Mechanisms of Chaos: Stable Instability

Reading for this lecture:

NDAC, Sec. 12.0-12.3, 9.3, and 10.5.
Mechanisms of Chaos ...

Unpredictability:
- Orbit complicated: difficult to follow
- Repeatedly convergent and divergent
- Net amplification of small variations

What geometry produces this?

**Stretch and fold:**
- Flow stretches state space
- But to be stable, must be done in a compact region
  - Must fold back into region
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**Baker’s transformation:**

kneading state space

![Baker's transformation diagram](image-url)
Mechanisms of Chaos ...  

Baker’s transformation ...  
kneading state space
Mechanisms of Chaos ...

Baker’s transformation ...

2D Baker’s Map:

\[(x_n, y_n) \in [0, 1] \times [0, 1]\]

\[x_{n+1} = 2x_n \pmod{1}\]

\[y_{n+1} = \begin{cases} \frac{1}{2}y_n, & x_n \leq \frac{1}{2} \\ \frac{1}{2} + \frac{1}{2}y_n, & x_n > \frac{1}{2} \end{cases}\]
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Baker’s transformation ...

Stability? \[ A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \]

Calculate:

\[ \lambda_1 = 2 \quad \text{Stretch} \quad \vec{v}_1 = (1, 0) \quad \text{Only horizontal} \]

\[ \lambda_2 = \frac{1}{2} \quad \text{Shrink} \quad \vec{v}_2 = (0, 1) \quad \text{Only vertical} \]

\[ \text{Tr}(A) = \frac{5}{2} \]

\[ \text{Det}(A) = 1 \quad \text{Area preserving: No attractor per se} \]

Independent of \( \vec{x} \)
Mechanisms of Chaos ...

Dissipative Baker’s Map:
Mechanisms of Chaos ...

Dissipative Baker’s Map ... again!

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Mechanisms of Chaos...

Dissipative Baker’s Map...

\[
x_{n+1} = 2x_n \pmod{1}
\]

\[
y_{n+1} = \begin{cases} 
ay_n, & x_n \leq \frac{1}{2} \\
\frac{1}{2} + ay_n, & x_n > \frac{1}{2}
\end{cases}
\]

\[
a \in \left[0, \frac{1}{2}\right]
\]
Mechanisms of Chaos ...

Dissipative Baker’s Map ...

Stability? \[ A = \begin{pmatrix} 2 & 0 \\ 0 & a \end{pmatrix} \]

Calculate:

\[ \lambda_1 = 2 \quad \vec{v}_1 = (1, 0) \]
\[ \lambda_2 = a \quad \vec{v}_2 = (0, 1) \quad \text{Independent of } \vec{x} \]

\[ \det(A) = 2a \quad \text{Dissipative: } a < 1/2 \]
\[ \text{Area contraction} \]

Attractor!

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Dissipative Baker’s Map Simulation: $a = 0.3$
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Dissipative Baker’s Map ...

Stability? \((x, y)\) versus \((x + \epsilon, y + \delta)\)

\[
\Delta x_1 = 2(x_0 + \epsilon) - 2x_0 = 2\epsilon
\]
\[
\Delta y_1 = a(y_0 + \delta) - a y_0 = a\delta
\]

\[
\Delta x_n = 2^n \epsilon \quad \text{Exponential Growth of Errors}
\]
\[
\Delta y_n = a^n \delta \quad \text{Exponential Stability}
\]
Dimension of a Set:

Number of boxes to cover set at given measurement resolution:

\[ \epsilon = \frac{1}{2}, \quad N = 4 \]
\[ \epsilon = \frac{1}{4}, \quad N = 16 \]
\[ \epsilon = \frac{1}{8}, \quad N = 64 \]

\[ N(\epsilon = \frac{1}{2^n}) = \left( \frac{1}{2^n} \right)^{-2} = 2^{2n} \]

\[ N(\epsilon) \propto \epsilon^{-2} \]

Generalizing

\[ N(\epsilon) \propto \epsilon^{-d} \]

Or (Definition) **dimension**: \[ d = \lim_{\epsilon \to 0} - \frac{\log N(\epsilon)}{\log \epsilon} \]
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Dimension of Dissipative Baker’s Attractor ...  

At iteration $n$:

$2^n$ strips of thickness $a^n$

How many boxes $N(\varepsilon)$ to cover attractor at resolution $\varepsilon$?

Take: $\varepsilon = a^n$

Number of boxes for each strip: $a^{-n}$

$$N(\varepsilon) = a^{-n} \times 2^n = \left(\frac{a}{2}\right)^{-n}$$
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Dimension of Dissipative Baker’s Attractor ...

Dimension:

\[
d = \lim_{\epsilon \to 0} - \frac{\log N(\epsilon)}{\log \epsilon} \\
= \lim_{n \to \infty} - \frac{\log(a/2)^{-n}}{\log a^n} \\
= 1 + \frac{\log \frac{1}{2}}{\log a}
\]

\( a = 0.3 \Rightarrow d = 1.576 \ldots < 2 \)!

Area preserving: as \( a \to \frac{1}{2} \), \( d \to 2 \)

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**Cat map (aka toral automorphism):** \((x, y) \in T^2\)

Intrinsic stretch/shrink directions

\[
\begin{pmatrix}
  x_{n+1} \\
  y_{n+1}
\end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad \text{(mod 1)}
\]

**Fixed point:** \(\vec{x}^* = (0, 0)\)
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Cat map (aka Toral automorphism) ...

\[
\begin{pmatrix}
  x_{n+1} \\
  y_{n+1}
\end{pmatrix} = \begin{pmatrix}
  2 & 1 \\
  1 & 1
\end{pmatrix} \begin{pmatrix}
  x_n \\
  y_n
\end{pmatrix} \pmod{1}
\]

Calculate:

\[
\begin{align*}
\lambda_1 &= \frac{3 + \sqrt{5}}{2} > 1 \quad \text{stretch} \\
\lambda_2 &= \frac{3 - \sqrt{5}}{2} < 1 \quad \text{shrink}
\end{align*}
\]

\[
\vec{v}_1 = \left( \frac{1 + \sqrt{5}}{2}, 1 \right) \\
\vec{v}_2 = \left( \frac{1 - \sqrt{5}}{2}, 1 \right)
\]

\[
\text{Tr}(A) = 3
\]

\[
\text{Det}(A) = 1 \quad \text{area preserving}
\]

Independent of \( \vec{x} \)
Mechanisms of Chaos ...
Mechanisms of Chaos ...
Mechanisms of Chaos ...

Hénon map: \((x, y) \in \mathbb{R}^2\)

\[
x_{n+1} = y_n + 1 - ax_n^2 \\
y_{n+1} = bx_n
\]

Stretch and fold depend on location

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Hénon map ...

Stretch & fold depend on location:

Jacobian:

\[
A = \begin{pmatrix}
-2ax_n & 1 \\
 b & 0
\end{pmatrix}
\]

Dissipative (and orientation reversing):

\[\text{Det}(A) = -b\]
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Hénon Attractor:

Control parameters: \((a, b) = (1.4, 0.3)\)
Self-similar:

Self-similar attractor = Dissipation + Instability
Mechanisms of Chaos ...

How does the stretch & fold mechanism work in ODEs?
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Rössler Chaotic Attractor:

Branched manifold:
Mechanisms of Chaos ...

Rössler Chaotic Attractor ...

Branched manifold unfurled and refurled
Mechanisms of Chaos ...

Rössler stability + instability:
Dot spreading demo (ds)

Time step = 0.03
Remembered trajectory = 10000
Orient
3000 e
nEns = 100000
IC = (0,-7,0)
radius = .1
1, 1, 1
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Lorenz stability + instability:
Dot spreading demo (ds)

Time step = .005
Remembered trajectory = 10000
Orient
1000 e
nEns = 60000
IC = (5,5,5)
radius = .05
1, 1, 1
Mechanisms of Chaos ...

Quantifying instability: Growth-of-error model:

\[ \| \delta(t) \| \sim \| \delta(0) \| \ e^{\lambda t} \]

Or
\[ \lambda \sim t^{-1} \ln \left( \frac{\| \delta(t) \|}{\| \delta(0) \|} \right) \]

Lyapunov Characteristic Exponent (LCE):

\[ \lambda = \lim_{t \to \infty} \lim_{\| \delta(0) \| \to 0} \frac{1}{t} \log_2 \frac{\| \delta(t) \|}{\| \delta(0) \|} \]

\( \delta(t) \) aligns with most unstable direction!

\( \lambda \): Exponential rate of growth of errors
Units: [bits per second]
Mechanisms of Chaos ...

**Measurement Resolution:** $\epsilon$

Number of scale factors to locate initial state: $I_0 = -\log_2 \epsilon$

Resolution loss rate (bits per second): $\lambda$

**Prediction horizon:** $t_{\text{unpredict}} \sim \frac{I_0}{\lambda}$

**Example:**

Loss rate: Factor of 2 each second: $\lambda = 1$

Measurement resolution: $\epsilon = 10^{-3}$

$I_0 = 10$ bits $\quad t_{\text{unpredict}} = 10$ seconds
Mechanisms of Chaos ...

**Measurement Resolution:** $\epsilon$

Number of scale factors to locate initial state: $I_0 = -\log_2 \epsilon$

Resolution loss rate (bits per second): $\lambda$

**Prediction horizon:** $t_{\text{unpredict}} \sim \frac{I_0}{\lambda}$

Example:

Loss rate: Factor of 2 each second: $\lambda = 1$
Measurement resolution: $\epsilon = 10^{-3}$

$I_0 = 10$ bits \hspace{1cm} $t_{\text{unpredict}} = 10$ seconds

Thousand times higher resolution: $\epsilon = 10^{-6}$

$I_0 = 20$ bits \hspace{1cm} $t_{\text{unpredict}} = 20$ seconds
Mechanisms of Chaos ...

Quantifying instability and stability ... n dimensions:

\[ \chi = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \quad \lambda_i \geq \lambda_{i+1} \]

\[
\lambda_i = \lim_{t \to \infty} \lim_{\|\delta \vec{x}_i(0)\| \to 0} \frac{1}{t} \log_2 \frac{\|\delta \vec{x}_i(t)\|}{\|\delta \vec{x}_i(0)\|}
\]

\[
\{\delta \vec{x}_1, \delta \vec{x}_2, \ldots, \delta \vec{x}_n\}, \quad \delta \vec{x}_i \cdot \delta \vec{x}_j = 0, \ i \neq j
\]

Lyapunov Characteristic Exponent Spectrum:
Mechanisms of Chaos ...

Quantifying instability and stability ...

LCE Spectrum and Submanifolds:

\[ \lambda_i < 0 \iff \text{stable manifold} \]
\[ \lambda_i > 0 \iff \text{unstable manifold} \]

LCE Spectrum: Key to characterizing attractors
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Dissipation rate:

Divergence of vector field:

\[ \nabla \cdot \vec{F}(\vec{x}) = \sum_{i=1}^{n} \frac{\partial \vec{F}}{\partial x_i} \bigg|_{\vec{x}} = \text{Tr}(A(\vec{x})) \]

\[ \mathcal{D} = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} dt \ \nabla \cdot \vec{F}(\vec{x}(t)) \]

Theorem: \[ \mathcal{D} = \sum_{i=1}^{n} \lambda_i \]

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**Dimension of attractor:**

\[ d = j + \sum_{i=1}^{j} \frac{\lambda_i}{|\lambda_{j+1}|} \]

\[ j \text{ largest integer such that } \sum_{i=1}^{j} \lambda_i \geq 0 \]

(Conjectured to be true in most cases.)
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**Entropy** of attractor (jumping ahead a little bit):

\[ h_\mu = \sum_{\lambda_i > 0} \lambda_i \]

Rate of information production.
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LCE Attractor Classification:

An attractor’s LCE signature:

\[ (\lambda_1, \lambda_2, \ldots, \lambda_n), \quad \lambda_i \geq \lambda_{i+1} \]

Constraints:

1. Attracting: \( D < 0 \)

   \[ \Rightarrow \sum_{i=1}^{n} \lambda_i < 0 \]  \quad (A)

   \Rightarrow \lambda < 0, \text{ for at least one } i

2. If not fixed point, flow along trajectory neutrally stable:

   \[ \lambda_i = 0, \text{ for at least one } i \]

3. If chaotic:

   \[ \lambda_i > 0, \text{ for at least one } i \]
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LCE Spectrum Attractor Classification ...

\((\text{sgn}(\lambda_1), \text{sgn}(\lambda_2), \ldots, \text{sgn}(\lambda_n))\)

<table>
<thead>
<tr>
<th>Dimension n</th>
<th>LCE Spectrum</th>
<th>Attractor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>2</td>
<td>(-,-)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>2</td>
<td>(0,-)</td>
<td>Limit Cycle</td>
</tr>
<tr>
<td>3</td>
<td>(-,-,-)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>3</td>
<td>(0,-,-)</td>
<td>Limit Cycle</td>
</tr>
<tr>
<td>3</td>
<td>(0,0,-)</td>
<td>Torus</td>
</tr>
<tr>
<td>3</td>
<td>(+,0,-)</td>
<td>Chaotic</td>
</tr>
<tr>
<td>4</td>
<td>(0,0,0,-)</td>
<td>3-Torus</td>
</tr>
<tr>
<td>4</td>
<td>(+,0,0,-)</td>
<td>Chaotic 2-Torus</td>
</tr>
<tr>
<td>4</td>
<td>(+,+0,-)</td>
<td>Hyperchaos</td>
</tr>
</tbody>
</table>

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Definition of **chaotic attractor:**

1. Attractor: $\Lambda \subset \mathcal{X}$
   - (a) Invariant set: $\Lambda = \phi_T(\Lambda)$.
   - (b) Attracts an open set $U \subset \mathcal{X}$: $\Lambda \subset U$

$$\Lambda = \lim_{T \to \infty} \phi_T(U)$$

   - (c) Minimal: no proper subset is also (a) & (b).

2. Aperiodic long-term behavior of a deterministic system with exponential amplification.
   - (2') Positive maximum LCE: $\lambda_{\max}(\Lambda) > 0$.
   - (2'') Positive metric entropy: $h_\mu(\Lambda) > 0$.
Mechanisms of Chaos ...
Reading for next lecture:

**NDAC, Sections 10.5-10.7.**