Bidirectional Computational Mechanics II

Reading for this lecture: CMR articles

TBA
PRATISP
IACP
IACPLCOCS
Bidirectional
Computational Mechanics

Bidirectionality
Excess entropy from $\varepsilon$-machine
Information diagrams for processes
Bidirectional machines
Bidirectional complexities
Bidirectionality
Bidirectionality

Causal shielding for forward and reverse states:

$$\Pr(\vec{X}, \overset{\to}{X} | S^+) = \Pr(\vec{X} | S^+) \Pr(\overset{\to}{X} | S^+)$$

$$\Pr(\vec{Y}, \overset{\to}{Y} | S^-) = \Pr(\vec{Y} | S^-) \Pr(\overset{\to}{Y} | S^-)$$

Both forward and reverse states equally good at shielding
... but for potentially different, though, related processes:

$$\overset{\to}{Y} = \overset{\sim}{X}$$

Direct relationship between forward and reverse states?
Bidirectionality

Excess entropy from $\varepsilon$-machine:

Process $\Pr(\overleftarrow{X}, \overrightarrow{X})$ is a communication channel from the past $\overleftarrow{X}$ to the future $\overrightarrow{X}$:
Bidirectionality

Excess entropy from $\varepsilon$-machine ...

Mutual information between the past and future

$$E = I[\mathbin{\leftarrow X}; \mathbin{\rightarrow X}]$$

A process’s channel capacity?

More like the effective channel utilization.

Now, how to get from given $\varepsilon$-machine?
Bidirectionality

Excess entropy from $\varepsilon$-machine ...

Theorem: $E = I[S^+; S^-]$

Proof sketch: $E = I[\vec{X}; \vec{X}]$

$= I[\epsilon^+(\vec{X}); \epsilon^-(\vec{X})]$

$= I[S^+; S^-]$

$\Rightarrow$ Need $Pr(S^+, S^-)$

New interpretation: Effective transmission capacity of channel between forward and reverse processes.
Bidirectionality

Mixed state presentation of time-reversed $\varepsilon$-machine:

$$\tilde{M}^+ = \mathcal{T}(M^+)$$
$$M^- = \mathcal{U}(\tilde{M}^+) \quad \text{(minimize!)}$$

yields conditional entropy btw forward and reverse states:

$$\Pr(S^+|S^-)$$
Bidirectionality

Switching maps between forward and reverse causal states:

**Forward-state simplex:** $\Delta^m \quad m = |S^+|$  
$\Pr(S_0^+ = \sigma_0, S_1^+ = \sigma_1, \ldots) \in \Delta^m$

**Reverse-state simplex:** $\Delta^n \quad n = |S^-|$  
$\Pr(S_0^- = \sigma_0, S_1^- = \sigma_1, \ldots) \in \Delta^n$

**Forward map:** $f : \Delta^n \rightarrow \Delta^m$  
$f(\sigma^-) = \Pr(S^+ | \sigma^-)$

**Reverse map:** $r : \Delta^m \rightarrow \Delta^n$  
$r(\sigma^+) = \Pr(S^- | \sigma^+)$
Bidirectionality

Switching maps ...

Uses:

Calculate with conditional & joint state dependencies:

\[ \Pr(S^+ | S^-) \]
\[ \Pr(S^+, S^-) \]

Recast quantifiers purely in terms of forward & reverse states.

New quantifiers.

A new representation ...
Bidirectionality

Example: Random Noisy Copy (RnC)

\[
\Pr(S^+) = \frac{1}{2} \begin{pmatrix} A & B & C \\ 1 & p & 1 - p \end{pmatrix}
\]
Bidirectionality

Example: Random Noisy Copy (RnC)

\[
\Pr(S^-) = \frac{1}{2} \begin{pmatrix}
D & E & F \\
1 & (1 - p)(1 - q) & p + q(1 - p)
\end{pmatrix}
\]
Bidirectionality

Example: Random Noisy Copy (RnC)

Forward switching map:
Conditional distribution, from previous MSP calculation

\[
\Pr(S^+ | S^-) = \begin{pmatrix}
D & 0 & 0 & 1 \\
E & 1 & 0 & 0 \\
F & 0 & \frac{p}{p+q(1-p)} & \frac{q(1-p)}{p+q(1-p)} \\
\end{pmatrix}
\]
Bidirectionality

Example: Random Noisy Copy (RnC)

Joint distribution:

$$\Pr(S^+, S^-) = \frac{1}{2} \times \begin{pmatrix} A & B & C \\ D & 0 & 0 \\ E & (1 - p)(1 - q) & 0 \\ F & 0 & p \\ \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 - p \\ q(1 - p) \end{pmatrix}$$

Componentwise calculation:

$$\Pr(S^+ = i, S^- = j) = \Pr(S^+ = i | S^- = j)\Pr(S^- = j)$$
Bidirectionality

Example: Random Noisy Copy (RnC)

Reverse switching map:
“Transpose” of joint distribution

\[
\Pr(S^{-}, S^{+}) = \frac{1}{2} \times \begin{pmatrix}
D & E & F \\
A & (1 - p)(1 - q) & 0 \\
B & 0 & p \\
C & 0 & q(1 - p)
\end{pmatrix}
\]

Componentwise:

\[
\Pr(S^{-} = i | S^{+} = j) = \frac{\Pr(S^{-} = i, S^{+} = j)}{\Pr(S^{+} = j)}
\]

\[
\Pr(S^{-} | S^{+}) = \begin{pmatrix}
D & E & F \\
A & 0 & (1 - p)(1 - q) & 0 \\
B & 0 & 0 & 1 \\
C & \frac{1}{1 - p} & 0 & q
\end{pmatrix}
\]
Bidirectionality

Example: Random Noisy Copy (RnC)

Reverse switching map ...

Normalize rows to get actual transition probabilities:

\[ \Pr(S^-|S^+) = \begin{pmatrix} D & E & F \\ A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & \frac{1}{1+q(1-p)} & 0 & \frac{q(1-p)}{1+q(1-p)} \end{pmatrix} \]
Bidirectionality

Example: Random Noisy Copy (RnC) ...

Excess entropy:

\[ E = I[S^+; S^-] \]
\[ = H[S^+] - H[S^+|S^-] \]
\[ = H \left[ \frac{1}{2} (1 \ p \ 1-p) \right] - H[\{B, C\}|F] \Pr(F) \]
\[ = H \left[ \frac{1}{2} (1 \ p \ 1-p) \right] - H \left( \frac{p}{p+q(1-p)} \right) (p + q(1 - p)) \]

\[
\Pr(S^+) = \frac{1}{2} \begin{pmatrix} A & B & C \\ 1 & p & 1-p \end{pmatrix}
\]

\[
\Pr(S^+|S^-) = E \begin{pmatrix} D & 0 & 1 \\ 0 & \frac{p}{p+q(1-p)} & \frac{q(1-p)}{p+q(1-p)} \end{pmatrix}
\]

\[
\Pr(S^-) = \frac{1}{2} \begin{pmatrix} D & E & F \\ 1 & (1-p)(1-q) & p + q(1-p) \end{pmatrix}
\]
Bidirectionality

Crypticities, recast:

Forward:

\[ \chi^+ = H[S^+|\vec{X}] \]
\[ = H[S^+|\epsilon^- (\vec{X})] \]
\[ = H[S^+|S^-] \]

Reverse:

\[ \chi^- = H[S^-|\overset{\leftarrow}{\vec{X}}] \]
\[ = H[S^-|\epsilon^+ (\overset{\leftarrow}{\vec{X}})] \]
\[ = H[S^-|S^+] \]
Bidirectionality

Example: Random Noisy Copy (RnC) ...

\[ \chi^+ = H[S^+|S^-] \]
\[ = H[S^+ = \{B, C\}|S^- = F] \]
\[ = H \left( \frac{p}{p+q(1-p)} \right) (p + q(1 - p)) \]

\[ \chi^- = H[S^-|S^+] \]
\[ = H[S^- = \{D, F\}|S^+ = C] \]
\[ = H \left( \frac{1}{1+q(1-p)} \right) \left( \frac{1 - p}{2} \right) \]
Bidirectionality

Causal irreversibility, recast:

$$\Xi \equiv C_\mu^+ - C_\mu^-$$

$$= H[S^+|S^-] - H[S^-|S^+]$$

using

$$C_\mu^+ = E + \chi^+$$

$$C_\mu^- = E + \chi^-$$
Bidirectionality

Example: Random Noisy Copy (RnC) ...

\[ \Xi = H[S^+|S^-] - H[S^-|S^+] \]
\[ = H \left[ \frac{1}{2} (1 \ p \ 1 - p) \right] - H \left[ \frac{1}{2} (1 - p)(1 - q) \ p + q(1 - p) \right] \]
Bidirectionality

Summary:

New meaning for excess entropy:

\[ E = I[S^+; S^-] \]

\( E \) can be directly (and accurately!) calculated from \( \varepsilon \)-machine.

New level of analytical calculation possible.

New algorithms for measuring intrinsic computation using bidirectional representations.
Information Diagrams for Processes
Now, ready for full information diagram of a process.

Requires both forward and reverse scanning of a process.
Information Diagrams for Processes

$\epsilon$-machine information Diagram

\[ H[S^+] = C_\mu^+ \]

\[ H[S^-] = C_\mu^- \]

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Information Diagrams for Processes

**ε-machine information Diagram**

\[ H[S^+] = C^+_\mu \]

\[ H[S^-] = C^-\mu \]

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Information Diagrams for Processes

$\epsilon$-machine information Diagram

\[ H[S^+] = C^+_{\mu} \]

\[ H[S^-] = C^-_{\mu} \]

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Information Diagrams for Processes

**ε-machine information Diagram**

\[ H[S^+] = C^+_\mu \]

\[ H[S^-] = C^-_\mu \]

\[ H[X | S^+] = h_\mu L \]

\[ H[X | S^-] \]

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$H[S^+] = C_\mu^+$

$H[S^-] = C_\mu^-$

$H[\hat{X} | S^+] = h_\mu L$

$H[\hat{X} | S^-] = h_\mu L$

$E = I[S^-; S^+]$

$H[\hat{X} | S^+] = h_\mu L$

$H[\hat{X} | S^-] = h_\mu L$
Bidirectional Machines
Bidirectional Machines

Summary:

So far, re-expressed key measures in terms of forward and reverse $\varepsilon$-machines

\[ E = I[S^-; S^+] \]
\[ \chi^+ = H[S^+|S^-] \]
\[ \chi^- = H[S^-|S^+] \]
\[ \Xi = H[S^+|S^-] - H[S^-|S^+] \]

How to more directly represent the interdependence between forward and reverse $\varepsilon$-machines?
Bidirectional Machines

Bidirectional Process Lattice:

<table>
<thead>
<tr>
<th>Past</th>
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<th>Future</th>
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Bidirectional Machines

Bidirectional Process Lattice:
Bidirectional Machines

Bidirectional Process Lattice:

Past   Present   Future

\[ S^+_4 \quad S^+_3 \quad S^+_2 \quad S^+_1 \quad S^+_0 \quad S^+_1 \quad S^+_2 \quad S^+_3 \quad S^+_4 \]

\[ X^-_4 \quad X^-_3 \quad X^-_2 \quad X^-_1 \quad X^0 \quad X^1 \quad X^2 \quad X^3 \]

\[ S^-_4 \quad S^-_3 \quad S^-_2 \quad S^-_1 \quad S^-_0 \quad S^-_1 \quad S^-_2 \quad S^-_3 \quad S^-_4 \]
Bidirectional Machines

Bidirectional Process Lattice:
Bidirectional Machines

Bidirectional Process Lattice:

You can choose to reverse direction. Path automaton over scan direction \{+, −\}. 

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Bidirectional Machines

Bidirectional Process Lattice:

You can choose to reverse direction.
Path automaton over scan direction \{+, -\}.

What describes this?
A bidirectional machine, with forward and reverse moves.
Previous forward & reverse machines are a subset of paths.
Bidirectional Machines

Bidirectional machine: $M^\pm$

Equivalence relation: $\sim^\pm$

$$\epsilon^\pm(\langle \vec{x} \rangle) = \left\{ \langle \vec{x}' \rangle = \langle \vec{x} \rangle \langle \vec{x}' \rangle : \langle \vec{x}' \rangle \in \epsilon^+(\langle \vec{x} \rangle) \text{ and } \langle \vec{x}' \rangle \in \epsilon^-(\langle \vec{x} \rangle) \right\}$$

Bidirectional states:

$$S^\pm = \Pr(\langle \vec{X} \rangle, \langle \vec{X} \rangle) / \sim^\pm \quad \text{A partition of } \langle \vec{X} \rangle$$

$$\subseteq S^+ \times S^-$$

Bidirectional Machine:

$$M^\pm = \{ S^\pm ; T(x), x \in A \}$$
Bidirectional Machines

The bidirectional causal state the process is in at time $t$ is

$$S_t^\pm = (\epsilon^+ (\underleftarrow{x}_t), \epsilon^- (\overrightarrow{x}_t))$$

$$\overrightarrow{x} = \overrightarrow{x}_t \overrightarrow{x}_t$$
Bidirectional Machines

Example: Even Process

\begin{align*}
\text{Forward } \varepsilon\text{-machine} \\
\Pr(S^+) &= \begin{pmatrix} A \\ \frac{1}{2-p} \end{pmatrix} \begin{pmatrix} B \\ \frac{1-p}{2-p} \end{pmatrix} \\
S^+ &= \{A, B\}
\end{align*}

\begin{align*}
\text{Reverse } \varepsilon\text{-machine} \\
\Pr(S^-) &= \begin{pmatrix} C \\ \frac{1}{2-p} \end{pmatrix} \begin{pmatrix} D \\ \frac{1-p}{2-p} \end{pmatrix} \\
S^- &= \{C, D\}
\end{align*}
Bidirectional Machines

Example: Even Process ...

\[ h_\mu = \frac{H(p)}{2 - p} \]
\[ C^+_\mu = H\left(\frac{1}{2 - p}\right) \]
\[ C^-_\mu = H\left(\frac{1}{2 - p}\right) \]
\[ \Xi = 0 \]
Bidirectional Machines

Example: Even Process ...

Forward switching map:

\[
\Pr(S^+ | S^-) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Reverse switching map:

\[
\Pr(S^- | S^+) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Identities!
Bidirectional Machines

Example: Even Process ...

Bidirectional machine:

$M^{\pm}$

$+|p|0$

$-|p|0$

$+|1-p|1$

$-|1-p|1$

$S^{\pm} = \{AC, BD\}$

$(\vec{X}, \vec{X}) :$

$(A, C) \sim (\{\ast(11)^{k}\}, \{(11)^{k}\ast}\})$

Even # 1s

$(B, D) \sim (\{\ast(11)^{k}1\}, \{1(11)^{k}\ast}\})$

Odd # 1s
Bidirectional Machines

Example: Even Process ...

\[ +|p|0 \quad -|p|0 \]

\[ +|1-p|1 \quad -|1-p|1 \]

\[ +|1|1 \quad -|1|1 \]

\[ S^+ \]

\[ X \]

\[ S^- \]

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Bidirectional Machines

Example: Even Process ...

\[ \chi^+(p) = \chi^-(p) = 0 \]

\[ \mathbf{E} = C^+_{\mu} = C^-_{\mu} \]

Sofic, non-Markov

But simple!

Microscopically reversible
Causally reversible
Explicit process: non-cryptic
Bidirectional Machines

Example: Golden Mean Process

**Forward $\varepsilon$-machine**

\[ M^+ \]

\[ p \mid 1 \rightarrow A \rightarrow B \rightarrow p \mid 1 \]

\[ 1 \mid 1 \rightarrow 1 - p \mid 0 \rightarrow A \rightarrow B \rightarrow 1 \mid 1 \]

**Pr($S^+$)** = \[
\begin{pmatrix}
A & B \\
\frac{1}{2-p} & \frac{1-p}{2-p}
\end{pmatrix}
\]

\[ S^+ = \{A, B\} \]

**Reverse $\varepsilon$-machine**

\[ M^- \]

\[ p \mid 1 \rightarrow C \rightarrow D \rightarrow p \mid 1 \]

\[ 1 \mid 1 \rightarrow 1 - p \mid 0 \rightarrow C \rightarrow D \rightarrow 1 \mid 1 \]

**Pr($S^-$)** = \[
\begin{pmatrix}
C & D \\
\frac{1}{2-p} & \frac{1-p}{2-p}
\end{pmatrix}
\]

\[ S^- = \{C, D\} \]
Bidirectional Machines

Example: Golden Mean Process ...

\[ h_\mu = H(p)/(2 - p) \]

\[ C^+_\mu = H(1/(2 - p)) \]

\[ C^-_\mu = H(1/(2 - p)) \]

\[ \Xi = 0 \]

Same as Even Process!
Bidirectional Machines

Example: Golden Mean Process ...

Forward switching map:

\[
\Pr(S^+ | S^-) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} p & 1 - p \\ 1 & 0 \end{pmatrix}
\]

Reverse switching map:

\[
\Pr(S^- | S^+) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} p & 1 - p \\ 1 & 0 \end{pmatrix}
\]

Not identities!

Ambiguity and loss of information on switching.
Bidirectional Machines

Example: Golden Mean Process ...

Bidirectional machine:

\[ \mathcal{M}^\pm = \{ AC, AD, BC \} \]

\[ (\overleftarrow{X}, \overrightarrow{X}) : \]
\[ (A, C) \sim (\{ *1 \}, \{ 1* \}) \]
\[ (A, D) \sim (\{ *1 \}, \{ 0* \}) \]
\[ (B, C) \sim (\{ *0 \}, \{ 1* \}) \]
Bidirectional Machines

Example: Golden Mean Process ...

\[ \begin{align*}
S^+ & \quad (A, D) \quad (B, C) \quad (A, C) \quad (A, D) \quad (B, C) \quad (A, D) \quad (B, C) \quad (A, D) \quad (B, C) \\
X & \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \\
S^- & \quad D \quad C \quad C \quad D \quad C \quad D \quad C \quad D \quad C
\end{align*} \]
Bidirectional Machines

Example: Golden Mean Process ...

Order-1 Markov:

\[
E = C^+_\mu - h_\mu
= H \left( \frac{1}{2 - p} \right) - \frac{H(p)}{2 - p}
\]

Subshift of finite-type
But simple:
  Causally reversible: \( \Xi = 0 \)
and not simple:
  Cryptic process: \( \chi^+ = \chi^- = \frac{H(p)}{2 - p} \)
Causal Irreversibility

Example: Random Insertion Process

Forward $\varepsilon$-machine:

$$\Pr(\mathcal{S}^+) = \begin{pmatrix} \frac{1}{p+2} & \frac{p}{p+2} & \frac{1}{p+2} \end{pmatrix}$$
Causal Irreversibility

Example: Random Insertion Process ...

Reverse $\varepsilon$-machine:

\[
\begin{align*}
\Pr(S^-) &= \begin{pmatrix}
\frac{1}{p+2} & \frac{1-pq}{p+2} & \frac{pq}{p+2} & \frac{p}{p+2} \\
\end{pmatrix}
\end{align*}
\]
Bidirectional Machines

Example: Random Insertion Process ...

Causally irreversible:

\[ M^+ \neq M^- \]

\[ C^+_\mu = \log_2(p + 2) - \frac{p \log_2 p}{p + 2} \]

\[ C^-_\mu = \log_2(p + 2) + \frac{H(pq) - p \log_2 p}{p + 2} \]

\[ C^+_\mu \neq C^-_\mu \]

\[ \Xi = \frac{H(pq)}{p + 2} \]
Bidirectional Machines

Example: Random Insertion Process ...

Forward switching map: \( p = q = \frac{1}{2} \)

\[
\begin{align*}
\Pr(S^+|S^-) &= \begin{pmatrix}
A & B & C \\
D & 0 & 0 & 1 \\
E & 2/3 & 1/3 & 0 \\
F & 0 & 1 & 0 \\
G & 1 & 0 & 0
\end{pmatrix}
\end{align*}
\]

Reverse switching map: \( p = q = \frac{1}{2} \)

\[
\begin{align*}
\Pr(S^-|S^+) &= \begin{pmatrix}
A & 0 & 1/2 & 0 & 1/2 \\
B & 0 & 1/2 & 1/2 & 0 \\
C & 1 & 0 & 0 & 0
\end{pmatrix}
\end{align*}
\]
Example: Random Insertion Process...

Joint distribution (general $p$ and $q$):

$$\Pr(S^+, S^-) = \frac{1}{(p+2)}$$

$$A = \begin{pmatrix} D & E & F & G \\ D & E & F & G \\ 0 & 1-p & 0 & p \\ 0 & p(1-q) & pq & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
Bidirectional Machines

Example: Random Insertion Process ...

Bidirectional machine: $p = q = \frac{1}{2}$

$M^\pm$

(c)

Diagram of the bidirectional machine showing transitions between states BF, AG, BE, CD, and AE with probabilities $p$ and $q$. The transitions are labeled with input symbols and transitions, illustrating the random insertion process.
Bidirectional Machines

Example: Random Insertion Process ...

\[ \chi^+ = \frac{1 - pq}{p + 2} H \left( \frac{1 - p}{1 - pq} \right) \]

\[ \chi^- = \frac{1 - pq}{p + 2} H \left( \frac{1 - p}{1 - pq} \right) + \frac{H(pq)}{p + 2} \]

\[ E = C^+_{\mu} - \chi^+ \]

\[ = \log_2(p + 2) - \frac{p \log_2 p}{p + 2} - \frac{1 - pq}{p + 2} H \left( \frac{1 - p}{1 - pq} \right) \]
Bidirectional Machines

Example: Random Insertion Process ...

Generally, cryptic, irreversible process \((p, q) \in [0, 1]^2\)

But ranges over:
- non-cryptic, reversible
- semi-cryptic, irreversible
- cryptic, reversible
- cryptic, irreversible

\[ p = q \]

\[ q = 0.99 \]

\[ q = 0.5 \]

\[ p = 0.01 \]

\[ p = 0.99 \]

\[ p = 1 - q \]

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Bidirectional Machines

Comments:

Naive combination of forward and reverse machines would be addition or product over states, bidirectional machine is neither.

Bidirectional machine:
Nonminimal
Nonunifilar
Not $\varepsilon$-machine.

Projections onto forward moves gives forward process, but might be nonminimal machine.
Projection onto reverse moves gives reverse process, but might be nonminimal machine.
Bidirectional Machines

Oddities of "prediction":

Predictive states are better retrodictors than they are predictors (by $\chi^+$).

Predictive states are better retrodictors than retrodictive states (by $\chi^-$).
Bidirectional Machines

Open questions:

What are bidirectional transient states?

Minimality?

Unifilar presentation?
Bidirectional Complexities
Bidirectional Complexities

**Statistical Complexity** of Bidirectional Machine:

\[ C^\pm_\mu \equiv H[S^\pm] = H[S^+, S^-] \]

The stored information required to optimally predict *and* retrodict.
Bidirectional Complexities

Excess entropy:

\[
\begin{align*}
E &= I[S^+; S^-] \\
&= H[S^+] + H[S^-] - H[S^+, S^-] \\
&= C^+_\mu + C^-_\mu - C^\pm_\mu
\end{align*}
\]
Bidirectional Complexities

Excess entropy ...

Only when $E = 0$:

$$C_{\mu}^{\pm} = C_{\mu}^{+} + C_{\mu}^{-}$$
Bidirectional Complexities

Bounds:

Before:

$$E \leq C_\mu \ (\equiv C_\mu^+)$$

Now, tighter bounds on excess entropy:

$$E \leq C_\mu^+$$

and

$$E \leq C_\mu^-$$
Bidirectional Complexities

Bounds ...

Bidirectional machine smaller than forward and reverse:

\[ C_{\mu}^{\pm} \leq C_{\mu}^{+} + C_{\mu}^{-} \]

\( M^{\pm} \) is efficient representation.

Forward \( \varepsilon \)-machine smaller than bidirectional:

\[ C_{\mu}^{+} \leq C_{\mu}^{\pm} \]

Reverse \( \varepsilon \)-machine smaller than bidirectional:

\[ C_{\mu}^{-} \leq C_{\mu}^{\pm} \]
Bidirectional Complexities

From I-diagram:

\[ C_{\mu}^{\pm} = E + \chi^{\pm} \]

where

Crypticity: \( \chi^{\pm} = \chi^{+} + \chi^{-} = H[S^+|S^-] + H[S^-|S^+] \)

Distance between measurements & model: \( C_{\mu}^{\pm} - E \)

Form of a true distance: \( d(X, Y) = H[X|Y] + H[Y|X] \)

Also, distance between forward and reverse processes.

Information inaccessibility:
Degree to which internal information is hidden.
Bidirectional Complexities

Bounds ...

\[ \chi^\pm \leq C^\pm_{\mu} \]

Truly cryptic process:

\[ \chi^\pm = C^\pm_{\mu} \quad (\text{All state information is in crypticity.}) \]

\[ E = 0 \]

Nothing can be learned about a process’s structure from measurements.
Summary

Bidirectionality
Excess entropy from $\epsilon$-machine
Information diagrams for processes
Bidirectional machines
Bidirectional complexities