shape of the plot of $f_P(N)$ vs. N. In this section, we describe a systematic survey of a wide range of lattices. The need for this survey was originally mentioned in section 12.2. In that discussion, the lattice-size dependence of $f_P(N)$ was cited as part of the argument that Λ^0 and $\Lambda^{0,0}$ were indeed fixed-point regular attractors. The particular assertion made there was that the set of asymptotically nonattracted states in the vicinity $V_1(\Lambda) = V_H(\Lambda)$ has zero size relative to V_1.

The survey results, discussed in detail below, indicate two ways in which the asymptotic fraction f_P varies with lattice size: (i) there is an overall decay envelope that bounds $f_P(N)$ from above and that approaches 0 as a power law in N; and (ii) the set of lattices with $f_P > 0$ is sparse. Either of these facts is sufficient to establish that the nonattracted set has zero size. As we will see, there is in fact a great deal more structure in the relation between $f_P(N)$ and N, largely reflecting the influence of number-theoretic properties of N on the detailed evolution in state space.

On each lattice size in $3 \leq N < 500$, we traced the evolution of the Hamming vicinity using the same technique as in section 13.3, taking ensembles of size $M = 1 \times 10^6$ for $3 \leq N \leq 180$, and $M = 2 \times 10^4$ for $180 < N < 500$. The asymptotic fraction $f_P(N)$ was measured at each value of N.

Fig. 13.6 shows a plot of $f_P(N)$ vs. N for all lattices in $3 \leq N < 500$. Note the fast decay in the maximal values of f_P. Beneath this upper bound, f_P fluctuates wildly with N. The apparent horizontal line at $f_P = 0$ is due to the thickness of the plotted points. Clearly, a great many lattices have $f_P = 0$.

As expected, the lattice-size dependence of f_P differs for N even or odd. Fig. 13.7 shows a log-log plot of $f_P(N)$ vs. even values of N. As noted above, the worst-case lattices, on which f_P is maximal, are given by $N_k = 3 \cdot 2^k - 2, k = 0, 1, 2, \ldots$. These lattices define an overall envelope that decays for large N roughly as

$$f_P(N_k) \propto N_k^{-\alpha} \tag{13.20}$$

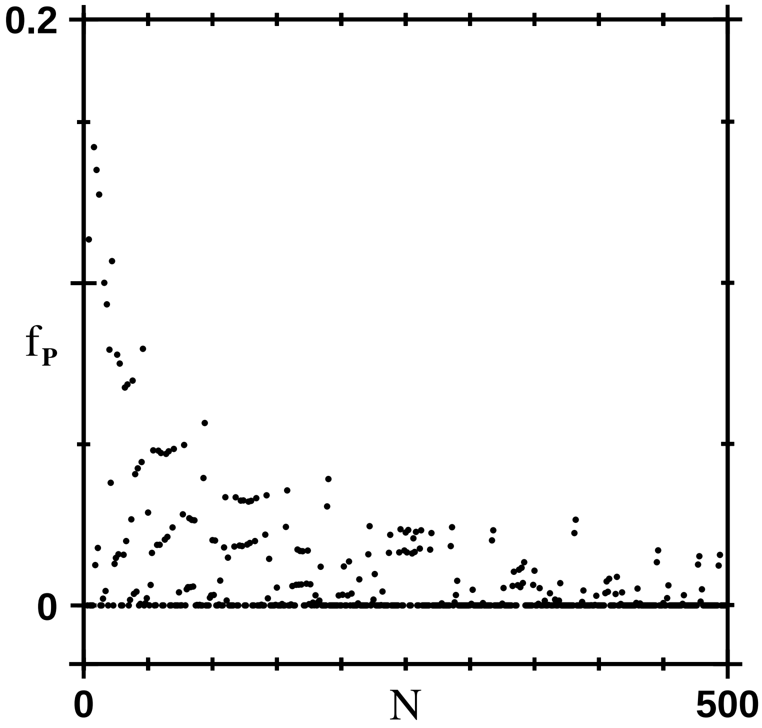

Fig. 13.6 Asymptotic nonattracted fraction $f_P(N)$ vs. lattice size N, for all N.

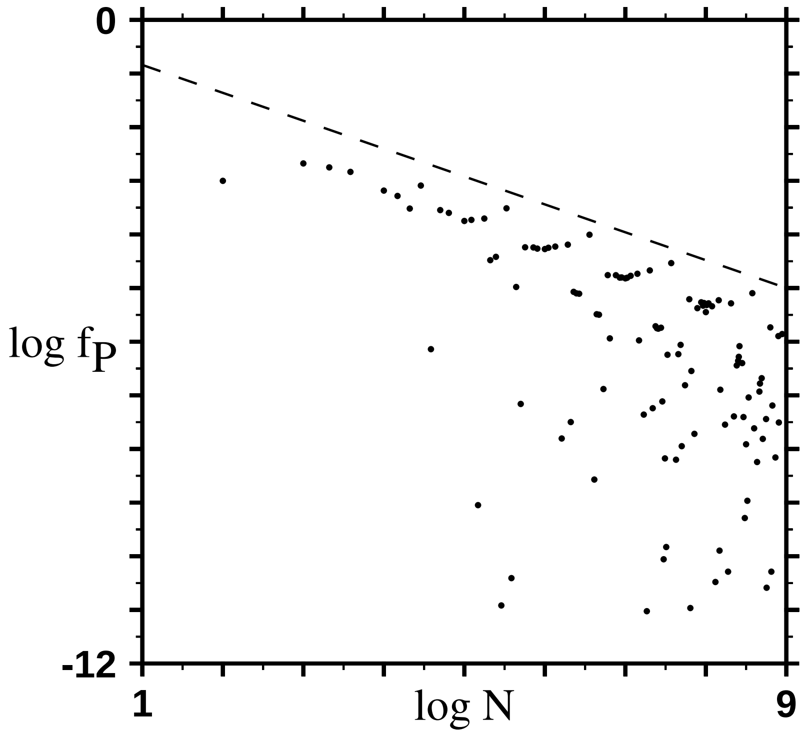

Fig. 13.7 Log-log plot of the asymptotic nonattracted fraction f_P vs. lattice size N, for N even. The diagonal line is slightly above the overall decay envelope. The data is taken from the same experiment at that of fig. 13.6.

with exponent $\alpha = 0.52 \pm 0.05$ determined by a least-squares fit. This decay envelope is indicated by the dashed line in fig. 13.7, which has slope $m = -0.52$. The line has been offset slightly above the worst-case values $f_P(N_k)$ to make the figure easier to read. In the upper right part of the figure we can distinguish similarly-shaped families of points drifting gradually up toward the envelope, with corresponding points in adjacent families separated by a factor of 2 in lattice size. Beneath these families there is a small gap, as can be seen by viewing the figure on edge along the direction of the decay envelope. Beneath the gap, the points are distributed in a more disorderly fashion, although some remnants of the factor-of-two repetition can be observed. These regularities in the dependence of f_P on N, and those noted below for odd-size lattices, are evidently the consequence of number-theoretic properties of N, and are exactly the type of question the algebraic analysis[65] should answer. They may also be indicative of some kind of structural regularity in the *machines* describing the nonattracted sets on the different lattices; such regularity in the machines would contribute to a fully language-theoretic proof that Λ^0 and $\Lambda^{0,0}$ are fixed-point regular attractors (see the discussion on page 125). We plan to investigate these issues in greater detail elsewhere as the need arises.

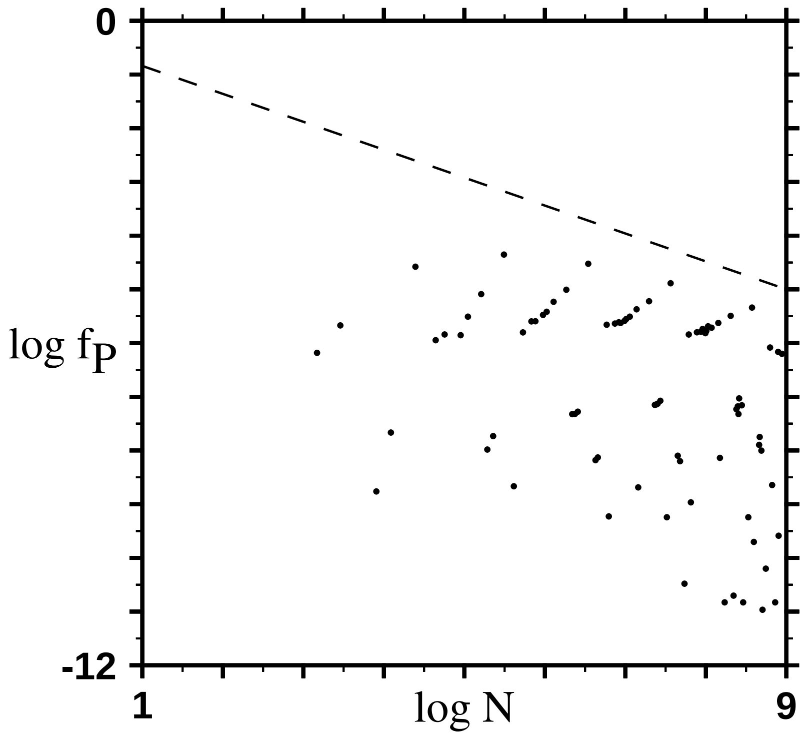

Fig. 13.8 Log-log plot of the asymptotic nonattracted fraction f_P vs. lattice size N, for N odd. The data is taken from the same experiment at that of fig. 13.6.

Fig. 13.8 shows the dependence of the asymptotic nonattracted fraction f_P on lattice size for odd N. Comparison of this plot with fig. 13.7 indicates that the value of f_P for a given odd N is strictly less than the value for even N of about the same size. The (slightly offset) decay envelope from fig. 13.7 is repeated here, showing that the odd lattices also fall beneath the worst-case values (eq. 13.20). In the plot, we can clearly identify the same qualitative features that were present in fig. 13.7; they are in fact considerably more pronounced. There are families of points strung out in roughly diagonal lines angling toward the decay envelope, separated by a factor of 2. Beneath these families there is a well-defined gap, much wider than the corresponding gap for N even, and beneath the gap the points are distributed in a less orderly fashion—though again, some regularity is still in evidence.

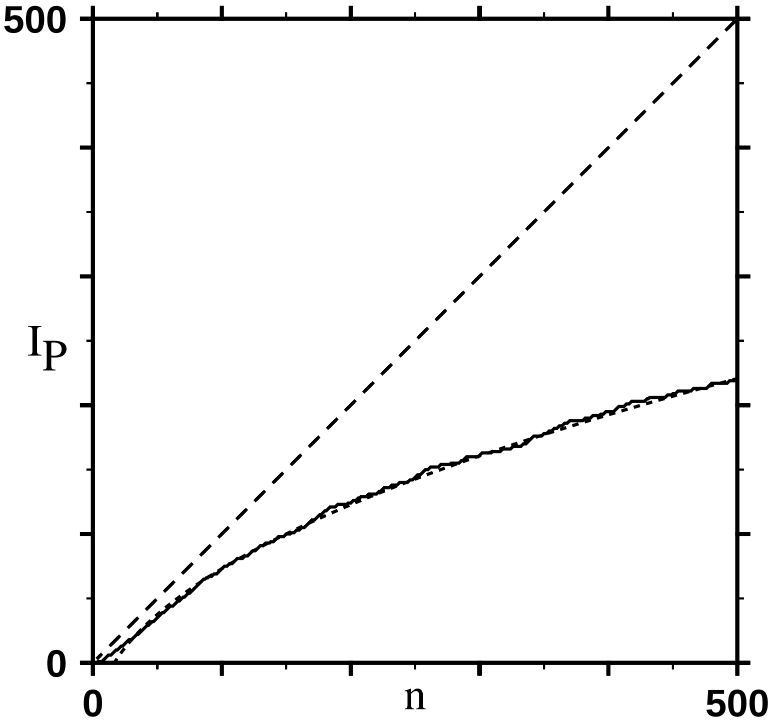

Fig. 13.9 $\mathcal{N}_P$ is sparse: the solid line shows $I_P(n)$ vs. n for the same data as in fig. 13.6. The dotted curve is a least-squares fit to a functional form given in the text. The dashed line is the identity.

To show that the set $\mathcal{N}_P$ of lattices on which $f_P > 0$ is sparse, we measured the number $I_P(n)$ of lattices of size $N \leq n$ in $\mathcal{N}_P$ as a function of n. By definition (see section 2.1), $\mathcal{N}_P$ is a sparse set if and only if

$$\lim_{n \to \infty} \frac{I_P(n)}{n} = 0. \tag{13.21}$$

The dependence of $I_P(n)$ on n is shown in fig. 13.9. The identity is shown as the dashed line with slope 1. The dotted line following the curve of $I_P(n)$ vs. n shows a least squares fit to a functional form of

$$I_P(n) = Cn^{1/2} + A \tag{13.22}$$

with $C = 11.86$ and $A = -43.7$. As the figure indicates, the fit is surprisingly good for $n > 25$. Assuming that this trend continues in the limit $n \to \infty$, then $\mathcal{N}_P$ is a sparse set.

The numerical results described above have forcefully shown that the fraction of Hamming vicinity states $\mathbf{s} \in V_H(\Lambda)$ that do not fall back into the attractor Λ comprise a set of zero size. In other words, to use the notation of section 8.2, we have seen that $S(R|V_1) = 0$, where R is the set of nonattracted states and $V_1 = V_H$. Further, the temporal decay profiles for noisy lattices indicate that the distinctions between classes of temporal evolution are destroyed by very small perturbations. Even the smallest level of noise washes out all deviations from the purely stochastic RMW model. Asymptotically nonattracted orbits, and the state space structures supporting them, appear to exist only for perfectly noise-free evolution; they vanish if even the smallest amount of noise is present. And the latter, presumably, is the physically relevant context for the evolution of ECA 18.

The conclusion is that for moderate to large lattices, the Hamming vicinities' nonattracted sets are both insignificant and unstable: Λ^0 and $\Lambda^{0,0}$ are regular attractors.

14 Outlook

In the preceding chapters, we have developed the elements of the computational mechanics of one-dimensional cellular automata and applied them to a number of different CA. This final chapter summarizes the state of the art—that is, what the methods we have developed can and can not do—and collects together several of the more important open questions and directions for future work that we have mentioned in various places in the text.

14.1 The State of the Art

We have seen how regular languages provide a powerful and extensive class of patterns appropriate to discrete spatially-extended systems such as CA. Regular languages and their associated finite automata are the mathematical underpinnings of the notion of pattern bases in CA. Pattern bases are the fundamental patterns emerging during a system's evolution, governing and organizing its behavior, and determining the form of its analysis. We have developed the idea of a regular domain as a dynamically-homogeneous space-time pattern of a CA. Regular domains are the pattern bases of a wide class of CA. They can describe the entire range of spatial patterns from simply ordered to complex to disordered, "turbulent" patterns. In terms of the domains, coherent structures—walls, dislocations and generally "particles"—are defined as points at which the domain pattern breaks down. The complexities of the domain languages and the computational properties of wall evolution give one measure of the intrinsic computational capability of typical initial conditions evolving under the CA rule.

Machine reconstruction makes it possible to discover experimentally the pattern bases of a given CA from samples of its behavior, rather than relying on visual identification or imposing on the system some preexisting pattern basis. Without this crucial technique, we would be forced to assume a pattern basis *a priori*. Machine reconstruction makes it possible to find pattern bases that are optimally suited to a particular system.

Each pattern basis is "complete" in the sense that an arbitrary configuration can be decomposed into a sequence of domains and walls. From any given set of domains, a domain filter can be constructed that performs this classification with at most two passes over the data. Domain filtration is at once a form of pattern recognition, a method of image compression, and a visualization tool for tracking the trajectories of domain walls. By filtering out the regularities captured by the domain languages, the filtered space-time diagram uncovers and emphasizes structural features in the system's evolution that are obscured in the unfiltered plot.

The filtered plot, which shows the evolution of the walls and dislocations separating domains, can then be treated as data in its own right. It can either be subjected to another step of spatial machine reconstruction, or used to develop an explicit model of wall evolution. The different walls are the "fundamental particles" of the system, and their equations of motion and of interaction can be found. In this way, a hierarchy of models of the system is constructed, with each new level built out of the structural features that were not recognized on the previous level.

Domains, regular vicinities and regular attractors are basic elements in the qualitative pattern dynamics of CA. This draws a connection between the powerful methods of the qualitative theory of dynamical systems and the evolution of patterns, as described by the domains and domain walls. The qualitative pattern dynamics of a CA is summarized by an attractor-basin portrait giving its regular attractors, their basins, and the basins' separatrices.

The FME algorithm provides an explicit method of calculating the evolution of a pattern—i.e., of an ensemble, rather than a single state. This makes it possible to trace analytically the pattern dynamics of the system, even when the patterns are described by positive entropy rate, "turbulent" processes. Tracking ensemble evolution also has the advantage of avoiding the highly nontrivial complications introduced by tracing the evolution of individual states. When a more detailed investigation is indicated, the "transients and cycles" or algebraic analysis can be used in conjunction with the pattern dynamics to get at the exact, lattice-size-dependent, individual state evolution. The FME algorithm can be applied to any system whose dynamic can be expressed in terms of the operation of a finite state transducer. This includes stochastic cellular systems, which were not touched on in this work, as well as CA and a variety of other deterministic cellular systems.

14.2 Open Questions

Some of the elements of the formal specification of dynamics in language space are not fully developed. For example, it has recently come to our attention[68] that the regular language topology is in fact the same as the discrete topology, in which *every* set is both open and closed. This is because a single CA state can be regarded as a regular language consisting of a single word, so individual CA states—singleton sets—are open sets in the regular language topology. This has far-reaching consequences for definitions of stability, proofs of attraction, and so forth. However, on further reflection, it is clear that a single finite CA state does not conform to the notion of pattern that we have been using. A pattern is more than a single state, it is an ensemble of structurally similar states. Further, a regular domain is defined as a process language, which contains words of all finite lengths. We therefore need to restrict the class of languages we use in the formal development to some subset of the regular languages. One possibility is to consider only those languages with an infinite number of words; another is to consider languages with words on some non-sparse set of lattices. In either case, the resulting "pattern space" is a subset of regular language space, which may avoid the problems caused by singleton sets.

Also, as we noted in section 8.2, the definition of attractor is incomplete. A fully satisfactory definition would include some form of minimality or transitivity. Another alternative is to include some form of stability under random noise, as was addressed in the empirical studues of section 13.4.

A more application-oriented development would be to create a larger range of choices for regular vicinities. The Hamming vicinity and the restricted Hamming vicinity implement one natural type of perturbation, but may not be particularly suited to the dynamics of any given system. Subbasin vicinities can be defined for systems of annihilating particles; but what about in general?

Finally, with regard to the specific CA that we have studied, there are a number of unanswered questions. The language-theoretic proofs of ECA 18's attractors on finite lattices are unfinished. We have indicated a way in which these might proceed, but the actual work remains to be done. The analysis of the finite-size scaling properties of nonattracted states in ECA 18 can probably be put on a rigorous foundation by algebraic techniques. Of the other rules that we studied, none of the regular attractors have been explicitly identified, although in some cases it is not difficult to predict what these must be. The detailed development of wall evolution equations has yet to be done for many of the rules. We have been content to indicate the direction that further work will most likely take.

14.3 Further Development

An immediate question for further work is the problem of finding domains from the rule table itself, or from the global machine. Currently, we must explicitly evolve the CA and reconstruct machines from the spatial configurations. Yet the necessary information is presumably implicit in the time-1 global language Ω_1, since it is the image of the entire extended state space $\mathcal{A}^*$ under the action of the dynamic. In the minimal machine $\mathrm{M}(\Omega_1)$, edges do double duty in the sense that a given edge may correspond to a cell in a domain or to a wall, depending on the previous edges taken. Perhaps there is some way to de-minimize or tease out $\mathrm{M}(\Omega_1)$ to make the domains evident. If the domains have already been found, the domain transducer T_Λ composed with $\mathrm{M}(\Omega_1)$ yields the desired machine, in which each edge corresponds to a wall cell or a cell in a domain. But in order to construct T_Λ in the first place, we must already know the domains.

One thing that was only briefly mentioned was the reconstruction of the dynamic from space-time data by performing transducer reconstruction on parent/child symbol pairs. The CA update rule transducer has a built-in spatial "delay" of r symbols, so that when parent symbol σ_t^{i+r} is read, the child symbol σ_{t+1}^i is emitted. This delay simply reflects the fact that the entire parent neighborhood must be read before the child symbol can be emitted. In reconstructing transducers from space-time data, the delay is chosen before the machine is reconstructed, in effect imposing another approximation parameter on the reconstruction.

Minimization of the update transducer is a new and possibly useful method of quantifying the complexity of the dynamic itself. While all CA rules of a given radius r and alphabet size k are naturally implemented with transducers having $2k^{2r} - 1$ states, some of these are not minimal. Two striking examples are the nilpotent rule, that maps all parent neighborhoods to the same symbol, e.g. 0, and the identity rule, that leaves all configurations unchanged. The minimal update transducers for both of these rules have just one state. The relationship between the size of the minimal update transducer and the system's space-time patterns is not clear, however. One preliminary result is that, because of the details of the machine composition operation, a transducer with fewer states imposes a lower ceiling on the growth rate of machines during ensemble evolution than does a transducer with more states.

As described in this work, the methods of computational mechanics have been applied to *spatial* patterns only. We have not touched on the analysis of the *temporal* patterns in the space-time evolution of CA. The method of investigating these is similar to the investigation of spatial patterns. Machine reconstruction is performed on temporal sequences, such as the timeseries of values of a given cell in the lattice, or the union of such timeseries over the different cells. The machines reconstructed in this way describe the temporal regularities in the data. They are necessarily related to the spatial machines through the CA's evolution, although the relationship is probably not a simple one.

The fuller description of the computational properties of a spatially-extended system is given by the *space-time* machine, of which the spatial and temporal machines are projections. Crutchfield[18] has suggested the following approach to space-time machine reconstruction. The machine is reconstructed by means of an extension of the machine reconstruction algorithm which uses a *path automaton* to convert the two-dimensional space-time plot into a one-dimensional symbol sequence. Informally, the path automaton is a "bug" that walks around on the space-time plane, emitting symbols as it encounters them. The output of the path automaton consists of both the cell value and the direction the path automaton was moving when the cell was encountered. Instead of a tree of symbol sequences, space-time machine reconstruction builds a *pyramid* with branches labeled by pairs giving cell value and scan direction. The most general path automaton chooses a sequence of moves at random. But path automata can have restrictions built into the path itself, so that, for example, purely spatial or temporal or oblique regularities are detected. More complex path automata can scan the space-time plane in any number of different ways.

The extension of computational mechanics to coupled map lattices can be done using a generating partition to discretize the real-valued site variables. In this way, temporal symbol sequences provide information about the site variables, and the space-time machine describes the action of both the spatial shift and the temporal shift—i.e., the CA rule—on the symbolic dynamics of the system.

Space-time reconstruction is also closely linked with the extension to higher-dimensional systems. The latter are of obvious importance, since for example the majority of natural systems are two- or three-dimensional. The difficulty in higher dimensions is that there is no longer an unambiguous ordering to the cells: the states of the system are not symbol sequences. Just as for the space-time machines, the path automaton may also provide a means of overcoming this problem.

All three of these developments—space-time machines, symbolic dynamics of coupled map lattices, and higher-dimensional systems—require essentially the same innovations in the existing framework. It is therefore to be expected that work on any one of them can be applied to the others with little or no adaptation. Extending the methodology to these new areas is perhaps the most pressing issue in the computational mechanics of spatially-extended dynamical systems.

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