An FTLE Analysis of Active Fluids

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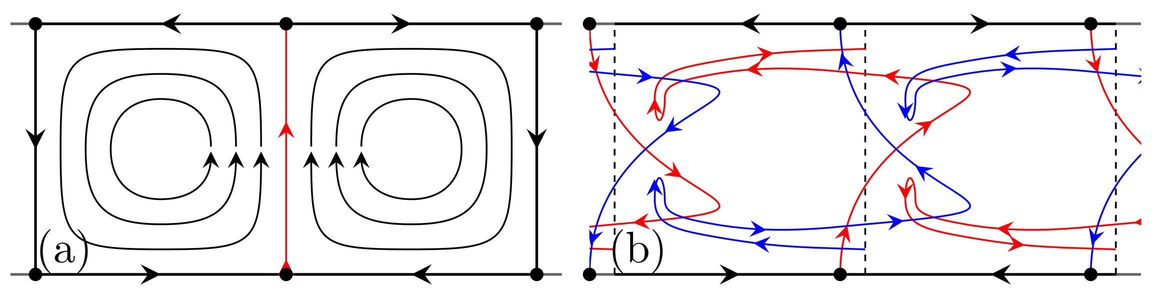
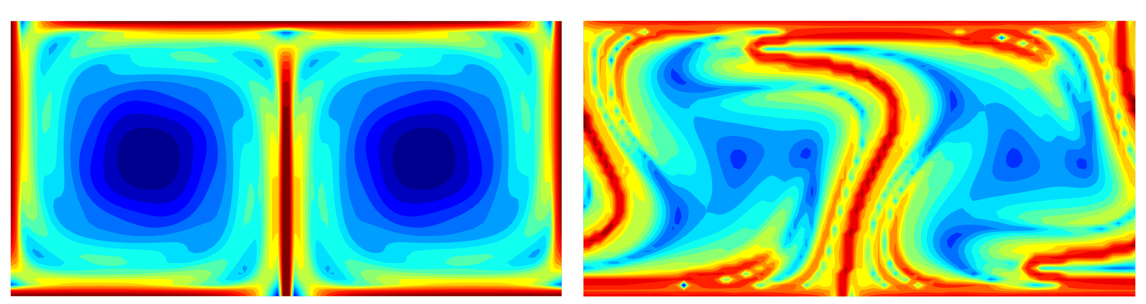
Passive fluids

Invariant manifolds are the organizing geometric objects that describe the transport of fluid or passive tracers within the fluid.



Time-independent flows have separatrices. Periodic perturbation splits these into stable and unstable invariant manifolds.

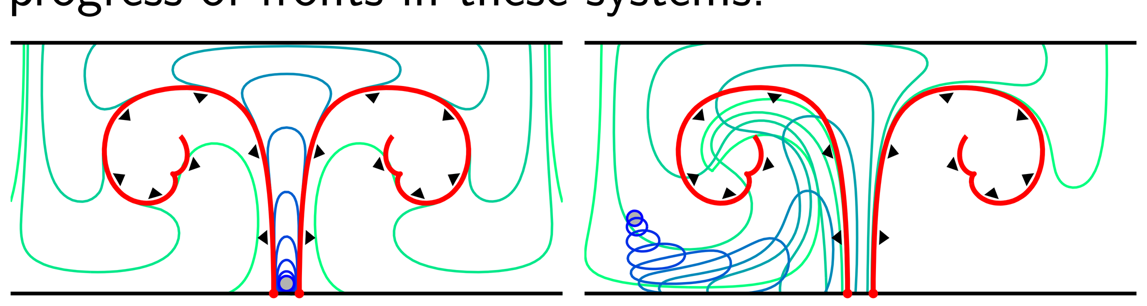
The finite-time-Lyapunov-exponent (FTLE) is a measure of local stretching or separation. Ridges in the FTLE field correspond to invariant manifolds.



Reverse time FTLE ridges correspond to unstable invariant manifolds in these time-independent (left) and time-periodic (right) flows.

Active fluid

Now consider an *active* fluid—one where fronts (of some some state change) propagate in a flow. It is known that burning invariant manifolds (BIMs) are the analogous *one-sided* barriers that describe the progress of fronts in these systems.



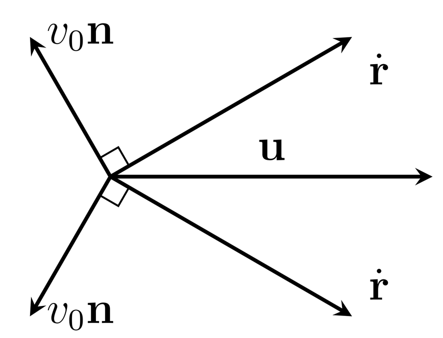
Fronts in time-independent alternating vortex flow. (left) Stimulation between BIMs progresses up and around bounded by a BIM on each side until a cusp is reached. (right) Stimulation is unhindered by oppositely oriented BIM, and converges upon the other.

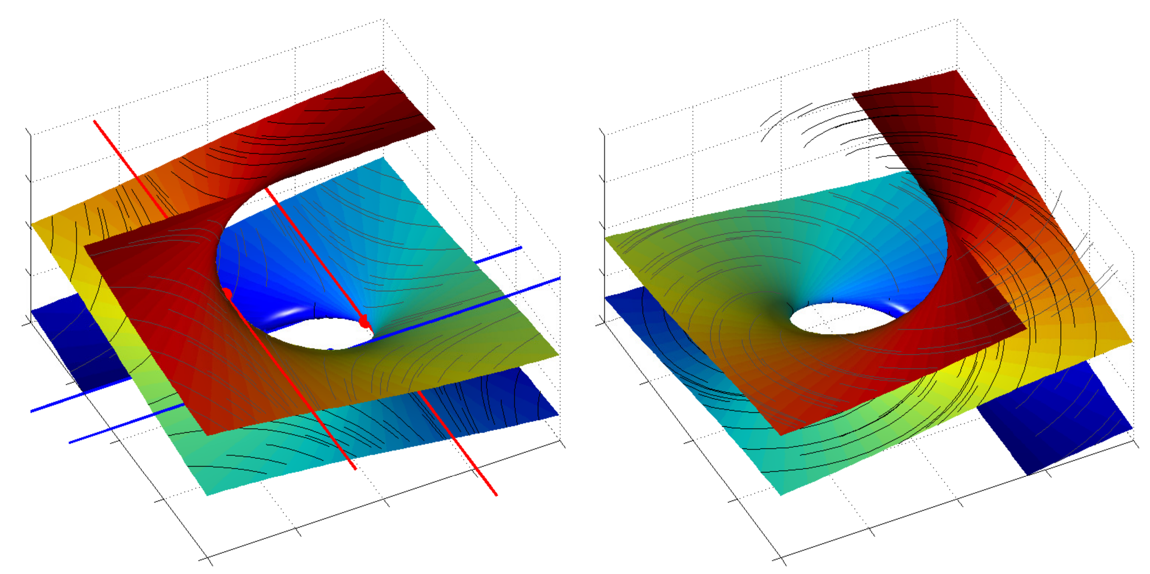
Challenge

Develop an analogous FTLE approach for active fluids.
Although the fluid is 2D, the front element phase space is 3D. Extracting 1D FTLE ridges is difficult. We desire a method which permits a quasi-2D analysis (analogous to the BIMs (projection)).

Invariant submanifold method

For time-independent flows, the front element dynamics has a 2D invariant submanifold defined by the relation $\dot{\mathbf{r}} \perp \hat{\mathbf{n}}$. This constraint results in a 2-valued graph over xy. We refer to the branches as w_+, w_-.

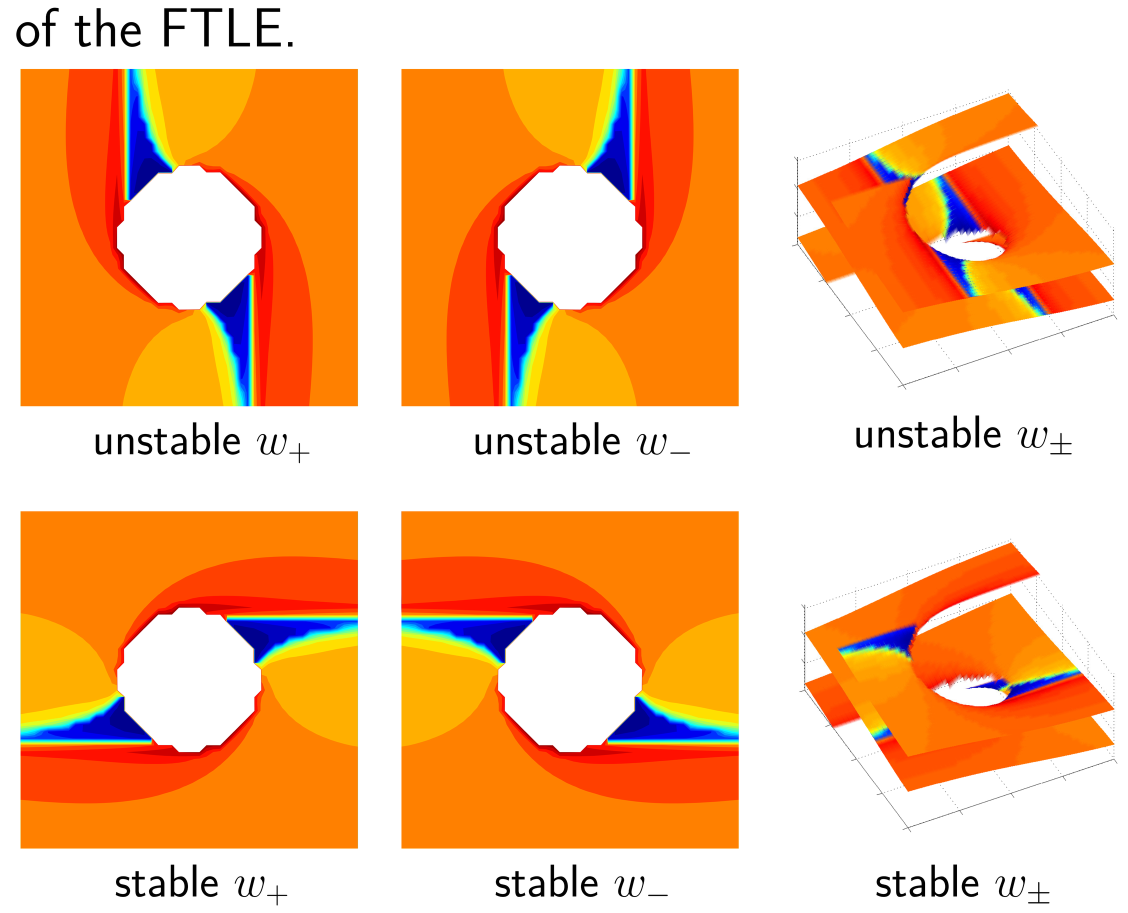




Front element dynamics restricted invariant manifold $w_\pm$ for hyperbolic (left) and elliptic (right) flows. BIMs are attached to saddle points around the hyperbolic flow fixed point.

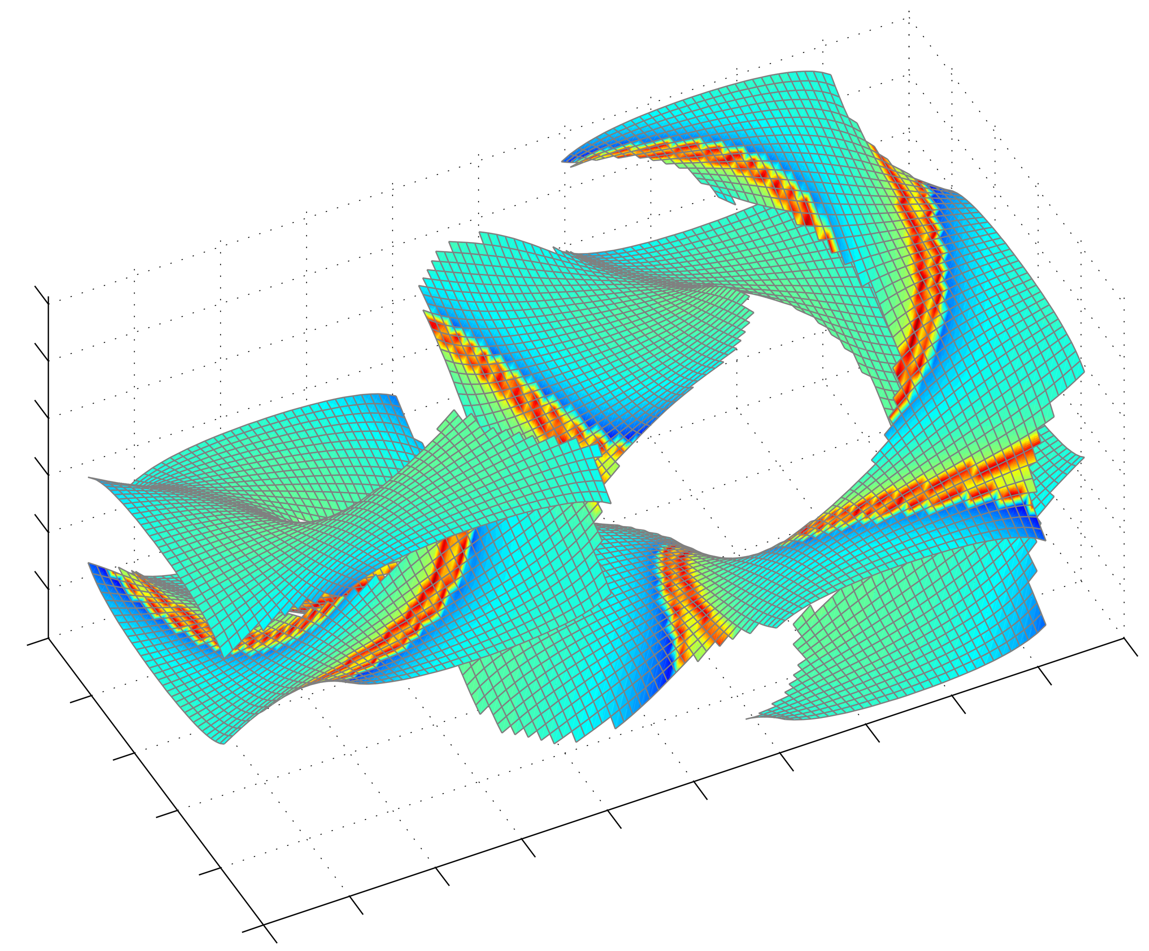
Hyperbolic $w_\pm$ analysis

Time-independent BIMs lie on $w_\pm$ surface, so we restrict our FTLE computation. We slice the $w_\pm$ manifold into two graphs allowing for a simple planar view of the FTLE.

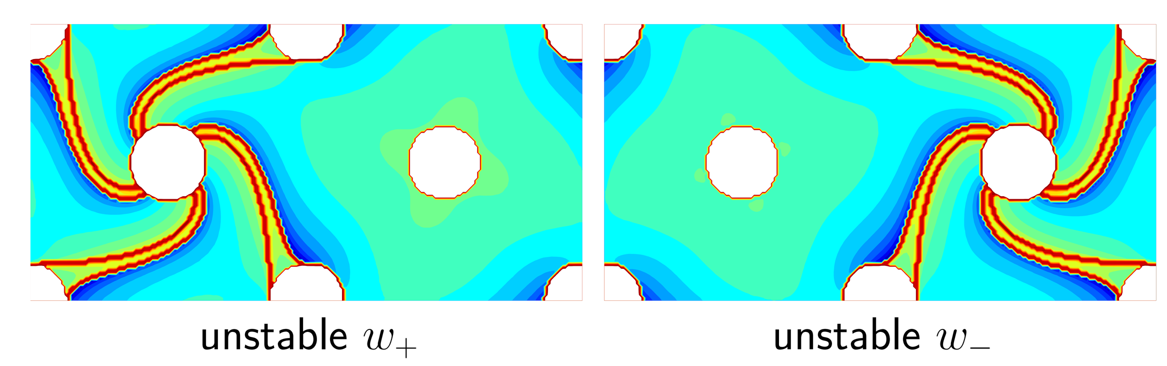


Vortex chain $w_\pm$ analysis

A similar analysis of the time-independent alternating vortex chain flow shows the FTLE computed on the $w_\pm$ submanifold and then split into two planar views.



Notice the local hyperbolic and elliptic structures.



The two layers capture CW vs CCW flow features.

Viewing the two layers separately makes comparison with the previous BIM computation straightforward. This method allows for an intuitive step from passive to active FTLE analysis.

Local FTLE maximization method

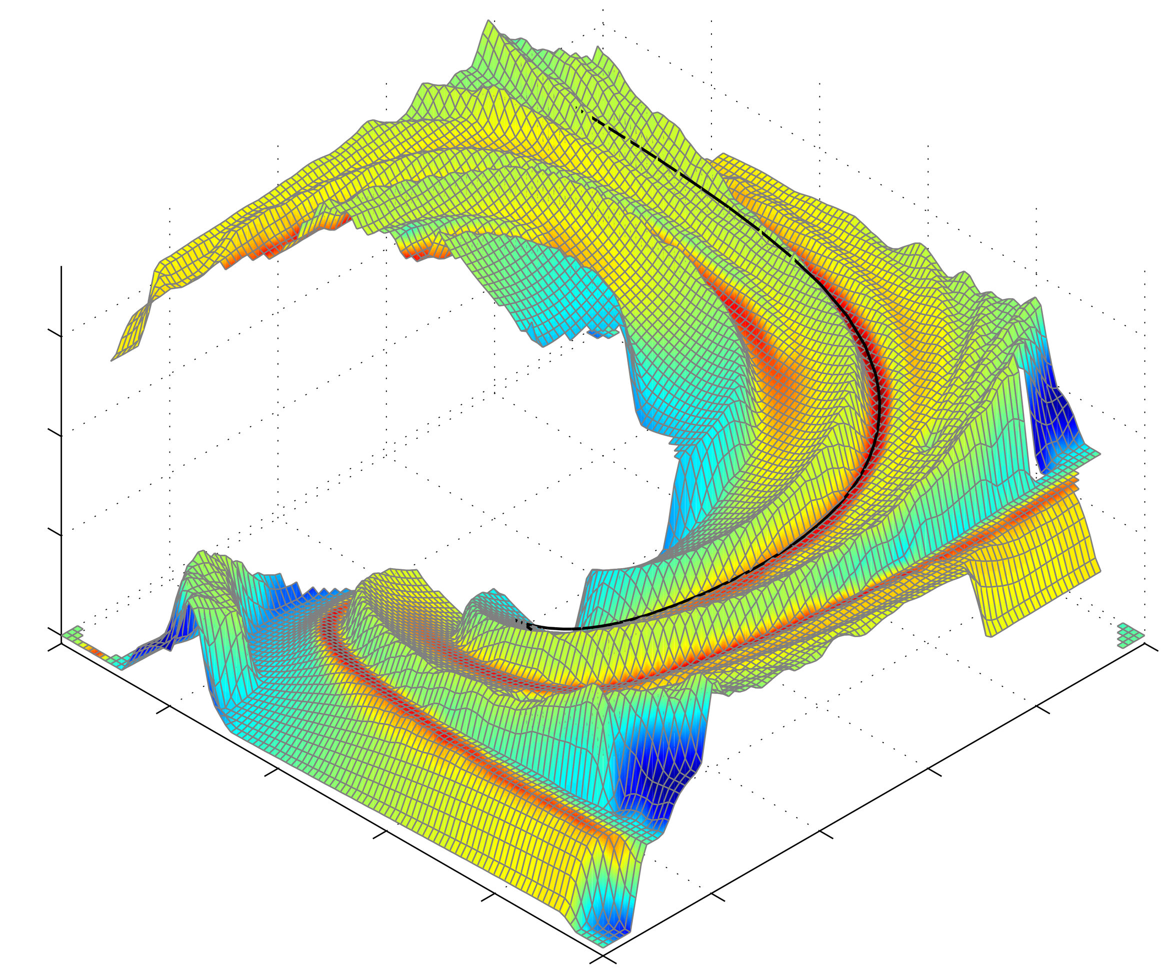
In order to describe time-periodic (and eventually time-aperiodic) flows, we introduce a second technique. Now the submanifold on which FTLE values are shown is not invariant.
This submanifold is defined by maximizing the FTLE value over θ,

$$\theta^*(x, y) \equiv \max_{\theta}\{\mathsf{FTLE}(x, y, \theta)\}.$$

Using continuation methods, we compute a (possibly multi-branch) submanifold θ^* and color it with the corresponding FTLE value.

Time-independent flow

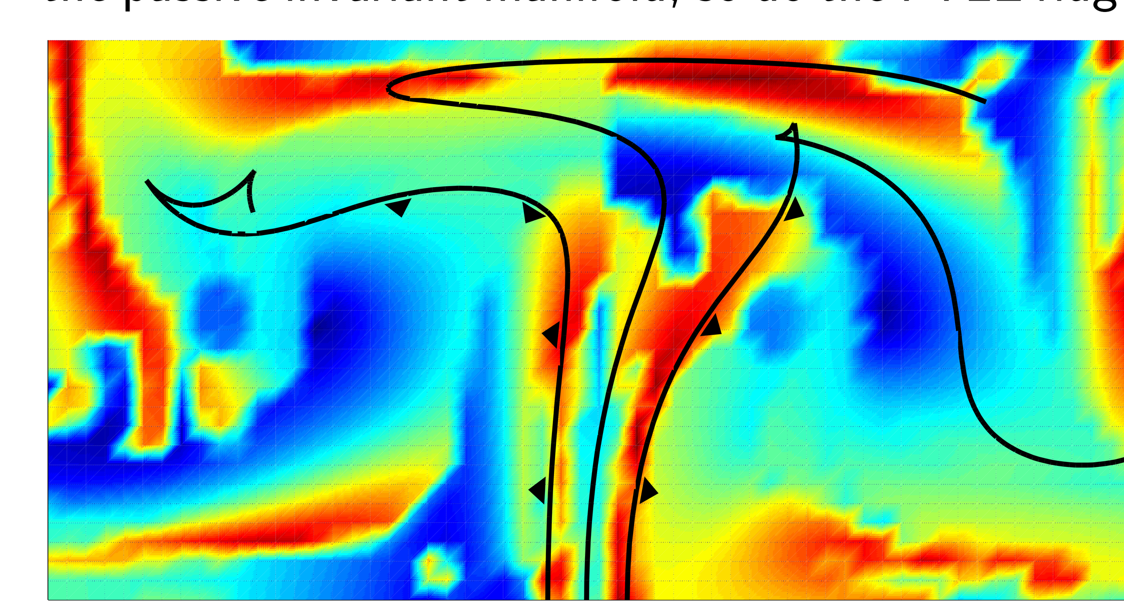
Here we analyze one cell of the alternating vortex chain flow.



θ^* surface colored with the corresponding FTLE values. The FTLE ridges correspond to independently computed BIMs.

Time-periodic flow

Here we add front propagation to the time-periodic vortex array seen earlier. Just as the BIMs split from the passive invariant manifold, so do the FTLE ridges.



FTLE ridges reveal left- and right-oriented BIMs. Multiple local maxima lead to computation challenges in continuation.

References

- Chaotic Transport in Dynamical Systems, S. Wiggins
- Barriers to front propagation in ordered and disordered flows, D. Bargteil and T. Solomon.
- Invariant barriers to reactive front propagation in fluid flows, Mahoney, et al.