From Determinism to Stochasticity

Reading for this lecture:

(These) Lecture Notes.
Cave: Sensory Immersive Visualization

A video or two:

• Cave Dynamics
• Protein Manipulation

Tour:
This Thursday (27 May)
Two Sessions: 4:00-5:00 PM & 5:00 - 6:00 PM

Where:
Rm. 3255 Earth & Physical Sciences Bldg
Top floor, northwest corner.
# Cave Tour Sign-Up

**Thursday 27 May**

![Image](image.png)

## Balanced Numbers

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<th>Session One: 4:00-5:00 PM</th>
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One-D Map Tool

Example Code
Now on course website
See Homework Page
Probability Theory of Dynamical Systems:

The Big Deal:

Deterministic systems can be:

“Unpredictable”
“Noisy”
“Random”
“Chaotic”

What is the role of probability?
Probability Theory of Dynamical Systems:

Probability Theory Review:

Continuous Random Variable: $X$

Takes values over continuous space: $\mathcal{X}$

Cumulative distribution function: $P(x) = \Pr(X \leq x)$

If continuous, then random variable is.

Probability density function: $p(x) = P'(x)$

Normalized: $\int_{-\infty}^{\infty} dx \ p(x) = 1$ or $\Pr(X < \infty) = 1$

Support of distribution: $\text{supp} X = \{ x : p(x) > 0 \}$
Continuous random variable $X$:

Uniform distribution on interval: $\mathcal{X} = \mathbb{R}$

Density: $p(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

Distribution: $\Pr(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$

Support: $\text{supp } X = [0, 1]$
Continuous random variable \( X \):

Gaussian: \( \mathcal{X} = \mathbb{R} \)

Density: \( p(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \)

Distribution: \( P(x) = \int_{-\infty}^{x} dy \ p(y) \equiv \text{Erf}(x) \)

Support: \( \text{supp} \ X = \mathbb{R} \)
Dynamical Evolution of Distributions:

Dynamical system: \( \{ \mathcal{X}, T \} \)

**State density:** \( p(x) \) \( x \in \mathcal{X} \)

Can evolve individual states and sets: \( T : x_0 \rightarrow x_1 \)

**Initial density:** \( p_0(x) \) Model of measuring a system

**Evolve a density?** \( p_0(x) \rightarrow_T p_1(x) \)
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Probability Theory of Dynamical Systems ...
Dynamical Evolution of Distributions ...

Conservation of probability:

\[ p_1(y)dy = p_0(x)dx \]

Perron-Frobenius Operator:

Locally: \( y = T(x) \)

\[ p_{n+1}(y) = \frac{p_n(x)}{|T'(x)|} \]

Globally:

\[ p_{n+1}(y) = \sum_{x \in T^{-1}(y)} \frac{p_n(x)}{|T'(x)|} \]
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Frobenius-Perron Equation:

\[ p_{n+1}(y) = \int dx \ p_n(x) \delta(y - T(x)) \]

Dirac delta-function:

\[ \delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \]

\[ \int dx \ \delta(x - c) f(x) = f(c) \]

\[ \int dx \ \delta(x) = 1 \]
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Example: Delta function initial distribution

Map: \( x_{n+1} = f(x_n) \)

Initial condition: \( x_0 \in \mathbb{R} \)

Initial distribution: \( p_0(x) = \delta(x - x_0) \)

\[
p_1(y) = \int dx \ p_0(x) \ \delta(y - f(x)) \\
= \int dx \ \delta(x - x_0) \ \delta(y - f(x)) \\
= \delta(y - f(x_0)) \\
= \delta(y - x_1) \\
\vdots \\
p_n(y) = \delta(y - x_n) \quad \text{... reduces to an orbit}
\]
Delta function IC: The easy case and expected result.

What happens when the IC has finite support?

\[ p_0(x) = \begin{cases} 
20, & |x - 1/3| \leq 0.025 \\
0, & \text{otherwise}
\end{cases} \]

Consider a set of increasingly more complicated systems and how they evolve distributions ...
Example:

**Linear circle map**

\[ x_{n+1} = 0.1 + x_n \pmod{1} \]

\[ f'(x) = 1 \]
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Example: Shift map

\[ x_{n+1} = f'(x) = 2 \]

Spreading: \[ f'(x) = 2 \]

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Example:

Tent map $a = 2.0$

\[ x_{n+1} \]

Spreading: $|f'(x)| = 2$
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Example:
Logistic map \( r = 4 \)

\[
\begin{align*}
\log P(x) = & 6 \\
\log P(x) = & 2 \\
\log P(x) = & 0 \\
\log P(x) = & -2 \\
\log P(x) = & -6
\end{align*}
\]

\[
t = 0 \quad t = 1 \quad t = 2 \\
t = 3 \quad t = 4 \quad t = 5 \\
t = 6 \quad t = 7 \quad t = 8
\]

\[
f'(x) = 4(1 - 2x)
\]

Spreading: \( x < 3/8 \) or \( x > 5/8 \)

Contraction: \( 3/8 < x < 5/8 \)

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Example:
Logistic map \( r = 3.7 \)

Peaks in distribution are images of maximum
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Time-asymptotic distribution: What we observe

How to characterize?

**Invariant measure:**
A distribution that maps “onto” itself
Analog of invariant sets

Stable invariant measures:
Stable in what sense?
Robust to noise or parameters or ???

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Invariant measures for 1D Maps:

Probability distribution (density $p^*(x)$) that is invariant:

1. Distribution’s support must be an invariant set:

$$\Lambda = f(\Lambda), \quad \Lambda = \text{supp } p^*(x) = \{x : p^*(x) > 0\}$$

II. Probabilities “invariant”:

Distribution a fixed point of Frobenius-Perron Equation

$$p^*(y) = \int dx \ p^*(x) \ \delta(y - f(x))$$

Functional equation: Find $p^*(\cdot)$ that satisfies this.
Example: Periodic-k orbit \( \{x_1, x_2, \ldots, x_k\} \) has density

\[
p(x) = \delta \left( \prod_{i=1}^{k} (x - x_i) \right)
\]

Is it invariant?

\[
p_1(y) = \int dx \ p(x) \delta(y - f(x))
\]

\[
= \int dx \ \delta \left( \prod_{i=1}^{k} (x - x_i) \right) \delta(y - f(x))
\]

\[
= \delta \left( \prod_{i=1}^{k} (y - f(x_i)) \right)
\]

\[
= \delta \left( \prod_{i=1}^{k} (y - x_{(i+1) \mod k}) \right)
\]

\[
= \delta \left( \prod_{i=1}^{k} (y - x_i) \right)
\]

Yes!

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Example: Shift map invariant distribution

Uniform distribution: $p(x) = 1, \ x \in [0, 1]$

$p_1(y) = p'_1(y) + p''_1(y)$

$p_1(x) = p_0(x) \quad (y \Rightarrow x)$
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Example: Shift map invariant distribution
Uniform distribution: \( p(x) = 1, \ x \in [0, 1] \)

Via Frobenius-Perron Equation: Two cases

**A:** \( 0 \leq x \leq 1/2 \)

\[
p'_1(y) = \int_0^{1/2} dx \ p_0(x) \delta(y - f(x)) = \int_0^{1/2} dx \ \delta(y - 2x) = \frac{1}{2}
\]

\[
p_1(y) = p'_1(y) + p''_1(y) = p_0(x) \quad (y \Rightarrow x)
\]

**B:** \( 1/2 < x \leq 1 \)

\[
p''_1(y) = \int_{1/2}^{1} dx \ p_0(x) \delta(y - f(x)) = \int_{1/2}^{1} dx \ \delta(y - 2x) = \frac{1}{2}
\]

\[
p_1(y) = p'_1(y) + p''_1(y) = p_0(x) \quad (y \Rightarrow x)
\]
Example: Tent map

\[ x_{n+1} = \begin{cases} 
  ax_n, & 0 \leq x_n \leq \frac{1}{2} \\
  a(1 - x_n), & \frac{1}{2} < x_n \leq 0 
\end{cases} \]

Fully two-onto-one: \( a = 2 \)

Uniform distribution is invariant: \( p(x) = 1, \ x \in [0, 1] \)

Proof from FP Equation: Two cases

First case: exactly that of shift map

Second case: \(|\text{slope}|\) is all that’s important

\[
1/2 < x \leq 1 \quad p''_1(y) = \int_{1/2}^{1} dx \ p_0(x) \delta(y - f(x))
\]

\[
= \int_{1/2}^{1} dx \ \delta(y - (2 - 2x)) = \frac{1}{2}
\]
Example: Tent map where two bands merge to one: \( a = \sqrt{2} \)

Invariant distribution:

\[
p(x) = \begin{cases} 
  p_0, & x_{\text{min}} \leq x \leq x^* \\
  p_1, & x^* < x \leq x_{\text{max}} \\
  0, & \text{otherwise}
\end{cases}
\]

\[
x_{\text{max}} = \frac{a}{2} \\
x_{\text{min}} = a(1 - a/2) \\
x^* = \frac{a}{1 + a}
\]

Equal areas: \( p_0(x^* - x_{\text{min}}) = p_1(x_{\text{max}} - x^*) \)

Normalization: \( p_0(x^* - x_{\text{min}}) + p_1(x_{\text{max}} - x^*) = 1 \)

\[
p_0 = \frac{1}{2(x^* - x_{\text{min}})} \quad p_1 = \frac{1}{2(x_{\text{max}} - x^*)}
\]
Example: Logistic map \( x_{n+1} = r x_n (1 - x_n) \)

Fully two-onto-one: \( r = 4 \)

Invariant distribution? \( p(x) = \frac{1}{\pi \sqrt{x(1-x)}} \)

Exercise.

Hint: Coordinate transform to tent map.
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Numerical Example: Tent map

Typical chaotic parameter: \( a = 1.75 \)

Two bands merge to one: \( a = \sqrt{2} \)
Numerical Example: Tent map

Typical chaotic parameter: $a = 1.75$

Two bands merge to one: $a = \sqrt{2}$
Numerical Example: Tent map

Typical chaotic parameter:
\[ a = 1.75 \]

Two bands merge to one:
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Probability Theory of Dynamical Systems ... 

Numerical Example: Logistic map \( x_{n+1} = r x_n (1 - x_n) \)

Typical chaotic parameter: \( r = 3.7 \)

Two bands merge to one: \( r = 3.6785735104283219 \)
Numerical Example: Logistic map \[ x_{n+1} = rx_n(1 - x_n) \]

Typical chaotic parameter: \( r = 3.7 \)

Two bands merge to one: \( r = 3.6785735104283219 \)

\[ -\log P(x) \]

\[ 0 \quad 1 \]

Monday, May 24, 2010
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Numerical Example: Logistic map \( x_{n+1} = rx_n(1 - x_n) \)

Typical chaotic parameter: \( r = 3.7 \)

Two bands merge to one: \( r = 3.6785735104283219 \)

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Numerical Example: Cusp map \( x_{n+1} = a(1 - |1 - 2x_n|^b) \)

\((a, b) = (1, 1/2)\)
Numerical Example: Cusp map \( x_{n+1} = a(1 - |1 - 2x_n|^b) \)

\((a, b) = (1, 1/2)\)
Issue: Many invariant measures in chaos:

An infinite number of unstable periodic orbits: Each has one. But none of these are what one sees, one sees the aperiodic orbits.

How to exclude periodic orbit measures?

Add noise and take noise level to zero; which measures are left?

Robust invariant measures.
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Reading for next lecture:

*Lecture Notes.*