

# Information Trimming: Sufficient Statistics, Mutual Information, and Predictability from Effective Channel States

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One of the most basic characterizations of the relationship between two random variables,  $X$  and  $Y$ , is the value of their mutual information. Unfortunately, calculating it analytically and estimating it empirically are often stymied by the extremely large dimension of the variables. One might hope to replace such a high-dimensional variable by a smaller one that preserves the relationship with the other. It is well-known that either  $X$  (or  $Y$ ) can be replaced by its *minimal sufficient statistic* about  $Y$  (or  $X$ ) while preserving the mutual information. While intuitively reasonable, it is not obvious or straightforward that both variables can be replaced *simultaneously*. We demonstrate that this is in fact possible: the information  $X$ 's minimal sufficient statistic preserves about  $Y$  is exactly the information that  $Y$ 's minimal sufficient statistic preserves about  $X$ . We call this procedure *information trimming*. As an important corollary, we consider the case where one variable is a stochastic process' past and the other its future. In this case, the mutual information is the channel transmission rate between the channel's effective states. That is, the past-future mutual information (the excess entropy) is the amount of information about the future that can be predicted using the past. Translating our result about minimal sufficient statistics, this is equivalent to the mutual information between the forward- and reverse-time causal states of computational mechanics. We close by discussing multivariate extensions to this use of minimal sufficient statistics.

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## I. INTRODUCTION

How do we elucidate dependencies between variables? This is one of the major challenges facing today's data-rich sciences, a task often stymied by the curse of dimensionality. One approach to circumventing the curse is to reduce each variable while still preserving its relationships with others. The maximal reduction—the minimal sufficient statistic—is known to work for a single variable at a time [1]. In the multivariate setting, though, it is not straightforward to demonstrate that, as intuition might suggest, all variables can be simultaneously replaced by their minimal sufficient statistics. Here, we prove that this is indeed the case in the two and three variable settings; therefore this technique is a form of lossless multivariate dimensionality reduction [2, 3].

The need for sufficient statistics arises in many arenas. Consider, for example, the dynamics of a complex system. Any dynamical system can be viewed as a communication channel that transmits (information about) its past to its future through its present [4]. Shannon information theory [1] tells us that we can monitor the

amount of information being transmitted through the present by the past-future mutual information—the *excess entropy* [5]. However, this excess entropy can rarely be calculated from past and future sequence statistics, since the sequences are semi-infinite. This makes calculating the excess entropy an ideal candidate for using sufficient statistics. The latter take the form of either a process' prescient states or its causal states [6]. Though known for some time [4], a detailed proof of this relationship was rather involved, as laid out in Ref. [7].

The proof of our primary result turns on analyzing the information-theoretic relationships among four random variables  $W$ ,  $X$ ,  $Y$ , and  $Z$ . All possible informational relationships—in terms of Shannon multivariate information measures—are illustrated in the information diagram [8, 9] (I-diagram) of Fig. 1. This Venn-like diagram decomposes the entropy  $H[X, Y, Z, W]$  of the joint random variable  $(X, Y, Z, W)$  into a number of *atoms*—informational units that cannot be further decomposed using the variables at hand. For example, take the region labeled  $f$  in Fig. 1; this region is the conditional entropy  $H[X | Y, Z, W]$ . Similarly, one has the four-variable mutual information  $k = I[X : Y : Z : W]$  and the condition mutual information  $d = I[W : Z | X, Y]$ . The analogy with set theory, while helpful, must be handled with care: Shannon informations form a *signed* measure. Any atom

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quantifying the information shared among at least three variables can be negative. In the context of our example, Fig. 1, atoms  $g, h, m, n,$  and  $k$  can be negative. Negative information has led to a great deal of investigation; see, for example, Refs. [10, 11].

Here, we are interested in what happens when  $W$  is a sufficient statistic of  $X$  about  $Y$  and  $Z$  is a sufficient statistic of  $Y$  about  $X$  [1]. We denote this  $W = X \searrow Y$  and  $Z = Y \searrow X$ . The resulting (reduced) I-diagram provides a useful and parsimonious view of the relations among the four variables. In particular, it leads us to the main conclusion that each variable can be *simultaneously* reduced to its sufficient statistic while maintaining the mutual informations. Our development proceeds as follows: Section II defines sufficient statistics and utilizes two of their properties to reduce the informational relationships among the variables. Section III discusses how this result applies to stochastic processes as communication channels. Section IV extends our results to the three variable case and makes a conjecture about broader applicability. Finally, Section VI outlines further directions and applications.

## II. SUFFICIENT STATISTICS

A *statistic* is a function  $f(\bullet)$  of random variable samples [1]. Let  $\mathbf{F}_X$  denote the set of all functions of a random variable  $X$ . These functions are also random variables. Given variables  $X$  and  $Y$ , a variable  $V$  forms a *Markov chain*  $X-V-Y$  if  $p(x, y) = \sum_v p(x)p(v|x)p(y|v)$ . Let  $\mathbf{M}_{\mathbf{X}\mathbf{Y}}$  denote the set of all variables that form a Markov chain with  $X$  and  $Y$ . A *sufficient statistic* of  $X$  about  $Y$  is an element of  $\mathbf{S}_{\mathbf{X}\rightarrow\mathbf{Y}} = \mathbf{F}_X \cap \mathbf{M}_{\mathbf{X}\mathbf{Y}}$ .<sup>1</sup> The *minimal sufficient statistic*  $X \searrow Y$  of  $X$  about  $Y$  is the minimal-entropy sufficient statistic:

$$X \searrow Y = \arg \min_{V \in \mathbf{S}_{\mathbf{X}\rightarrow\mathbf{Y}}} H[V]. \quad (1)$$

It is unique up to isomorphism [12].

The minimal sufficient statistic can be directly constructed from variables  $X$  and  $Y$ . Consider the function  $f(\bullet)$  mapping  $x$  to the conditional distribution  $p(Y|X = x)$ ; then  $X \searrow Y \sim f(X)$  [13, 14]. Put more colloquially,  $X \searrow Y$  aggregates the outcomes  $x$  that induce the same conditional distribution  $p(Y|X = x)$ . This is an equivalence class over  $X$ , where the probability of each class is

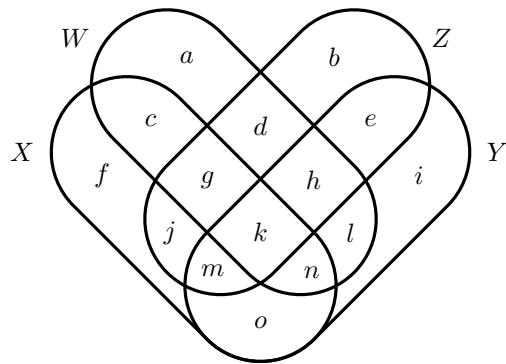


FIG. 1. Information diagram (I-diagram) for four random variables  $X, W, Z,$  and  $Y$ . Each is depicted as a stadium shape and the information atoms are obtained by forming all possible intersections. Individual atoms are identified with lowercase letters.

the sum of the probabilities of the outcomes contained in that class.

### A. Sufficient Statistic as a Function

Our first step in reducing Fig. 1 is to consider the fact that  $W = X \searrow Y$  is a function of  $X$ .<sup>2</sup> Any  $W = f(X)$  if and only if  $H[W | X] = 0$  [12]. Furthermore, conditional entropies  $H[W | \bullet]$  are never increased by conditioning on additional variables [1]. Since conditional entropies are nonnegative [1], conditioning  $W$  on variables in addition to  $X$  can only yield additional zeros. In terms of the information atoms, the relations:

$$\begin{aligned} H[W | X] &= a + d + h + l = 0 \\ H[W | X, Y] &= a + d = 0. \\ H[W | X, Z] &= a + l = 0. \\ H[W | X, Z, Y] &= a = 0, \end{aligned}$$

imply  $a = d = h = l = 0$ . A symmetric argument implies that  $b = d = g = j = 0$ . Each of these zeros is marked with an asterisk in Fig. 2.

### B. Sufficient Statistic as a Markov Chain

Variables  $X, V,$  and  $Y$  form a Markov chain  $X - V - Y$  if and only if  $I[X : Y | V] = 0$ . Said informally,  $V$  statistically shields  $X$  and  $Y$ , rendering them conditionally

<sup>1</sup> Our definition here is equivalent to that provided in, e.g., Ref. [1], but in a form that more directly emphasizes the properties we exploit over the next two subsections.

<sup>2</sup> By  $Y = f(X)$ , we mean for all  $x, |\{y : p(Y = y|X = x) > 0\}| = 1$ .

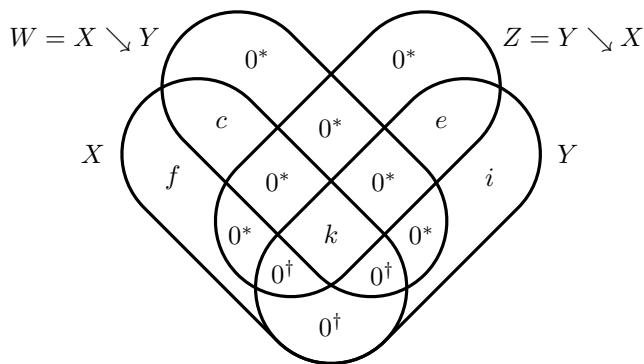


FIG. 2. I-diagram for sufficient statistics: The vanishing information atoms implied by a sufficient statistic being a function of a random variable are labeled  $0^*$ . Those vanishing atoms implied by a sufficient statistic forming a Markov chain are marked with  $0^\dagger$ .

independent. Applied to variable  $W$  we find:

$$\begin{aligned} I[X : Y | W] &= 0 \\ m + o &= 0, \end{aligned}$$

and similarly for  $Z$ ,

$$\begin{aligned} I[X : Y | Z] &= 0 \\ n + o &= 0. \end{aligned}$$

Since  $o = I[X : Y | W, Z]$  is a conditional mutual information,  $o$  is nonnegative by the standard Shannon inequality [1].

Thus far,  $m$  and  $n$  are not individually constrained and so could be negative. However, consider  $I[X : Z | W] = j + m$ , another conditional mutual information, which is therefore also nonnegative. It is already known that  $j = 0$ , therefore  $m$  is nonnegative. Clearly, then,  $m$  and  $o$  are individually zero.

Analogously, we find that  $n$  is nonnegative and conclude that  $n$  and  $o$  are individually zero. These vanishing atoms are marked with  $0^\dagger$  in the simplified I-diagram in Fig. 2.

From this reduced diagram we can easily read that:

$$\begin{aligned} k &= I[X : Y] \\ &= I[X : Z] \\ &= I[W : Y] \\ &= I[W : Z] \\ &= I[X : W : Z] \\ &= I[X : W : Y] \\ &= I[X : Z : Y] \\ &= I[W : Z : Y] \\ &= I[X : W : Z : Y]. \end{aligned} \quad (2)$$

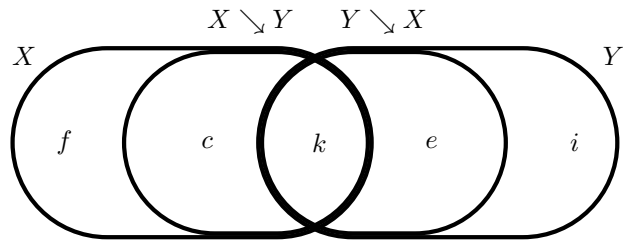


FIG. 3. Minimal I-diagram containing only nonvanishing atoms in Fig. 2.

Furthermore, one can remove the atoms that vanish to arrive at the reduced I-diagram of Fig. 3. It now contains only five of the original fifteen atoms. This demonstrates the procedure of *information trimming* for two arbitrary random variables.

### III. STOCHASTIC PROCESSES AS CHANNELS

We find useful application of this result in the analysis of stationary stochastic processes. Computational mechanics [6] is an information-theoretic framework for analyzing structured stochastic processes. There, a process is considered a channel that communicates its (semi-infinite) past  $X_{-\infty:0}$  to its (semi-infinite) future  $X_{0:\infty}$  through the present [4, 15]. (The following suppresses  $\infty$  when indexing.) An important process property—*excess entropy*—is the mutual information  $\mathbf{E} = I[X_{:0} : X_{0:}]$  between the past and future.  $\mathbf{E}$  is the amount of uncertainty in the future than can be removed using knowledge of the past (and vice versa).

At first blush, it is not clear how to proceed in computing a mutual information between two infinite-dimensional random variables such as these. The answer lies in the concept of causal states. Causal states play a central role as the minimal effective states of a process' channel. The *forward-time causal states* encapsulate the minimal amount of information from the past required for predicting the future. More precisely, the random variable  $\mathcal{S}_0^+$  is the minimal sufficient statistic of the past about the future. Analogously, the *reverse-time causal states*  $\mathcal{S}_0^-$  embody the minimal sufficient statistic of the future about the past—the states needed for optimally retrodicting the past from the future.

By making the following substitutions:  $X \rightarrow X_{:0}$ ,  $W \rightarrow \mathcal{S}_0^+$ ,  $Z \rightarrow \mathcal{S}_0^-$ , and  $Y \rightarrow X_{0:}$  in Eq. (2), we immediately see that the excess entropy (past-future mutual informa-

tion) has several alternate expressions:

$$\mathbf{E} \equiv \mathbf{I}[X_{:0} : X_0] \quad (3)$$

$$\begin{aligned} &= \mathbf{I}[X_{:0} : \mathcal{S}_0^-] \\ &= \mathbf{I}[\mathcal{S}_0^+ : X_0] \\ &= \mathbf{I}[\mathcal{S}_0^+ : \mathcal{S}_0^-] . \end{aligned} \quad (4)$$

The last identity is the most useful form: The excess entropy is the mutual information between the forward-time and reverse-time causal states. As such, this provocatively suggests a communication channel between the forward- and reverse-causal-state processes—a channel that determines the amount information being transmitted through the present. See also Fig. 1 in Ref. [4], analogous to Fig. 3.

We can interpret this operationally. Consider a past  $x_{:0}$ , the particular forward-time causal state  $\mathcal{S}_0^+$  it induces, and an instance  $x_0$  of the future following this state. This future analogously induces a reverse-time causal state  $\mathcal{S}_0^-$ . Considering the above channel between forward- and reverse-time states, the forward state  $\mathcal{S}_0^+$  corresponds to a distribution over reverse-time causal states  $\mathcal{S}_0^-$ . Sampling a state from this distribution results in a state that gives as much information (retrodictivity) about the past as the particular reverse state determined by the future.

Continuing, there are a number of related multivariate mutual information [10] identities that follow directly:

$$\begin{aligned} \mathbf{E} &= \mathbf{I}[X_{:0} : \mathcal{S}_0^+ : \mathcal{S}_0^-] \\ &= \mathbf{I}[X_{:0} : \mathcal{S}_0^+ : X_0] \\ &= \mathbf{I}[X_{:0} : \mathcal{S}_0^- : X_0] \\ &= \mathbf{I}[\mathcal{S}_0^+ : \mathcal{S}_0^- : X_0] \\ &= \mathbf{I}[X_{:0} : \mathcal{S}_0^+ : \mathcal{S}_0^- : X_0] . \end{aligned}$$

Furthermore, making use of the vanishing information atoms, we find that the following Markov chains exist:

$$\begin{aligned} &X_{:0} - \mathcal{S}_0^+ - \mathcal{S}_0^- - X_0 , \\ &\mathcal{S}_0^+ - X_{:0} - \mathcal{S}_0^- - X_0 , \\ &X_{:0} - \mathcal{S}_0^+ - X_0 - \mathcal{S}_0^- , \text{ and} \\ &\mathcal{S}_0^+ - X_{:0} - X_0 - \mathcal{S}_0^- . \end{aligned}$$

Causal states are, as noted, *minimal* sufficient statistics. This minimality is not necessary in the above development. As defined in Ref. [12], a *prescient state*  $\widehat{\mathcal{R}}$  is one for which  $\mathbf{I}[X_{:0} : X_0 : \widehat{\mathcal{R}}_0] = 0$  and  $\widehat{\mathcal{R}}$  is a function of the past. In contrast to the causal states, prescient states need not be minimal. And so, with little else said, the analogous results follow for predictive and retrodictive prescient states. For example, we have  $\mathbf{E} = \mathbf{I}[\widehat{\mathcal{R}}^- : \widehat{\mathcal{R}}^+]$ .

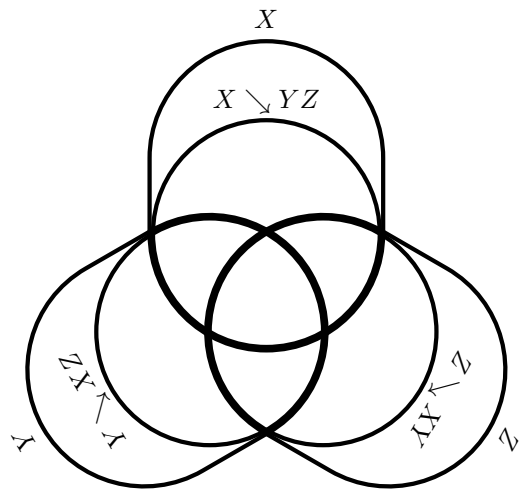


FIG. 4. Minimal I-diagram involving three variables and their minimal sufficient statistics. This differs from a standard 3-variable I-diagram by the addition of three atoms:  $\mathbf{H}[X | X \searrow YZ]$ ,  $\mathbf{H}[Y | Y \swarrow XZ]$ , and  $\mathbf{H}[Z | Z \swarrow XY]$ .

If we were to lift the restriction that prescient states are functions of the past (or the future), the resulting forward and reverse generative [16] states may interact in their “gauge” informations. That is, the atom labeled  $d$  in Fig. 1 may be nonzero; for more on this, see Ref. [17]. The utility of our mutual information identities is then unclear.

The excess entropy, and related information measures, are widely-used diagnostics for complex systems, having been applied to detect the presence of organization in dynamical systems [18–21], in spin systems [22, 23], in Markov random fields [24], in neurobiological systems [25–27], in long-memory processes [28], and even in human language [29, 30].

With these application domains in mind, we should call out the analytical benefits of using causal states, along the lines analyzed here. The benefits are particularly apparent in Refs. [27, 28], for example. While closed-form expressions for excess entropy of finite-state processes have existed for several years [4, 15], it is only recently that it has been analyzed for truly complex (infinite-state) processes [27, 28]. In this work, identifying and then framing calculations around the causal states led to substantial progress. The detailed results here show why this is true: as sufficient statistics, causal states capture the essential structural information in a process. Similar benefits should also accrue when developing empirical estimation and inference algorithms for related information measures.

#### IV. MULTIVARIATE EXTENSIONS

The results can be extended to multivariate systems as well as to alternative measures of shared information. Consider a system of three variables  $X$ ,  $Y$ , and  $Z$ . The I-diagram of interest involves six variables:  $X$ ,  $Y$ ,  $Z$ , and their sufficient statistics about the other variables:  $X \searrow YZ$ ,  $Y \searrow XZ$ , and  $Z \searrow XY$ . This I-diagram contains  $2^6 - 1 = 63$  atoms. It can be substantially simplified along the lines of the previous section. First, note that if  $A$ ,  $B$ ,  $C$ , and  $D$  form the Markov chain  $A - B - CD$ , then we also have the chains  $A - B - C$  and  $A - B - D$ . Second, recall our primary result that  $I[X : Y] = I[X \searrow YZ : Y \searrow XZ]$  and note there are similar relations for the pairs  $(X, Z)$  and  $(Y, Z)$ . Combining these two observations and the methods employed in Section II allows one to determine that 53 atoms are identically 0. This reduction results in the I-diagram of Fig. 4.

Remarkably, the structure of this reduced I-diagram allows us to immediately conclude that the *total correlation*  $T[X : Y : Z]$  [31], *dual total correlation*  $B[X : Y : Z]$  [32], *co-information*  $I[X : Y : Z]$  [33, 34], *CAEKL mutual information*  $J[X : Y : Z]$  [35], and any other multivariate generalization of the mutual information remains unchanged under substitution of sufficient statistics. That is:

$$\begin{aligned} T[X : Y : Z] &= T[X \searrow YZ : Y \searrow XZ : Z \searrow XY] , \\ B[X : Y : Z] &= B[X \searrow YZ : Y \searrow XZ : Z \searrow XY] , \\ I[X : Y : Z] &= I[X \searrow YZ : Y \searrow XZ : Z \searrow XY] , \text{ and} \\ J[X : Y : Z] &= J[X \searrow YZ : Y \searrow XZ : Z \searrow XY] . \end{aligned}$$

We conjecture that this behavior holds for any number of variables. That is, replacing each variable by its sufficient statistic about the others does not perturb the informational interactions among the variables. Nor does it induce any additional interactions among the sufficient statistics. And so, any multivariate mutual information will be invariant. We further conjecture that this is true of any common information, such as the *Gács-Körner common information* [36, 37], the *Wyner common information* [38, 39], and the *exact common information* [40].

#### V. EXAMPLE

We now provide an illustrative example of information trimming. In this example, shown in Fig. 5, we analyze a distribution over two variables,  $X$  and  $Y$ , each with an alphabet of size 3. There, the distinction between  $X = 0$  and  $X = 2$  is irrelevant for the statistics of  $Y$  while

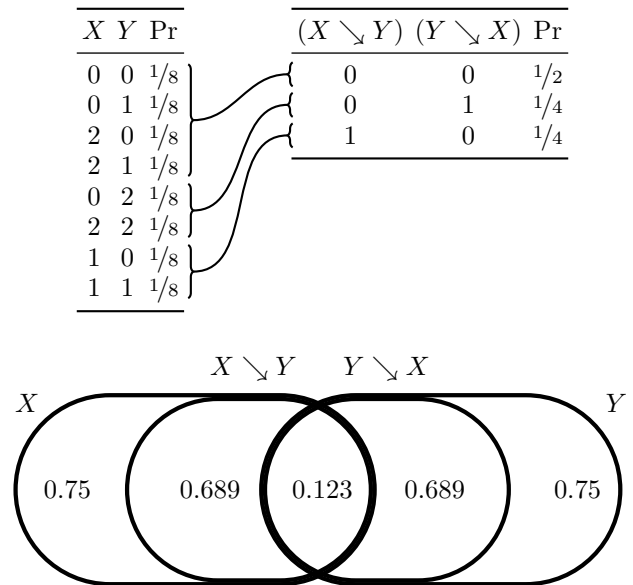


FIG. 5. Information trimming example: The distinction between  $X = 0$  and  $X = 2$  is irrelevant for the statistics of  $Y$ . Similarly, the distinction between  $Y = 0$  and  $Y = 1$  is irrelevant for the statistics of  $X$ . The informationally-trimmed version sheds this 1.5 bit of information.

the distinction between  $Y = 0$  and  $Y = 1$  is irrelevant for the statistics of  $X$ . Trimming the distribution sheds 1.5 bit of irrelevant information. It is worth noting that there remains conditional entropy in the distribution—it is generically not possible to capture just the mutual information as a random variable [36].

#### VI. CONCLUDING REMARKS

We demonstrated that it is proper to replace each variable with a sufficient statistic about the others without altering information-theoretic interactions among the variables. This is a great asset in many types of analysis and provides a principled method of performing lossless dimensionality reduction. As an important specific application, we demonstrated how the causal states of computational mechanics allow for the efficient computation of the excess entropy.

Our proof method centered around the use of an I-diagram and its atoms. Steps in our proof, such as identifying that the atom labeled  $m$  is nonnegative via its containment in  $I[X : Z | W]$ , are greatly aided by this graphical tool. Despite this, we believe that a superior proof of these results exists—a proof that does not depend on demonstrating atom-by-atom that all but a select few are zero. Such a proof would, hopefully, apply generically and directly to an  $N$ -variable system, hold for



the menagerie of multivariate generalizations of the mutual information, and perhaps apply even to the common informations.

Finally, we note that this procedure is available in the dit (discrete information theory) Python package [41] as `dit.algorithms.info_trim()`.

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