From Determinism to Stochasticity

Reading for this lecture:

(These) Lecture Notes.

Outline of next few lectures:
- Probability theory
- Stochastic processes
- Measurement theory
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Probability Theory of Dynamical Systems:
Probability Theory Review:

Discrete Random Variable (RV): $X$

Events (Alphabet): $\mathcal{X} = \{1, 2, \ldots, k\}$

Realization: $x \in \mathcal{X}$

Probability mass function ("distribution"): $\Pr(x) = \Pr\{X = x\}$

$0 \leq \Pr(x) \leq 1$, $x \in \mathcal{X}$

Normalized: $\sum_{x \in \mathcal{X}} \Pr(x) = 1$
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Probability Theory of Dynamical Systems:  
Probability Theory Review ...  

Discrete random variables:  

1. Biased coin: $\mathcal{X} = \{H, T\}$  

$$
\begin{align*}
\Pr(H) &= 1/3 \\
\Pr(T) &= 2/3
\end{align*}
$$

2. Sequence: No pairs of 0s  

$$
\mathcal{X} = \{000, 001, 010, 011, 100, 101, 110, 111\}
$$

$$
\Pr(s^3) = \begin{cases} 
0 & 000, 001, 100 \\
\frac{1}{3} & 101 \\
\frac{1}{6} & \text{otherwise}
\end{cases}
$$
Continuous Random Variable: \( X \)
Takes values over continuous event space: \( \mathcal{X} \)

Cumulative distribution function: \( P(x) = \Pr(X \leq x) \)
\[
0 \leq \Pr(x) \leq 1, \; x \in \mathcal{X}
\]

If continuous, then random variable is.

Probability density function: \( p(x) = P'(x) \quad 0 \leq p(x), \; x \in \mathcal{X} \)
\[
p(x)dx = \Pr(X < x + dx) - \Pr(X < x)
\]

Normalization: \( \Pr(X < \infty) = 1 \quad \text{or} \quad \int_{-\infty}^{\infty} dx \; p(x) = 1 \)

Support of distribution: \( \text{supp}X = \{x : p(x) > 0\} \)
Continuous random variable $X$:

Uniform distribution on interval: $\mathcal{X} = \mathbb{R}$

Density: $p(x) = \begin{cases} 
1, & 0 \leq x \leq 1 \\
0, & \text{otherwise}
\end{cases}$

Distribution: $\Pr(x) = \begin{cases} 
0, & x < 0 \\
x, & 0 \leq x \leq 1 \\
1, & x > 1
\end{cases}$

Support: $\text{supp } X = [0, 1]$
Continuous random variable $X$:

Gaussian: $X = \mathbb{R}$

Density: $p(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Distribution: $P(x) = \int_{-\infty}^{x} dy \ p(y) \equiv \text{Erf}(x)$

Support: $\text{supp} \ X = \mathbb{R}$
Probability Theory of Dynamical Systems: Probability Theory Review ...

Discrete RVs: $X$ over $\mathcal{X}$ & $Y$ over $\mathcal{Y}$

Joint distribution: $\Pr(X, Y)$

Marginal distributions:

$$\Pr(X) = \sum_{y \in \mathcal{Y}} \Pr(X, y)$$

$$\Pr(Y) = \sum_{x \in \mathcal{X}} \Pr(x, Y)$$
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Factor joint distribution:

\[ \Pr(X, Y) = \Pr(X|Y)\Pr(Y) \]
\[ \Pr(X, Y) = \Pr(Y|X)\Pr(X) \]

Conditional distributions:

\[ \Pr(Y|X) = \frac{\Pr(X, Y)}{\Pr(X)} , \quad \Pr(X) \neq 0 \]
\[ \Pr(X|Y) = \frac{\Pr(X, Y)}{\Pr(Y)} , \quad \Pr(Y) \neq 0 \]
Statistical independence: $X \perp Y$

$$\Pr(X, Y) = \Pr(X)\Pr(Y)$$

Conditional independence ("shielding"): $X \perp_{Z} Y$

$$\Pr(X, Y|Z) = \Pr(X|Z)\Pr(Y|Z)$$
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Dynamical Evolution of Distributions:

Dynamical system: \( \{ \mathcal{X}, \mathcal{T} \} \)

State density: \( p(x) \quad x \in \mathcal{X} \)

Can evolve individual states and sets: \( \mathcal{T} : x_0 \rightarrow x_1 \)

Initial density: \( p_0(x) \) \quad E.g., model of measuring a system

Evolve a density? \( p_0(x) \rightarrow_{\mathcal{T}} p_1(x) \)
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Dynamical Evolution of Distributions ...

Conservation of probability:

\[ p_1(y) dy = p_0(x) dx \]

Perron-Frobenius Operator:

Locally: \[ y = \mathcal{T}(x) \]

\[ p_{n+1}(y) = \frac{p_n(x)}{|\mathcal{T}'(x)|} \]

Globally:

\[ p_{n+1}(y) = \sum_{x \in \mathcal{T}^{-1}(y)} \frac{p_n(x)}{|\mathcal{T}'(x)|} \]
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Dynamical Evolution of Distributions ...

Frobenius-Perron Equation:

\[ p_{n+1}(y) = \int dx \, p_n(x) \delta(y - \mathcal{T}(x)) \]

Dirac delta-function:

\[ \delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \]

\[ \int dx \, \delta(x - c) f(x) = f(c) \]

\[ \int dx \, \delta(x) = 1 \]
Example: Delta function initial distribution

Map: $x_{n+1} = f(x_n)$

Initial condition: $x_0 \in \mathbb{R}$

Initial distribution: $p_0(x) = \delta(x - x_0)$

$p_1(y) = \int dx \ p_0(x) \ \delta(y - f(x))$

$= \int dx \ \delta(x - x_0) \ \delta(y - f(x))$

$= \delta(y - f(x_0))$

$= \delta(y - x_1)$

$\vdots$

$p_n(y) = \delta(y - x_n) \quad \text{... reduces to an orbit}$
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Dynamical Evolution of Distributions ...

Delta function IC: The easy case and expected result.

What happens when the IC has finite support?

\[ p_0(x) = \begin{cases} 
20, & |x - 1/3| \leq 0.025 \\
0, & \text{otherwise} 
\end{cases} \]

Consider a set of increasingly more complicated systems and how they evolve distributions ...
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Dynamical Evolution of Distributions ...

Example:
Linear circle map

\[ x_{n+1} = 0.1 + x_n \pmod{1} \]

\[ f'(x) = 1 \]
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Dynamical Evolution of Distributions...

Example:
Shift map

\[ f'(x) = 2 \]

Spreading:

Monday, February 24, 14
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Example:
Tent map \( a = 2.0 \)

\[
\begin{align*}
T \text{ent map } a &= 2.0 \\
\log P(x) &= -2 \\
0 \leq x &\leq 1 \\
\end{align*}
\]

Spreading: \( |f'(x)| = 2 \)
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Dynamical Evolution of Distributions ...

Example:  
Logistic map $r = 4$

$$x_{n+1} = f(x_n) = 4(1 - 2x)$$

- Spreading: $x < 3/8$ or $x > 5/8$
- Contraction: $3/8 < x < 5/8$

Lecture 10: Natural Computation & Self-Organization, Physics 256A (Winter 2014); Jim Crutchfield
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Probability Theory of Dynamical Systems ... 

Dynamical Evolution of Distributions ... 

Example: 
Logistic map $r = 3.7$

Peaks in distribution are images of maximum
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Time-asymptotic distribution: What we observe

How to characterize?

**Invariant measure:**
A distribution that maps “onto” itself
Analog of invariant sets

Stable invariant measures:
Stable in what sense?
Robust to noise or parameters or ???
Probability distribution (density $p^*(x)$) that is invariant:

1. Distribution’s support must be an invariant set:

$$\Lambda = f(\Lambda), \quad \Lambda = \text{supp } p^*(x) = \{x : p^*(x) > 0\}$$

11. Probabilities “invariant”:
Distribution a fixed point of Frobenius-Perron Equation

$$p^*(y) = \int dx \, p^*(x) \, \delta(y - f(x))$$

Functional equation: Find $p^*(\cdot)$ that satisfies this.
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Example: Periodic-k orbit \( \{x_1, x_2, \ldots, x_k\} \) has density

Is it invariant?

\[
p_1(y) = \int dx \ p(x) \delta(y - f(x))
\]

\[
= \int dx \ \delta \left( \prod_{i=1}^{k} (x - x_i) \right) \delta(y - f(x))
\]

\[
= \delta \left( \prod_{i=1}^{k} (y - f(x_i)) \right)
\]

\[
= \delta \left( \prod_{i=1}^{k} (y - x_{(i+1) \text{mod} \ k}) \right)
\]

\[
= \delta \left( \prod_{i=1}^{k} (y - x_i) \right) \quad \text{Yes!}
\]
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Example: Shift map invariant distribution

Uniform distribution: \( p(x) = 1, \ x \in [0, 1] \)

\[
p_0(y) \\
p_1(y) = p_0'(y) + p_0''(y) \\
p_1(x) = p_0(x) \quad (y \Rightarrow x)
\]
Probability Theory of Dynamical Systems ...

Example: Shift map invariant distribution

Uniform distribution: \( p(x) = 1, \ x \in [0, 1] \)

Via Frobenius-Perron Equation: Two cases

A: \( 0 \leq x \leq 1/2 \)

\[
p'_1(y) = \int_0^{1/2} dx \ p_0(x) \delta(y - f(x)) \\
= \int_0^{1/2} dx \ \delta(y - 2x) \\
= \frac{1}{2}
\]

B: \( 1/2 < x \leq 1 \)

\[
p''_1(y) = \int_{1/2}^{1} dx \ p_0(x) \delta(y - f(x)) \\
= \int_{1/2}^{1} dx \ \delta(y - 2x) \\
= \frac{1}{2}
\]

\[
p_1(y) = p'_1(y) + p''_1(y) \\
= p_0(x) \ (y \Rightarrow x)
\]
Example: Tent map
\[ x_{n+1} = \begin{cases} 
ax_n, & 0 \leq x_n \leq \frac{1}{2} \\
\frac{a}{2} + \frac{a}{2} (1 - x_n), & \frac{1}{2} < x_n \leq 1
\end{cases} \]

Fully two-onto-one: \( a = 2 \)

Uniform distribution is invariant: \( p(x) = 1, \ x \in [0, 1] \)

Proof from FP Equation: Two cases

First case: exactly that of shift map

Second case: \(|\text{slope}|\) is all that’s important
\[ 1/2 < x \leq 1 \]
\[ p''_1(y) = \int_{1/2}^{1} dx \ p_0(x) \delta (y - f(x)) \]
\[ = \int_{1/2}^{1} dx \ \delta (y - (2 - 2x)) = \frac{1}{2} \]
Example: Tent map where two bands merge to one: \( a = \sqrt{2} \)

Invariant distribution:

\[
p(x) = \begin{cases} 
p_0, & x_{\text{min}} \leq x \leq x^* \\
p_1, & x^* < x \leq x_{\text{max}} \\
0, & \text{otherwise}
\end{cases}
\]

\[
x_{\text{max}} = \frac{a}{2}
\]

\[
x_{\text{min}} = a(1 - \frac{a}{2})
\]

\[
x^* = \frac{a}{1 + a}
\]

Equal areas: \( p_0(x^* - x_{\text{min}}) = p_1(x_{\text{max}} - x^*) \)

Normalization: \( p_0(x^* - x_{\text{min}}) + p_1(x_{\text{max}} - x^*) = 1 \)

\[
p_0 = \frac{1}{2(x^* - x_{\text{min}})} \\
p_1 = \frac{1}{2(x_{\text{max}} - x^*)}
\]
Example: Logistic map \( x_{n+1} = rx_n(1 - x_n) \)

Fully two-onto-one: \( r = 4 \)

Invariant distribution? \( p(x) = \frac{1}{\pi \sqrt{x(1 - x)}} \)

Exercise.
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Numerical Example: Tent map

Typical chaotic parameter: \( a = 1.75 \)

Two bands merge to one: \( a = \sqrt{2} \)
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Typical chaotic parameter: \( a = 1.75 \)

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Numerical Example: Tent map

Typical chaotic parameter:

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Numerical Example: Logistic map \( x_{n+1} = rx_n(1 - x_n) \)

Typical chaotic parameter: \( r = 3.7 \)

Two bands merge to one: \( r = 3.6785735104283219 \)
Numerical Example: Logistic map \( x_{n+1} = r x_n (1 - x_n) \)

Typical chaotic parameter: \( r = 3.7 \)

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Numerical Example: Logistic map \[ x_{n+1} = r x_n (1 - x_n) \]

Typical chaotic parameter: \( r = 3.7 \)

Two bands merge to one: \( r = 3.6785735104283219 \)
Numerical Example: Cusp map $x_{n+1} = a(1 - |1 - 2x_n|^b)$

$(a, b) = (1, 1/2)$
Numerical Example: Cusp map \( x_{n+1} = a(1 - |1 - 2x_n|^b) \)

\((a, b) = (1, 1/2)\)
Issue: Many invariant measures in chaos:

An infinite number of unstable periodic orbits: Each has one. But none of these are what one sees, one sees the aperiodic orbits.

How to exclude periodic orbit measures?

Add noise and take noise level to zero; which measures are left?

Robust invariant measures.
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Reading for next lecture:

*Lecture Notes.*