Practical Computational Mechanics

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Presented to the Dynamics of Learning Group, SFI, 17 July 2001.

I will review a relatively new approach to structural complexity that defines a process's causal states and gives a procedure for finding them. It turns out that the causal-state representation—an ϵ-machine—is the minimal one consistent with accurate prediction. The claim that this representation capture's all of a process's structure derives from ϵ-machine optimality and uniqueness and on how ϵ-machines compare to alternative representations. For example, one can directly relate measures of randomness and structural complexity obtained from ϵ-machines to more familiar ones from ergodic and information theories.

These notes show the basics of ϵ-machine reconstruction, illustrating the "batch" algorithm on a number of examples, both finite and infinite state.

Agenda

- Basics
- ϵ-Machine Reconstruction
- Entropy and Complexity
- Examples for Everyone
- The Wild Side

Processes

Observations $\overleftrightarrow{S} = \overleftarrow{S}\,\overrightarrow{S}$

$$past \quad \rightarrow \quad future$$

$$\ldots s_{-L} s_{-L+1} \ldots s_{-1} s_0 | s_1 \ldots s_{L-1} s_L \ldots, \quad s_i \in \mathcal{A}$$

Entropy rate (degree of unpredictability)

$$h_\mu = H(\Pr(s_i | s_{i-1} s_{i-2} s_{i-3} \ldots))$$

How much past information transmitted to future? (**Excess Entropy**)

$$\mathbf{E} = H\left(\frac{\Pr(\overleftrightarrow{S})}{\Pr(\overleftarrow{S})\Pr(\overrightarrow{S})}\right)$$

Note: Property that describes "raw" sequences.

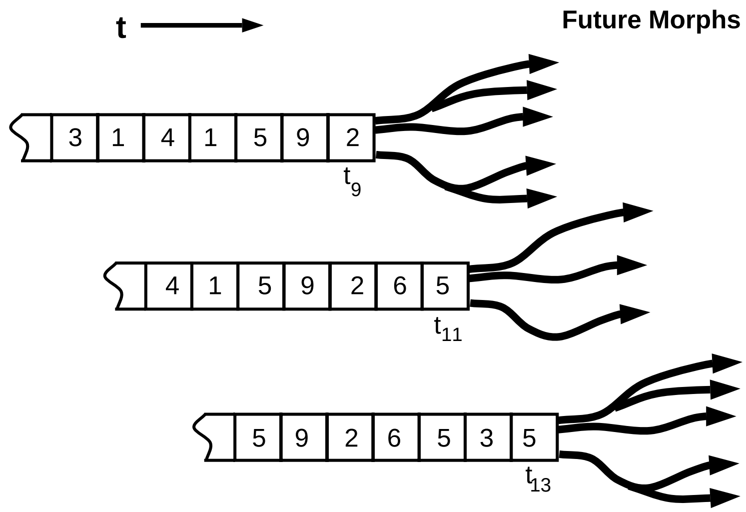
Prediction Game

Causal States and ϵ-Machines

Morph = Set of possible (future) behaviors, given knowledge of past

Causal State = Same future morph at different times

Causal state = identical conditions of ignorance/knowledge



Probabilistic Morph = $\Pr(\text{Future}|\text{Past})$

Causal State = Equivalence class of probabilistic morphs

$$t_i \sim t_j \Leftrightarrow \Pr(\text{Future}|\text{Past at } t_i) = \Pr(\text{Future}|\text{Past at } t_j)$$

Set of causal states: $\mathcal{S} = \{\text{Classes of histories induced by relation}$

Transition dynamic: $\mathcal{T} : \mathcal{S} \rightarrow \mathcal{S}$

ϵ-Machine = $\{\mathcal{S}, \mathcal{T}\}$

Always unique start state: $\mathcal{S}_0 \in \mathcal{S}$.

Entropy rate (degree of unpredictability)

$$h_\mu = H\left[\Pr\left(\mathcal{S} \underset{s \in \mathcal{A}}{\longrightarrow} \mathcal{S}'|\mathcal{S}\right)\right]$$

Statistical Complexity (size of ϵ-machine)

$$C_\mu = H[\Pr(\mathcal{S})]$$

Mathematical Foundations

(JPC and C. R. Shalizi, "Thermodynamic Depth of Causal States: Objective Complexity via Minimal Representations", Physical Review E **59** (1999) 275-283; C. R. Shalizi and JPC: "Computational Mechanics: Pattern and Prediction, Structure and Simplicity" (1999) June.

Lemma 1: Given $\mathcal{S}$, past and future are conditionally independent.

Proposition: ϵ-Machines are Semi-Groups.

Lemma 2: ϵ-Machines are Deterministic.

Definition: ϵ-Machine Reconstruction

$$\Pr(\vec{s}) \rightarrow \{\mathcal{S}, \mathcal{T}\} \ .$$

Theorem 1: Causal States are Maximally Predictive,

$$H\left[\vec{s}^L|\mathcal{R}\right] \geq H\left[\vec{s}^L|\mathcal{S}\right] \ , \text{ where } \mathcal{R} \ = \ \text{rival states.}$$

Corollary: Causal States are Sufficient Statistics.

Theorem 2: Causal States are Minimal,

$$C_\mu\left(\widehat{\mathcal{R}}\right) \geq C_\mu(\mathcal{S}) \ , \quad \text{where } \widehat{\mathcal{R}} \ = \text{prescient rivals.}$$

Theorem 3: Causal States are Unique,

$$C_\mu\left(\widehat{\mathcal{R}}\right) = C_\mu(\mathcal{S}) \quad \Rightarrow \quad \widehat{\mathcal{R}} = f(\mathcal{S}) \ \wedge \ \mathcal{S} = g\left(\widehat{\mathcal{R}}\right) \ .$$

Theorem 4: ϵ-Machines are Minimally Stochastic,

$$H\left[\widehat{\mathcal{R}}'|\widehat{\mathcal{R}}\right] \geq H[\mathcal{S}'|\mathcal{S}] \ .$$

Theorem 5: Statistical Complexity upper bounds Excess Entropy,

$$\mathbf{E} \leq C_\mu(\mathcal{S}) \ \ .$$

1. $\mathbf{E} \neq$ "stored information"; C_μ is.
2. $\mathbf{E}$ is *apparent* information ... from sequences, not internal states.
3. Theorem 5 $\Rightarrow$ we must build models.

ϵ-Machine Reconstruction

History: Geometry from a Time Series

- Steps
  a. Parse Tree of depth D
  b. Morphs of depth L
  c. Causal States
  d. Causal Transitions
  e. Symbol Transition Matrices
  f. Connection Matrix
- Topological reconstruction, first
- Examples
  a. Constant Process: 1^*
  b. Fair Coin: $(0 + 1)^*$
  c. Period-2: $(10)^*$
- Comments
  a. Complexity-Entropy Diagram
  b. How to set D and L? L = D/2.
  c. Recurrent v. transient states
- Probabilistic reconstruction

Randomness and Structure: Entropy v. Complexity

- Topological entropy:

$$h_0 = \log_2 \lambda$$

- Topological complexity:

$$C_0 = \log_2 |\mathcal{S}|$$

- Entropy rate:

$$h_\mu = -\sum_{v \in \mathcal{S}} p_v \sum_{s \in \mathcal{A}} p_{v \underset{s}{\rightarrow} v'} \log_2 p_{v \underset{s}{\rightarrow} v'}$$

- Statistical complexity:

$$C_\mu = -\sum_{v \in \mathcal{S}} p_v \log_2 p_v$$

- Eigenvalues and eigenvectors.
- For above examples.

Examples for All

- A Period-3 Process: $(101)^*$.
- Golden Mean Process: No consecutive 0s.
- Noisy Period-2 Process: $(1\Sigma)^*$, where $\Sigma = \{0, 1\}$.
- Even Process: 1s occur in blocks of even number, bordered by 0s.

What to See and Do

- Reconstruct ϵ-Machine
  a. Parse Tree and Morphs
  b. State-Transition Diagram
  c. Symbol Transition Matrices
- Calculate entropy rate and statistical complexity
- Interpret ϵ-machine structure: What do the states mean in each case?

The Wild Side

- Morse-Thue Sequence (Onset of Chaos via Period-Doubling): $0 \rightarrow 11$ and $1 \rightarrow 10$.
- Simple Nondeterministic Source.
- More General Hidden Markov Processes.