

# The Many Roads to Synchrony: Natural Time Scales and Their Algorithms

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We consider two important time scales—the Markov and cryptic orders—that monitor how an observer synchronizes to a finitary stochastic process. We show how to compute these orders exactly and that they are most efficiently calculated from the  $\epsilon$ -machine, a process’s minimal unifilar model. Surprisingly, though the Markov order is a basic concept from stochastic process theory, it is not a probabilistic property of a process. Rather, it is a topological property and, moreover, it is not computable from any finite-state model other than the  $\epsilon$ -machine. Via an exhaustive survey, we close by demonstrating that infinite Markov and infinite cryptic orders are a dominant feature in the space of finite-memory processes. We draw out the roles played in statistical mechanical spin systems by these two complementary length scales.

**Keywords:** Markov chain, hidden Markov model, Markov order, cryptic order, synchronization,  $\epsilon$ -machine

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## I. INTRODUCTION

Stochastic processes are frequently characterized by the spatial and temporal length scales over which correlations exist. In physics, the range of correlations is a structural property giving, for example, the distance over which significant energetic coupling exists among a system’s degrees of freedom [1]. In time series analysis, knowing the temporal scale of correlations is key to successful forecasting [2]. In biosequence analysis, the decay of correlations along DNA base pairs determines in some measure the difficulty faced by a replicating enzyme as it “decides” to begin transcribing a gene [3]. In multiagent systems, one of an agent’s first goals is to detect useful states in its environment [4]. The common element in these is that the correlation scale determines how quickly an observer—analyst, forecaster, enzyme, or agent—*synchronizes* to a process; that is, how it comes to know a relevant structure of the stochastic process.

We recently showed that there are a number of distinct, though related, length scales associated with synchronizing to stationary stochastic processes [5]. Here, we show that these length scales are *topological*, depending only

on the underlying graph topology of a canonical representation of the stochastic process. This reveals deep ties between the structure of a process’s minimal sufficient statistic and synchronization of an observer. We also recently introduced another class of synchronization length scales based, not on state-based models, but on the convergence of sequence statistics [6]. We briefly compare these to the Markov and cryptic orders.

Specifically, we investigate measures of synchronization and their associated lengths scales for hidden Markov models (HMMs)—a particular class of processes with an internal (hidden) Markovian dynamic that produces an observed sequence. We focus on two such measures—the Markov order and the cryptic order—and show through a series of incremental steps how they can be efficiently and accurately computed from the process’s minimal sufficient statistic, the  $\epsilon$ -machine.

Our development proceeds as follows. After briefly outlining the required background in Sec. II, we introduce the two primary measures of interest in Sec. III and demonstrate their calculation via naive methods in Sec. IV. Reflecting on a surprising finding in Sec. V, Sec. VB shows to how alleviate several weaknesses in the naive approach. Then, borrowing relevant data structures from formal language theory, Sec. VC resolves the last of the issues. Together these steps provide an efficient algorithm for exactly calculating the Markov order when it is finite and for determining when it is infinite. Building on this new understanding, Sec. VI goes on to show how to compute the second time scale—the cryptic order—through similar means. We then briefly

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touch upon other time scales and their relative bounds in Sec. VII. Leveraging the computational efficiency, we survey the Markov and cryptic orders among  $\epsilon$ -machines in Sec. VIII and conclude that infinite correlation is a dominate property in the space of memoryful stationary processes. The implication is that most, if not all, observers cannot synchronize exactly [7]. To illustrate how these time scales apply in practice, Sec. IX characterizes correlations in one-dimensional spin systems. Finally, we conclude by discussing how these time scales compare to other measures of interest and by suggesting applications where they and their algorithms will prove useful.

## II. BACKGROUND

We assume the reader has introductory knowledge of information theory and finite-state machines, such as that found in the first few chapters of Ref. [8] and Ref. [9], respectively. Our development makes particular use of  $\epsilon$ -machines, a natural representation of a process that makes many properties directly and easily calculable; for a review see Ref. [10]. A cursory understanding of symbolic dynamics, such as found in the first few chapters of Ref. [11] is useful for several results.

We denote subsequences in a time series as  $X_{a:b}$ , where  $a < b$ , to refer to the random variable sequence  $X_a X_{a+1} X_{a+2} \cdots X_{b-1}$ , which has length  $b - a$ . We drop an index when it is infinite. For example, the *past*  $X_{-\infty:0}$  is denoted  $X_{:0}$  and the *future*  $X_{0:\infty}$  is denoted  $X_{0:}$ . We generally use  $w$  to refer to a *word*—a sequence of symbols drawn from an alphabet  $\mathcal{A}$ . We place two words,  $u$  and  $v$ , adjacent to each other to mean concatenation:  $w = uv$ . We define a *process* to be a joint probability distribution over  $X_{:} = X_{:0} X_{0:}$ .

A *presentation* of a given process is any state-based representation that generates the process. A process's  $\epsilon$ -*machine* is its unique, minimal unifilar presentation [12]. The recurrent states of a process's  $\epsilon$ -machine are known as the *causal states* and, at time  $t$ , are denoted  $\mathcal{S}_t$ . The causal states are the minimal sufficient statistic of  $X_{:0}$  about  $X_{0:}$ . For a thorough treatment on presentations see Ref. [5].

## III. PROBLEM STATEMENT

When confronted with a process, one of the most natural questions to ask is, How much memory does it have? Is it like a coin or a die, with no memory? Does it alternate between two values, requiring that the process remember its phase? Does it express patterns that are arbitrarily long, requiring an equally long memory? This

type of memory is quantified by the Markov order:

$$R \equiv \min \{ \ell \mid \Pr(X_0 | X_{-\ell:0}) = \Pr(X_0 | X_{:0}) \} . \quad (1)$$

To put it colloquially, how many prior observations must one remember to predict as well as remembering the infinite past? Markov chains have  $R = 1$  by their very definition. Hidden Markov models, though their internal dynamics are Markovian ( $R = 1$ ), their observed behavior can range from memoryless ( $R = 0$ ) to infinite ( $R = \infty$ ). A major goal in the following is to show how to compute a process's  $R$  efficiently and accurately given its  $\epsilon$ -machine. In this vein it is prudent to recast Eq. (1) using causal states:

$$\begin{aligned} \Pr(X_0 | X_{-R:0}) &= \Pr(X_0 | X_{:0}) \\ \implies X_{:0} &\sim_{\epsilon} X_{-R:0} \\ \implies \mathbb{H}[\mathcal{S}_0 | X_{-R:0}] &= 0 \\ \implies R &= \min \{ \ell \mid \mathbb{H}[\mathcal{S}_0 | X_{-\ell:0}] = 0 \} \\ &= \min \{ \ell \mid \mathbb{H}[\mathcal{S}_{\ell} | X_{0:\ell}] = 0 \} . \end{aligned} \quad (2)$$

In effect, since the past  $R$  observations predict just as well as the infinite past, the causal states are a function of length- $R$  pasts.

The second primary length scale we discuss is the *cryptic order*  $k_{\chi}$  [13]. Its definition builds from Eq. (2):

$$k_{\chi} \equiv \min \{ \ell \mid \mathbb{H}[\mathcal{S}_{\ell} | X_{0:}] = 0 \} . \quad (3)$$

The difference between the two is that cryptic order is conditioned on the *infinite* future, as opposed to a finite one. This provides our interpretation of the cryptic order:  $k_{\chi}$  is the number of causal states that cannot be *retrodicted*. That is, no matter how many future symbols we know, the first  $k_{\chi}$  internal states the process visited cannot be inferred.

## IV. NAIVE APPROACH

To illustrate a direct method of determining a process's Markov and cryptic orders, we appeal to yet another form of their definitions [5]:

$$\begin{aligned} R &= \min \{ \ell \mid \mathbb{H}[X_{0:\ell}] = \mathbf{E} + \ell h_{\mu} \} \\ k_{\chi} &= \min \{ \ell \mid \mathbb{H}[X_{0:\ell}, \mathcal{S}_{\ell}] = \mathbf{E} + \ell h_{\mu} \} , \end{aligned} \quad (4)$$

where  $\mathbf{E} = \mathbb{I}[X_{:0}, X_{0:}]$  is known as the *excess entropy* and  $h_{\mu} = \mathbb{H}[X_0 | X_{:0}]$  is known as the *entropy rate* [14]. The intuition for these is identical to those above: Once we reach Markov (cryptic) order, we predict as accurately as possible. It is worth noting that these definitions only hold for *finitary* ( $\mathbf{E} < \infty$ ), stationary processes.

These definitions lead to a simple way of determining a process's Markov and cryptic orders. To compute the Markov order, we calculate the entropy  $H[X_{0:\ell}]$  of longer and longer blocks of contiguous observations until it begins to grow linearly. We call this function of  $\ell$  the *block entropy curve*. The first  $\ell$  at which  $H[X_{0:\ell}]$  matches its linear asymptote is the Markov order. To compute the cryptic order, we perform a similar test, but rather than calculating the entropy of blocks of observations alone, we calculate the entropy  $H[X_{0:\ell}, \mathcal{S}_\ell]$  of those blocks along with the causal states that are induced by those observations. We call this function of  $\ell$  the *block-state entropy curve*. The cryptic order is the length at which the block-state entropy curve reaches its asymptotic linear behavior. This view of the two orders is shown in Fig. 1. The data for the block entropy and block-state entropy curves shown there comes from the *Phase-Slip Backtrack* (PSB) Process shown in Fig. 2.

It is important to point out the weaknesses of this approach. They are at least fourfold, one must (i) know  $h_\mu$  exactly, (ii) know  $\mathbf{E}$  exactly, (iii) be able to differentiate the block entropies being *exactly on* the asymptote from *less than machine precision away from* the asymptote, and (iv) be able to “guess” when  $R$  or  $k_\chi$  are infinite in order to terminate the calculation. The first two are not prohibitive. The entropy rate  $h_\mu$  can be computed exactly from any unifilar model of the process, and so its calculation can be done fairly easily [15]. Similarly, the excess entropy  $\mathbf{E}$  can be computed if the joint distribution over both a unifilar, gauge-free model of the process and a unifilar, gauge-free model of the reverse of the process is on hand [16].

The last two weaknesses do not have such direct solutions. How are we to know if our entropy calculation at length  $\ell$  is exactly equal to  $\mathbf{E} + \ell h_\mu$ ? Or, instead, are the curve and linear asymptote so close that finite-precision estimates cannot differentiate them? Compounding this, what if  $H[X_{0:\ell}]$  has not equaled  $\mathbf{E} + \ell h_\mu$  by  $\ell = 10^6$ ? Can one assume that it ever will? Perhaps the process is Markov order  $R = 10^8$ . These are the two particular weaknesses that need to be overcome.

## V. MARKOV ORDER IS TOPOLOGICAL

We start with the somewhat surprising observation that Markov order is not a probabilistic property, as seemingly suggested by Eq. (1), but rather a topological one. The first hint at this comes, though, in an empirical study. The question then becomes just how is this so. By way of answering it, we solve the fundamental problems noted with the naive approach to Markov order. Several examples serve to drive home the idea and illustrate the

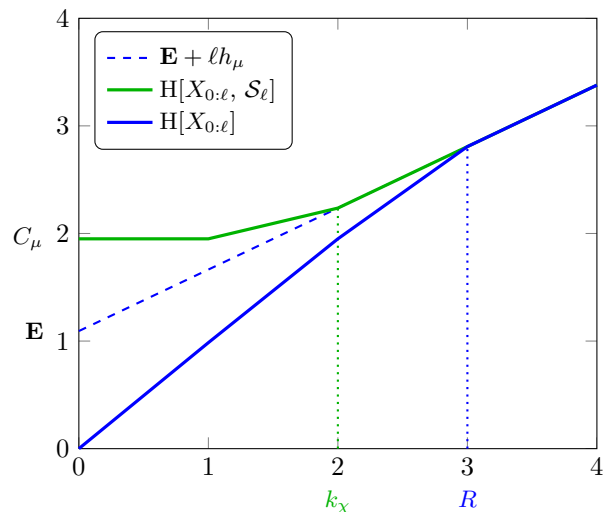


FIG. 1. Block entropy and block-state entropy for the PSB Process of Fig. 2: The block entropy curve reaches its asymptotic behavior ( $\mathbf{E} + \ell h_\mu$ ) at  $\ell = 3$ , indicating a Markov order  $R = 3$ . The block-state entropy curve reaches the same asymptote at  $\ell = 2$  and so the process is cryptic order  $k_\chi = 2$ .

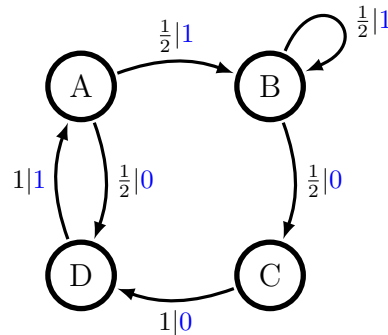


FIG. 2. The Phase-Slip Backtrack (PSB) Process: Edges are labeled  $p|s$  where  $p$  is the probability of an edge being followed and  $s$  is the symbol emitted upon traversing it.

calculation methods.

### A. An Observation

The first step forward in solving the two main problems encountered in the naive Markov order method is to take a step back. Rather than considering the particular process generated by the machine in Fig. 2, we study the family of processes generated when its transition probabilities are varied while the structure remains the same. This family is shown by the parametrized machine of Fig. 3. If we compute block and block-state entropy curves for a random ensemble of processes from this family, plot the derivative of those curves and subtract-

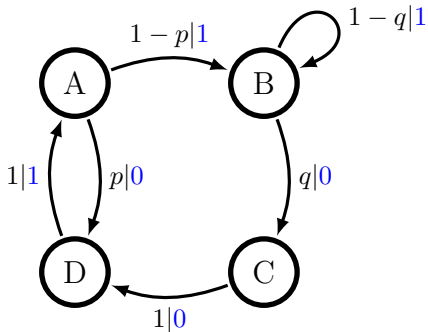


FIG. 3. Phase-Slip Backtrack Process with parametrized transition probabilities.

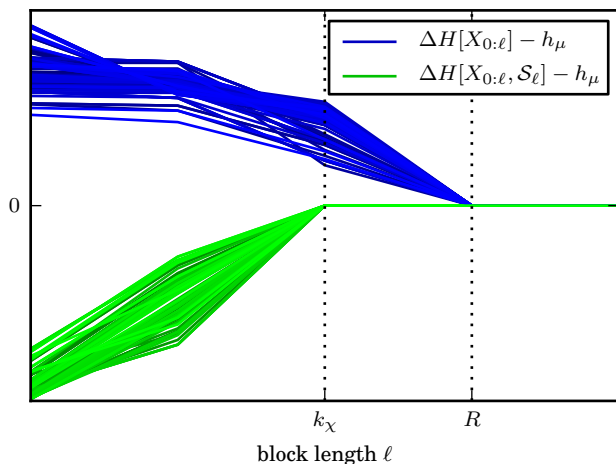


FIG. 4. Entropy convergence curves versus block length  $\ell$  for Fig. 3’s family of processes with several dozen random values for  $p$  and  $q$ . The linear asymptotic behavior ( $h_\mu$ ) has been subtracted out of each curve. (See inset.) The Markov order  $R$  and cryptic order  $k_\chi$  are the lengths  $\ell$  at which the blue (darker) and green (lighter) lines, respectively, reach zero. Thus, both orders are independent of the generating machine’s probability parameters.

ing out their asymptotic behavior, we arrive at the block and block-state entropy convergence shown in Fig. 4.

As it dramatically demonstrates, the Markov and cryptic orders are *independent* of the transition probabilities in the machine’s structure. Thus, any pattern relevant for prediction is encoded by the  $\epsilon$ -machine’s topology.

### B. Synchronizing Words

On careful inspection of Eq. (2), however, it is not surprising that the Markov order is a topological property. A conditional entropy  $H[X|Y]$  vanishes only if  $X$  is a deterministic function of  $Y$ . In our case,  $H[S_R|X_{0:R}] = 0$

means that each length- $R$  word determines a unique state of the model. We say that each word of length  $R$  is *synchronizing* [5]. (Later, we consider only *prefix-free* synchronizing words—those which have no initial subword that also synchronizes.) If one observes a process having no inkling as to which state its hidden Markov model began in, then after observing  $R$  symbols the exact state will be known.

This provides an improved method of determining the Markov order. Enumerate all words of increasing length noting which have synchronized and which have not. When all the words at the current length have synchronized, then that length is the Markov order  $R$ . This procedure has been completed for the PSB Process in Fig. 5. It can be verified that at lengths 0, 1, and 2 it is possible to still have ambiguity as to which state this system is in. For example, if the two symbols 10 are observed, the system may be in either state  $C$  or state  $D$ . One more observation is required to disambiguate which it is. Therefore, as observed previously, the Markov order for this process is  $R = 3$ . This method is improved by lexicographically enumerating words of increasing length *until* they synchronize to a single state. The longest such word—a prefix-free synchronizing word—is the Markov order  $R$ , since by that point every shorter word will have synchronized and, therefore, the causal states will be determined uniquely by words of that length.

This method addresses several weaknesses of the naive approach. Now, neither  $\mathbf{E}$  nor  $h_\mu$  are needed, nor do we need to concern ourselves with the details of comparing nearly equal numerical values. However, the method relies on enumerating prefix-free synchronizing words, and it is quite possible for a process to have an infinite number of prefix-free synchronizing words. In these situations, it is not feasible to enumerate them all, hoping to identify the longest. To address this problem, we turn to formal language theory [9].

### C. State Subset Construction

The remaining problem is to find the longest prefix-free synchronizing word without having to enumerate them all. This can be accomplished with a standard algorithm from the theory of finite automata. We construct an object known as the *power automaton* (PA), so-named since its states are elements of the power set of a given automaton’s states.

Construction of the power automaton begins with a single state: the set of all states from the  $\epsilon$ -machine. This is the PA’s *start* state. Then recursively, for each state in the PA and each symbol, consider all  $\epsilon$ -machine states that can be reached by any  $\epsilon$ -machine state within

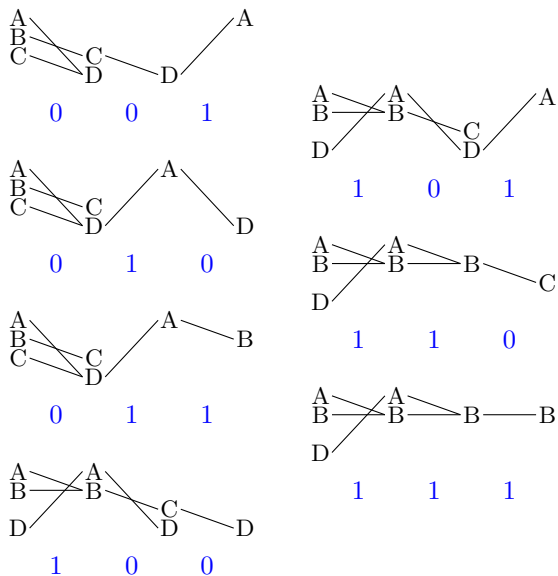


FIG. 5. All observable words of length 3 for the PSB Process. Each word has been annotated with the paths through which that word invokes synchrony. It is not until the observation of three symbols that in all cases there is only a single possible state. There are, however, some words which induce synchrony more quickly.

the current PA state on the currently considered symbol. A new PA state consisting of the set of  $\epsilon$ -machine successor states is added, along with a directed edge from the current to the new PA state, labeled with the current symbol. Once the successors to each PA state have been determined, there will be a subgraph of the PA that is isomorphic to the recurrent  $\epsilon$ -machine. This subgraph is the PA's *recurrent* component. When the  $\epsilon$ -machine generates an ergodic process, this subgraph is the only strongly connected component with no outgoing edges. The remainder of the PA consists of *transient* states.

Synchronizing words are associated with particular PA paths. Each path begins in the start state and traverses edges in PA's transient portion. Eventually, the path continues to a PA recurrent state. Prefix-free synchronizing words have paths that end as soon as they reach a recurrent PA state. To find the longest prefix-free synchronizing word, we weight each edge in PA's transient part with the value  $-1$  and each edge in the recurrent part with  $0$ . With these modifications, the Bellman-Ford algorithm can be employed to discover the path of least weight from the start state to any recurrent state. Due to the weighting, the path of least weight is the longest.

The alternative Floyd-Warshall algorithm can also be used; see Ref. [17] for details regarding both. We choose the Bellman-Ford algorithm for two reasons. First, it works on graphs with negative weight and, second, it detects negative-weight cycles. A negative weight cycle

here implies that the longest path is arbitrary (infinite) in length.

This specifies a complete method for computing the Markov order efficiently and accurately from a model of a process. First, construct the power automaton. Then, weight the edges according to their status as transient or recurrent. Last, find the path of least weight from the start to a recurrent state. It runs in  $O(\mathcal{A}^{2N})$  time, which is exponential but finite. And, it depends only on integer calculations. In this way, it circumvents all the computational difficulties encountered in the naive approach. Thus, if one can infer an accurate model from observations of a system, the problem of computing that system's Markov order is solved.

This method also provides a solution to weakness (iv) of the naive algorithm (Sec. IV). When finite, the Markov order depends on the longest path through the transient states of the power automaton, and for an  $n$  state recurrent  $\epsilon$ -machine, there are at most  $f(n) := 2^n - n - 1$  transient states (subtracting  $n$  recurrent states and also the empty set). Since loops in the transient structure imply infinite Markov order, it follows that the longest possible path is one which visits each of the transient states. Thus, if the Markov order has not been found by  $L = f(n)$ , then it is safe to conclude that the Markov order is infinite. Since, the Markov order bounds the cryptic order, the same bound works for the cryptic order. It is an open problem to find a tight upper bound for the Markov order in terms of the number of states and the number of symbols in the alphabet.

#### D. Examples

A variety of qualitatively different behaviors can be exhibited by the Markov order algorithm. Here, we illustrate the typical cases. Applying it to the PSB Process, the algorithm produces the fairly simple transient structure consisting of three nodes—PA states  $ABCD$ ,  $AB$ , and  $CD$ —seen in Fig. 6. There are two longest paths starting from PA start state  $ABCD$  and ending in a recurrent node:  $ABCD \xrightarrow{1} AB \xrightarrow{0} CD \xrightarrow{1} A$ , which is traversed with the word 101, and  $ABCD \xrightarrow{1} AB \xrightarrow{0} CD \xrightarrow{0} D$ , traversed with the word 100. This means that the longest prefix-free synchronizing words are 101 and 100, both of length three, and therefore PSB Process's Markov order is  $R = 3$ .

The second process we analyze is shown in Fig. 7. It has a slightly more complicated transient structure than that of the PSB Process. Of particular note is the self-loop on PA state  $AB$ . This loop exists because  $\epsilon$ -machine states  $A$  and  $B$  transition to each other on producing a 0. As a consequence, we cannot determine the state until

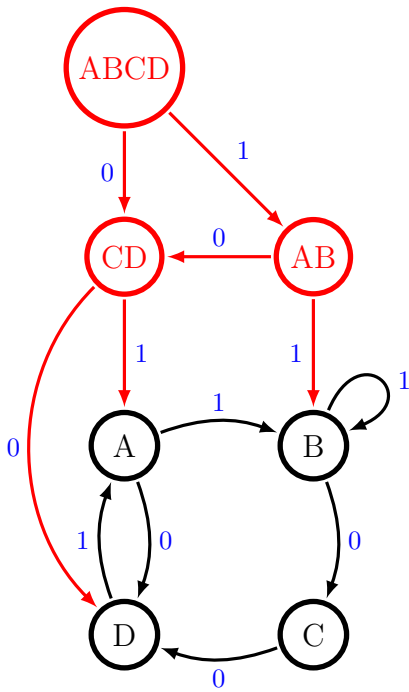


FIG. 6. PSB Process power automaton. The longest path beginning from state  $ABCD$ , traversing transient (red) edges, and ending in a recurrent (black) state is of length 3:  $ABCD \xrightarrow{1} AB \xrightarrow{0} CD \xrightarrow{1} A$  (or  $\xrightarrow{0} D$ ).

observing a 1. This inability to synchronize on some words results in a non-Markovian process; that is,  $R = \infty$ . The Bellman-Ford algorithm terminates as soon as it detects the corresponding negative-weight cycle.

Our third example is the Nemo Process, shown in Fig. 8. Its transient structure is particularly simple: a single state representing all the recurrent states. Since the recurrent states simply permute upon observing a 0, the word  $0000\dots$  never allows one to determine in which state the system is. This is indicated by the self-loop on PA state  $ABC$ . This once again means that the process is non-Markovian and has  $R = \infty$ . This condition is detected by the algorithm as well.

## VI. CRYPTIC ORDER

We now turn to calculate a process's cryptic order  $k_\chi$ . Recall that Eq. (3) involves a condition on the *infinite* future. With probability one, each infinite future synchronizes for exactly synchronizing  $\epsilon$ -machines [7]. We can then consider the problem of calculating  $k_\chi$  to be that of determining as much of a state history as possible, given a prefix-free synchronizing word and the state to which it synchronized. The maximum number of states we cannot retrodict is then the cryptic order.

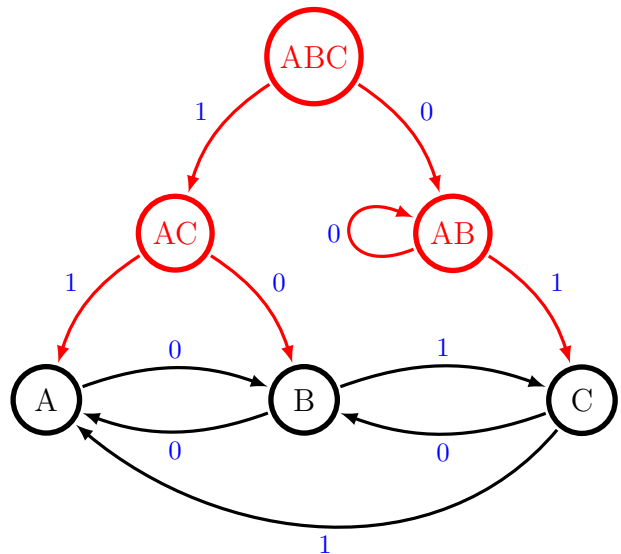


FIG. 7. Typical complications in the PA for a finite-state non-Markovian Process. The signature is the loop  $AB \xrightarrow{0} AB$  in the transient structure. This means there is the possibility of an arbitrarily long series of observations that never synchronize and that, in turn, cause Markov order to diverge. Generically, loops in the transient structure can consist of more than one PA state.

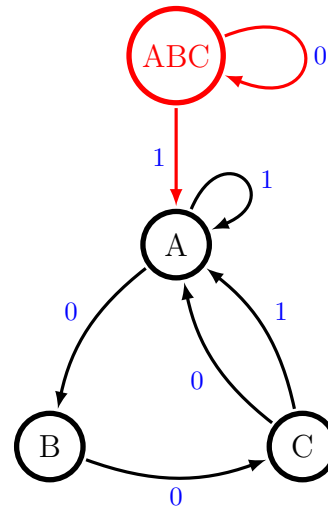


FIG. 8. Like Fig. 7's process, the Nemo Process here is non-Markovian. The Nemo Process makes this perhaps clearer, however, since the recurrent states permute into each other upon observing a 0. The transient structure makes this explicit:  $ABC$  maps back to itself on a 0.

### A. Calculation

Figure 9 depicts how the cryptic order is determined. Only the paths in Fig. 5 that survive all the way to synchrony (at the Markov order) are reproduced. From these, we determine how many symbols into each word

we must parse (from the left) before the  $\epsilon$ -machine is in one state only. The maximum such length is the cryptic order  $k_\chi$ .

As with the Markov order, we need only consider prefix-free synchronizing words. However, we are again faced with the prospect that there may be an infinite number of prefix-free synchronizing words. Fortunately, a better method is available, and it too begins by constructing the power automaton. Now, we examine the “veracity” of each transient edge. Take as an example the edge  $ABC \xrightarrow{1} A$  in Fig. 8. It states that upon producing a 1 from the superposition of states  $A$ ,  $B$ , and  $C$ , the system can only transition to state  $A$ . For the cryptic order, we now condition on the fact that we are in state  $A$  and ask what states could have transitioned to  $A$  on a 1. Upon inspection, it's clear that the system could have only transitioned from states  $A$  or  $C$  on a 1. The core of the cryptic order algorithm is to inspect each transient edge in the power automaton in this manner, updating the PA's structure to “honestly” reflect the process's dynamics. In this instance, we create a state  $AC$  that transitions to state  $A$  on a 1 instead of transitioning from  $ABC$  on a 1.

After creating a state, the automaton must be made consistent. To do this, subset construction is applied to include any newly added states. Generally, this creates new edges as well. And, these too must be analyzed by the cryptic order algorithm. Once every edge has been inspected, some transient structure will remain. Once again, the longest path is the key, and the same edge-weighting method (Bellman-Ford) is employed to find it and so give the cryptic order.

## B. Examples

The ways in which the cryptic order algorithm modifies the power automaton are diverse. Each example from Sec. VD above illustrates a different behavior.

First, consider its behavior on the PSB Process (Fig. 6), the final result of which is shown in Fig. 10. The edge  $CD \xrightarrow{0} D$  in Fig. 6 can be removed since it does not represent a path that is true. To see why, note that to get to  $D$  on a 0, one must come from either state  $A$  or state  $C$ . However, since we are assuming  $CD$ , the process must be in either state  $C$  or  $D$ . The intersection of those two sets is state  $C$  and it is, therefore, the only possible state the system could have *actually* been in. Thus,  $CD \xrightarrow{0} D$  is a misrepresentation from the cryptic order perspective and, in fact, it corresponds to the edge  $C \xrightarrow{0} D$ , which already exists in the PA. So, the edge  $CD \xrightarrow{0} D$  is removed.

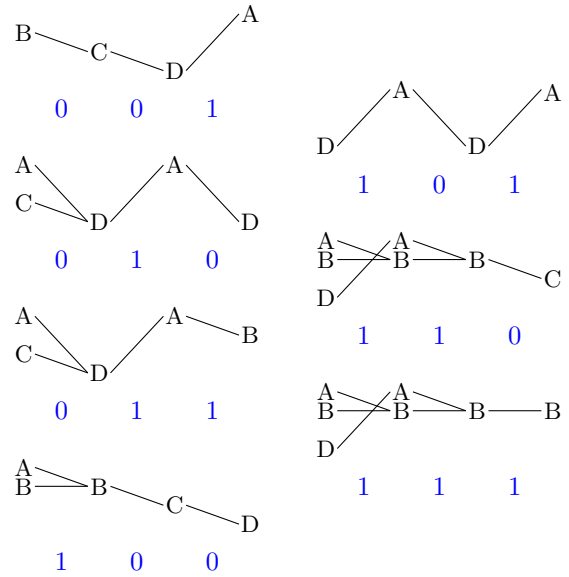


FIG. 9. Key paths for determining cryptic order  $k_\chi$ : We start with the paths in Fig. 5, except we remove paths that do not survive to the end of the sync word. The surviving paths give us the cryptic order: They each identify a single state by length  $\ell = 2$  and so  $k_\chi = 2$ .

This is not all, however. We must maintain the path's provenance. The edges that came in to  $CD$  must be redirected to  $C$  (add edges  $ABCD \xrightarrow{0} C$  and  $AB \xrightarrow{0} C$ ), since those are the edges that would have been traversed immediately prior to  $CD \xrightarrow{0} D$ . Note that these edges are later removed in this recursive algorithm and so do not appear in Fig. 10. In the end, we see that the longest path from a start state to the recurrent states is 2 and, therefore,  $k_\chi = 2$ , one less than the Markov order  $R = 3$ .

Next, consider the example from Fig. 7. The final output of the cryptic order algorithm is shown in Fig. 11. This process's PA consists of two major branches: One with a maximum depth of 2 and the other containing a loop. The cryptic order algorithm discovers that the branch with a loop is completely retrodictable.  $AB \xrightarrow{1} C$  is actually  $B \xrightarrow{1} C$ , and this creates edges  $AB \xrightarrow{0} B$  and  $ABC \xrightarrow{0} B$ , again to maintain provenance. The first of these newly added edges is also retrodictable:  $AB \xrightarrow{0} B$  can only be  $A \xrightarrow{0} B$ . The second,  $ABC \xrightarrow{0} B$ , is in fact  $AC \xrightarrow{0} B$ . Along this branch of the transient structure, we are thus only unable to retrodict the word 01, of which the 1 can be retrodicted, simply leaving us with  $AC \xrightarrow{0} B$ . The previous branch is more easily analyzed, leaving us with  $BC \xrightarrow{1} AC \xrightarrow{0} B$ , the later part of which was already in the PA from analyzing the other branch. This leaves a longest path of length 2, making  $k_\chi = 2$ . Thus, we see that this process is an example with infinite

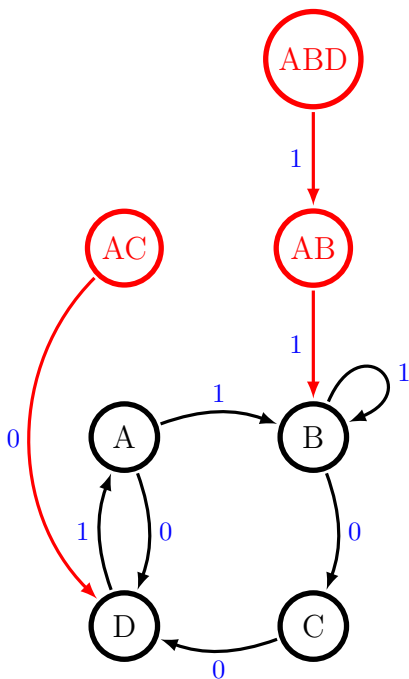


FIG. 10. Cryptic order algorithm applied to the PSB Process: The power automaton in Fig. 6 suggests that the word 11 could originate in any of  $A$ ,  $B$ ,  $C$ , or  $D$ . Careful inspection of the recurrent structure, though, shows that  $C$  cannot be the originator of 11, whereas the other three states can. The cryptic order algorithm accounts for such constraints. The longest path from a transient state to a recurrent state is  $ABD \xrightarrow{1} AB \xrightarrow{1} B$  and, therefore,  $k_\chi = 2$ .

Markov order, but finite cryptic order.

The last example to consider is the Nemo Process. Recall that it is infinite Markov, as observed in Fig. 8. Applying the cryptic order algorithm results in the structure shown in Fig. 12. In this case, the transient structure grows under the algorithm. The edge  $ABC \xrightarrow{1} A$ , connecting the transient to the recurrent structure in the power automaton, is modified by the algorithm since  $B$  cannot transition to  $A$  on a 1. The state  $AC$  is created and connected to  $A$ . Completing the power automaton structure from this state results in states  $AB$  and  $BC$  being added, forming the cycle  $AC \xrightarrow{0} AB \xrightarrow{0} BC \xrightarrow{0} AC$ . The algorithm terminates when the cycle is detected in this way. The cycle is valid as far as the cryptic order is concerned: Each of its states can be transitioned to from the recurrent state associated with the prior state in the cycle. The cycle results in an arbitrarily long path and, therefore,  $k_\chi = \infty$ .

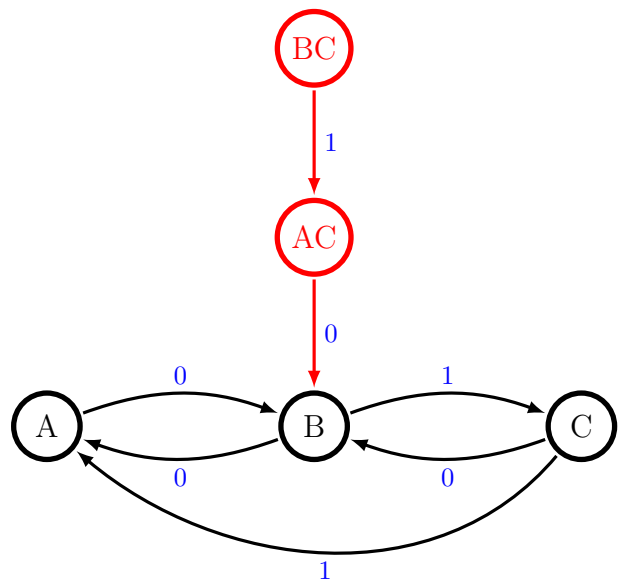


FIG. 11. Cryptic order analysis of Fig. 7's process: The transient structure branch shown there— $ABC \xrightarrow{0} (AB \xrightarrow{0} AB)^* \xrightarrow{1} C$ , with the arbitrarily long synchronizing word  $00^*1$ —can be perfectly retrodicted. Moreover, only a fragment of the left branch of the transient structure remains. This fragment has a length of 2, and so  $k_\chi = 2$ .

## VII. OTHER NATURAL TIME SCALES

Paralleling the interpretation of the Markov and cryptic orders as the block lengths at which an associated information measure reaches its asymptotic behavior, this section briefly defines several new time scales associated with the multivariate information measures recently introduced in Ref. [6] to dissect the information in a single measurement.

The first order  $k_I$  is the length at which the multivariate mutual information  $I[X_0; X_1; \dots; X_{N-1}]$  reaches its asymptotic behavior. Unfortunately, no bounds are known for this order.

The next collection of time scales—denoted  $k_R$ ,  $k_B$ ,  $k_Q$ , and  $k_W$ —are the lengths at which the residual entropy  $r_\mu$ , bound information  $b_\mu$ , enigmatic information, and local exogenous information each reach their respective asymptotes [6]. Furthermore, these four orders are equal, due to the linear interdependence of their respective measures. It turns out that there are lower and upper bounds for these with respect to the Markov order, which can be easily explained. Consider Fig. 8 in Ref. [6]: By definition  $H[X_{:0}]$  can be replaced with  $H[X_{-R:0}]$  and, if the process is stationary,  $H[X_{1:}]$  with  $H[X_{1:R+1}]$ . It is therefore reasonable that one requires at least  $R$  symbols and most  $2R$  symbols to accurately dissect  $H[X_0]$ . In fact, numerical surveys that we have carried out agree



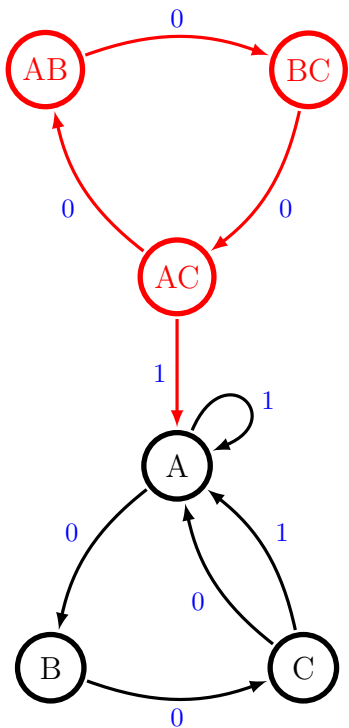


FIG. 12. Cryptic order analysis of the Nemo Process: Its power automaton (Fig. 8) contains the edge  $ABC \xrightarrow{1} A$ . However, upon closer inspection only states  $A$  and  $C$  can transition to  $A$  on a 1. This creates the  $AC$  state. When emitting a 0,  $AC$  becomes  $AB$  and on a second 0 that becomes  $BC$ . A third 0 completes the cycle. The edges indicate legitimate transitions as well: States that actually lead to  $AC$  on a 0 are  $BC$  and those that lead to  $BC$  are  $AB$ , and so on. This leads to a cycle in the cryptic order algorithm’s calculated transient structure. Therefore, one concludes that  $k_\chi = \infty$ .

with these limits.

Finally, a sequel analyzes the elusive information  $\sigma_\mu$ , showing that the Markov order  $R$  equals the length  $k_{\sigma_\mu}$  at which the present measurement block  $X_{0:\ell}$  renders the past and future conditionally independent.

While we have defined these orders and provided bounds, it remains to be seen if there exist efficient methods to calculate them, let alone topological interpretations for each.

### VIII. SURVEY

We illustrate the above results and algorithms, and their usefulness, by empirically answering several simple, but compelling questions about the space of finitary processes. In particular, how typical are infinite Markov order and infinite cryptic order?

Restricting ourselves to topological  $\epsilon$ -machines—those  $\epsilon$ -machines with a distinct set of allowed transitions and

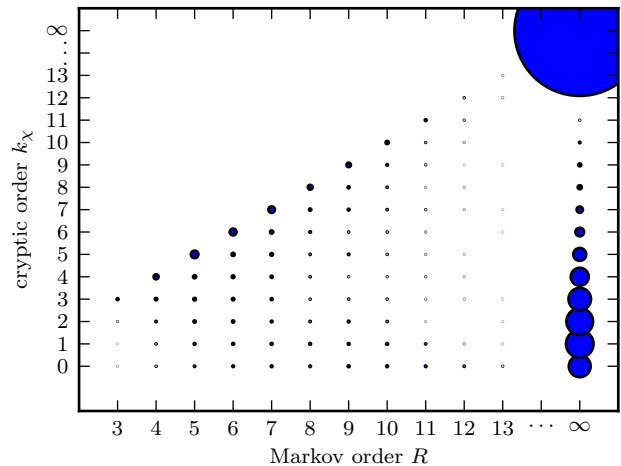


FIG. 13. Distribution of Markov order  $R$  and cryptic order  $k_\chi$  for all 1, 132, 613 six-state, binary-alphabet, exactly-synchronizing  $\epsilon$ -machines. Marker size is proportional to the number of  $\epsilon$ -machines within this class at the same  $(R, k_\chi)$ .

equiprobable transition probabilities—we enumerate all binary-alphabet processes with a given number of states to which one can exactly synchronize. Ref. [18] details their definition, the enumeration algorithm, and how it gives a view of the space of structured stochastic processes. For each of these  $\epsilon$ -machines, we compute its Markov and cryptic orders. The result for all of the 1, 132, 613 six-state  $\epsilon$ -machines is shown in Fig. 13.

The number of  $\epsilon$ -machines that share a  $(R, k_\chi)$  pair is encoded by the size of the circle at that  $(R, k_\chi)$ . The vast majority of processes—in fact, 98%—are non-Markovian at this state-size (6 states). Furthermore, most (85%) of those non-Markovian processes are also  $\infty$ -cryptic. However, this does not imply that synchronization is difficult; quite the contrary: synchronization occurs exponentially quickly [19]. What this does mean is that with growing state size it becomes predominately likely that a given process has particular sequences which will not induce state synchronization.

Also of interest are the “forbidden”  $(R, k_\chi)$  pairs within the space of 6-state topological  $\epsilon$ -machines. For example,  $\epsilon$ -machines with  $k_\chi = 4, 5, 8, 10, 11$  do not occur with  $R = 13$ . Also, processes with infinite Markov order and finite cryptic order appear to have a maximum cryptic order of  $k_\chi = 11$ , despite the fact that larger finite cryptic orders exist for finite Markov-order processes.

### IX. SPIN CHAINS AND BEYOND

Although our primary goal was to precisely define length scales, several being new, and to present efficient calculation methods for them, it will be helpful to briefly

draw out the physical meaning of Markov and cryptic orders by analyzing their role in spin chains and related systems. (A sequel will delve into this topic in greater depth.)

To start, recall that Ref. [20] showed that the Markov order  $R$  of an  $\epsilon$ -machine representing a (one-dimensional) Ising spin system is upper bounded by the interaction range specified in a system's Hamiltonian. Consider first the ferromagnetic, one-dimensional, nearest-neighbor Ising model at different temperatures  $T$ . The  $\epsilon$ -machines for this family of systems are shown in Fig. 14. As just noted, since the system has nearest-neighbor interactions, the Markov order should be  $R = 1$ . This is straightforward to see from the first  $\epsilon$ -machine, which is for a finite temperature  $T$ . Without an observation there are two possible causal states the system could be in:  $\uparrow$  or  $\downarrow$ . Once a single spin has been observed, however, the causal state is known exactly. This changes markedly at the temperature limits, though. At  $T = 0$ , the system is in a ground configuration of either all up spins or all down spins. Without an external field to break this symmetry an observation must be made to determine in which of these states it is and so the Markov order is still  $R = 1$ . In the presence of an external field, however, there is only a single ground state—that aligned with the field—and no observation is required to know in which state the system is. Thus,  $R = 0$ . At  $T = \infty$  the system collapses to a single causal state where the next spin is entirely determined by thermal fluctuations and so the Markov order is  $R = 0$ .

As a second case, consider the antiferromagnetic, one-dimensional, nearest-neighbor Ising model, which is similar enough that it makes for a useful contrast; see Fig. 15. The finite temperature and high temperature limits are identical to the ferromagnetic case, but the low temperature case differs. At  $T = 0$  the spin system forms a perfect crystal of alternating spins and so one must make a single observation to know in which spatial-phase the crystal is. Then, the entire structure is known exactly. Thus, the Markov order is  $R = 1$ . This situation is not a broken symmetry as in ferromagnetic low-temperature case. Even with a nonzero external field, an observation is still required to know in which causal state the system is.

Overall, now that we can directly determine intrinsic lengths in configurations, we see that the coupling range specified by a Hamiltonian need not be an intrinsic property of realized configurations. The simple extremes above make this easy to understand. At infinite temperature each system configuration is equally likely: the Hamiltonian range has no effect on which configurations are realized. At zero temperature only the ground states are expressed and these need not explore all the

possible configurations allowed by the Hamiltonian. Both of these situations mask the coupling range specified by the Hamiltonian. Due to this, the Markov order  $R$  captures the effective coupling range and need not match that specified by the Hamiltonian.

In Ising spin chains, the cryptic order equals the Markov order. (This is due most directly to the fact that spin blocks are in one-to-one correspondence with the  $\epsilon$ -machine causal states. In addition, one must add the caveat that the  $\epsilon$ -machine be ergodic.) This equality need not be the case, however, even in simple physical systems. We note how restrictive Hamiltonian-specified dynamics are via two (again, 1D) examples of infinite Markov order, but finite cryptic order, that arise from finite specification. In the first class of systems, even though one starts with strictly local interactions—configurations with finite Markov order specified by a Hamiltonian with finite coupling range—a 1D system can anneal to one with effectively infinite-range interactions, as shown in Ref. [21]. (See the  $\epsilon$ -machine in Fig. 2 there.) In this particular case, the annealed state is non-Markovian, exhibiting infinite-range structure and Markov order  $R = \infty$ . Notably, the annealed configurations for this example have finite cryptic order  $k_\chi = 4$ . For the second class of systems we just briefly note that these unusual length-scale properties are not restricted to classical systems. They also arise in quantum systems. See the analyses in Refs. [22] and [23].

Finally, since the results here emphasize properties intrinsic to realized configurations, let's turn the question around. Given a single typical instance from the ensemble of allowed configurations, how much can be inferred about the Hamiltonian? Though the topological techniques described above do not provide coupling amplitudes and the like, they do give the maximum range of effective interactions. What does one do, though, without a Hamiltonian or some other system specification? It turns out that a variety of methods exist for inferring hidden Markov models from a sample. And, since any hidden Markov model can be converted to an  $\epsilon$ -machine [16], from there the Markov and cryptic orders can be directly computed. And so, the above methods can be applied to a wide range of theoretically modeled or experimentally realized physical systems.

## X. CONCLUSION

We began by defining two different measures of memory in complex systems. The first, the Markov order  $R$ , is the length of time one must observe a system in order to make accurate predictions of its behavior. The second, the cryptic order  $k_\chi$ , quantifies the ability to retrodict

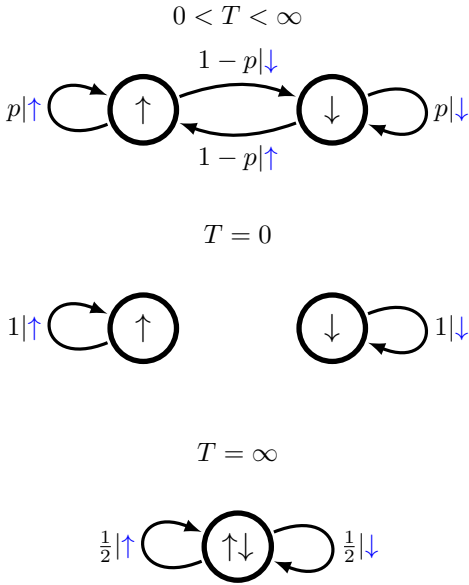


FIG. 14.  $\epsilon$ -Machines for a one-dimensional ferromagnetic Ising model as a function of temperature  $T$ , where  $p = \frac{1}{2}(1 + \tanh \beta)$ , the external field  $B = 0$ , and  $J = k_B = 1$ .

a system's internal dynamics. We showed that despite their statistical nature, these time scales are topological properties—properties of the synchronizing words of a process's  $\epsilon$ -machine.

We demonstrated how to compute these length scales for hidden Markov models, most of which can be motivated in terms of the synchronization properties of the underlying process. Interestingly, we found that one of the most fundamental and important properties—the Markov order  $R$ —is computable using *only* the process's  $\epsilon$ -machine. When calculated with non- $\epsilon$ -machines, the algorithms yield related quantities, such as the synchronization order. For more details, see the Appendices. In addition, the  $\epsilon$ -machine provides an exact method for computing the cryptic order. From these results, we constructed very efficient algorithms for their calculation.

In the empirical setting, we now see that one should first infer the  $\epsilon$ -machine and then, from it, calculate the Markov and cryptic orders. There are a number of methods of inferring an  $\epsilon$ -machine from data; e.g., Ref. [24, and citations therein]. In the theoretical setting, given some formal description of a process—such as a Hamiltonian or general hidden Markov model—one can analytically calculate a process's  $\epsilon$ -machine. In any case, as soon as one has the  $\epsilon$ -machine the preceding gives exact results.

To appreciate what is typical about these length scales, we surveyed the range of Markov and cryptic orders in

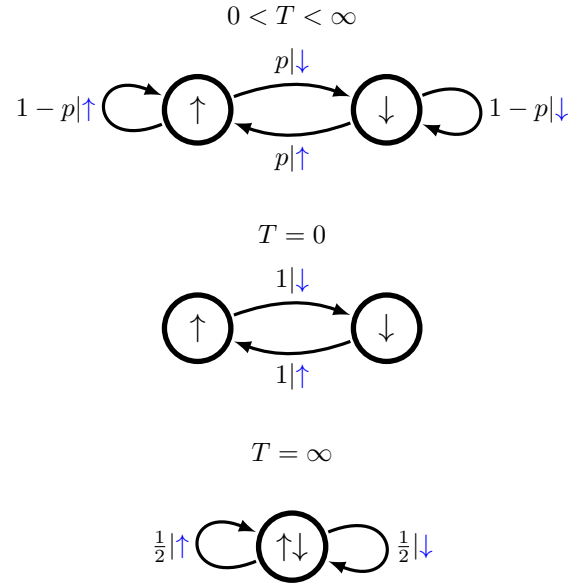


FIG. 15.  $\epsilon$ -Machines for a one-dimensional antiferromagnetic Ising model as a function of temperature  $T$ . Here,  $p = \frac{1}{2}(1 + \tanh \beta)$ , the external field  $B = 0$ , and  $J = k_B = 1$ .

the space of all structured binary processes represented by  $\epsilon$ -machines with six states. The main result was rather surprising, infinite Markov and cryptic orders dominate. Thus, the topological analysis leads one to conclude that synchronization, even to finite-state stochastic processes, is generically difficult. However, from a probabilistic view it is exponentially fast [19, 25]. A way to resolve this seeming contradiction is to conjecture that the topological properties are driven by sequences whose relative proportion vanishes with increasing length. The survey also revealed a variety of interesting ancillary properties that pose a number of open questions, presumably combinatoric and group-theoretic in nature.

We closed analyzing the role these scales play in classical (and briefly quantum) spin systems, drawing out the physical interpretations. We emphasized, in particular, the difference between the interaction range specified by a Hamiltonian and the effective range of correlation in realized spin configurations. This led us to propose calculating the orders to put constraints on spin systems whose Hamiltonians are unknown.

Finally, appendices prove key claims above, discuss other related measures of synchronization, survey the synchronization time and synchronization entropy, and provide step-by-step details for each algorithm.

## ACKNOWLEDGMENTS

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Agency (DARPA) Physical Intelligence project. The views, opinions, and findings contained here are those of the authors and should not be interpreted as representing the official views or policies, either expressed or implied, of ARO, DARPA, or the Department of Defense.

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## Appendix A: Definitions

Here, we provide additional results on length scales and synchronization and prove a number of claims made in the main text. First, we lay out the definitions needed and then give several key results that follow. Building on these, we delineate the central algorithms and conclude with a comparison of synchronization time and synchronization entropy, notions perhaps more familiar than other measures used and their length scales.

## 1. Minimal Synchronizing Words

For the synchronization problem, we consider an observer who begins with a correct model (a presentation) of a process. The observer, however, has no knowledge of the process's internal state. The challenge is to analyze how an observer's knowledge of the internal state changes as more and more measurements are observed.

At first glance, one might say that the observer's knowledge should never decrease with additional measurements, corresponding to a never-increasing state uncertainty, but this is generically not true. In fact, it is possible for the observer's knowledge (measured in bits) to oscillate with each new measurement. The crux of the issue is that additional measurements are being used to inquire about the *current* state rather than the state at some fixed moment in time.

It is helpful to identify the set of words that take the observer from the condition of total ignorance to exactly knowing a process's state. First, we introduce what we mean by synchronization in terms of lack of state uncertainty. Second, we define the set of minimal synchronizing words.

**Definition 1.** A word  $w$  of length  $L$  is synchronizing if the Shannon entropy over the internal state, conditioned on  $w$ , is zero:

$$\text{Sync}(w) \Leftrightarrow H[\mathcal{S}_\ell | X_{0:\ell} = w] = 0, \quad (\text{A1})$$

where  $\text{Sync}(w)$  is Boolean function.

**Definition 2.** A presentation's set of minimal synchronizing words is the set of synchronizing words that have no synchronizing prefix:

$$\mathcal{L}_{\text{sync}} \equiv \{w \mid \text{Sync}(w) \text{ and } \neg \text{Sync}(u) \text{ for all } u : w = uv\}.$$

**Remark.**  $\mathcal{L}_{\text{sync}}$  is a prefix-free, regular language. If each word is associated with its probability of being observed, we obtain a prefix-free code encoding each path to synchrony—a word in  $\mathcal{L}_{\text{sync}}$ —with the associated probability of synchronizing via that path. These codes are generally nonoptimal in the familiar information-theoretic sense.

## 2. Synchronization Order

According to Sec. A1, one is synchronized to a process's presentation after seeing word  $w$  if there is complete certainty in the state. We now expand this view slightly to ask about synchronization over all words of a

particular length. Equivalently, we examine synchronization to an ensemble of process realizations.

**Definition 3.** The synchronization order  $k_{\mathcal{S}}$  [5] is the minimum length for which every allowed word is a synchronizing word:

$$k_{\mathcal{S}} \equiv \min\{\ell \mid H[\mathcal{S}_\ell | X_{0:\ell}] = 0\}. \quad (\text{A2})$$

As for the Markov and cryptic orders,  $k_{\mathcal{S}}$  is considered  $\infty$  when the condition does not hold for any finite  $\ell$ .

## Appendix B: Results

We now provide several results related to these length scales that shed light on their nature, introducing connections and simplifications that make their computation tractable.

**Proposition 1.** The synchronization order is:

$$k_{\mathcal{S}} = \max\{R, k_{\chi}\}. \quad (\text{B1})$$

**Proof.** First, note that:

$$H[\mathcal{S}_\ell | X_{0:\ell}] = H[X_{0:\ell}, \mathcal{S}_\ell] - H[X_{0:\ell}]. \quad (\text{B2})$$

Since the block-state entropy upper bounds the block entropy, the conditional entropy above can only reach its asymptotic value once both terms have individually reached their asymptotic behavior. The latter are controlled by  $k_{\chi}$  and  $R$ , respectively.  $\square$

This result reduces the apparent diversity of length scales, eventually allowing one to calculate the Markov order via the synchronization order, which itself is directly computable.

**Proposition 2.** For  $\epsilon$ -machines:

$$R = k_{\mathcal{S}}. \quad (\text{B3})$$

**Proof.** Applying the causal equivalence relation  $\sim_{\epsilon}$  to Def. 1 we find:

$$\Pr(X_{0:} | X_{:0}) = \Pr(X_{0:} | X_{-\ell:0}) \implies X_{:0} \sim_{\epsilon} X_{-R:0}. \quad (\text{B4})$$

This further implies that the causal states  $\mathcal{S}$  are completely determined by  $X_{-R:0}$ :

$$H[\mathcal{S}_0 | X_{-R:0}] = 0. \quad (\text{B5})$$

This statement is equivalent to the Markov criterion.  $\square$

**Remark.** This provides an alternate proof that the cryptic order  $k_\chi$  is bounded above by the Markov order  $R$  in an  $\epsilon$ -machine via a simple shift in indices:

$$H[\mathcal{S}_0|X_{-R:0}] = 0 \quad (\text{B6})$$

$$\implies H[\mathcal{S}_R|X_{0:R}] = 0 \quad (\text{B7})$$

$$\implies H[\mathcal{S}_R|X_{0:}] = 0 . \quad (\text{B8})$$

This proposition gives indirect access to the Markov order via a particular presentation—the  $\epsilon$ -machine. Since the Markov order is not defined as a property of a presentation it would generally be unobtainable, but due to unique properties of the  $\epsilon$ -machine, it can be accessed through the synchronization order.

There is a subclass of  $\epsilon$ -machines to which one synchronizes in finite time; these are the *exact*  $\epsilon$ -machines of Ref. [7].

**Proposition 3.** *Given an exact  $\epsilon$ -machine with finite Markov order  $R$ , the subshift of finite type that underlies it has a “step” [11] equal to  $R$ .*

**Corollary 1.** *Given an exactly synchronizing  $\epsilon$ -machine, the underlying sofic system is a subshift of finite type if and only if  $R$  is finite.*

**Remark.** *A process with infinite Markov order can have a presentation whose underlying sofic system is a subshift of finite type.*

These results draw out a connection with length scales of sofic systems from symbolic dynamics [11]. Subshifts of finite type have a probability-agnostic length scale analog of the Markov order known as the “step”. In the case of  $\epsilon$ -machine presentations, they are in fact equal.

We will now prove that two of the lengths defined—the cryptic and synchronization orders—are topological. That is, they are properties of the presentation’s graph topology and are independent of transition probabilities, so long as changes to the probabilities do not remove transitions and do not cause states to merge. Additionally, due to Prop. 2, the Markov order is topological. All three are topological since they depend only on the length at which a conditional entropy vanishes, not on how it vanishes.

**Theorem 1.** *Synchronization order  $k_S$  is a topological property of a presentation.*

**Proof.** *Beginning from Def. 3, there is length  $\ell = k_S$  at which:*

$$H[\mathcal{S}_\ell|X_{0:\ell}] = \sum_{w \in \mathcal{A}^\ell} \Pr(w) H[\mathcal{S}_\ell|X_{0:\ell} = w] = 0 .$$

Thus,  $H[\mathcal{S}_\ell|X_{0:\ell} = w] = 0$  for all  $w \in \mathcal{A}^\ell$ , the set of length- $\ell$  words with positive probability. Since every word of length  $\ell$  is synchronizing,  $\ell$  is certainly greater than the synchronization order. As synchronizing words are synchronizing regardless of their probability of occurring, the synchronization order  $k_S$  is topological.  $\square$

**Corollary 2.** *Markov order  $R$  is a topological property of an  $\epsilon$ -machine.*

**Proof.** *Since  $k_S$  is a topological property by Thm. 1 and since an  $\epsilon$ -machine’s  $R = k_S$  by Prop. 2, the Markov order is topological.*  $\square$

**Theorem 2.** *Cryptic order  $k_\chi$  is a topological property of a presentation.*

**Proof.** *Beginning from Def. 3, there is a length  $\ell = k_\chi$  at which:*

$$\begin{aligned} 0 &= H[\mathcal{S}_\ell|X_{0:}] \\ &\stackrel{(1)}{=} \sum_{x_0 \in \mathcal{A}^\infty} \Pr(x_0) H[\mathcal{S}_\ell|X_{0:} = x_0:] \\ &\stackrel{(2)}{=} \sum_{w \in \mathcal{L}_{\text{sync}}} \Pr(w, \sigma_w) H[\mathcal{S}_\ell|X_{0:|w|} = w, \mathcal{S}_{|w|} = \sigma_w] . \end{aligned}$$

Here, step (1) simply expands the conditional entropy. Step (2) is true provided that the sum is over minimal synchronizing words and  $\sigma_w$  is the state to which one synchronizes via  $w$ . This final sum is zero only if the sum vanishes term-by-term. Thus, given a word that synchronizes and the state to which it synchronizes, each term provides a cryptic-order candidate—the number of states that could not be retrodicted from that state and word. Finally, the longest such cryptic order candidate is the cryptic order for the presentation.  $\square$

Restated, the cryptic order  $k_\chi$  is topological as it depends only on the minimal synchronizing words, which are topological by definition.

## Appendix C: Algorithms

We are now ready to turn to computing the various synchronization length scales given a presentation. While all of the algorithms to follow have compute times that are exponential in the number of machine states, we find them to be very efficient in practice. This is particularly the case when compared to naive algorithms to compute these properties. For example, computing synchronization, Markov, or cryptic orders by testing successively longer blocks of symbols is exponential in the length of the longest block tested. Worse, in the case of non-Markovian and  $\infty$ -cryptic processes the naive algorithm

will not halt. In addition, the naive implementation of Thm. 2 given in the proof to compute the cryptic order has a compute time of  $O(2^{2^N})$ , whereas the one presented below is a simple exponential of  $N$ .

Unsurprisingly, given the results provided in Sec. B, we begin with the minimal synchronizing words as they are the underpinnings of the synchronization and cryptic orders. The algorithms make use of standard procedures. Most textbooks on algorithms provide the necessary background; see, for example, Ref. [17].

### 1. Minimal Synchronizing Words

We construct a deterministic finite automaton (DFA) that recognizes  $\mathcal{L}_{\text{sync}}$  of a given presentation  $\mathcal{M} = (Q, E)$ , where  $Q$  are the states and  $E$  are the edges. This is done as follows:

#### Algorithm 1.

1. Begin with the recurrent presentation  $\mathcal{M}$ .
2. Construct  $\mathcal{M}$ 's power automaton  $2^{\mathcal{M}}$ , producing a DFA  $\mathcal{T} = 2^{\mathcal{M}}$ .
3. Set the node in  $\mathcal{T}$  that corresponds to all  $\mathcal{M}$ 's states as  $\mathcal{T}$ 's start state.
4. Remove all edges between singleton states of  $\mathcal{T}$ . (These are the edges from  $\mathcal{M}$ .)
5. Set all singleton states of  $\mathcal{T}$  as accepting states.

Now, we enumerate  $\mathcal{L}_{\text{sync}}$  via an ordered breadth-first traversal of  $\mathcal{T}$ , outputting each accepted word.

### 2. Synchronization Order

Thanks to Eq. (1) we see that  $k_{\mathcal{S}}$  is the shortest length  $\ell$  that encompasses all of  $\mathcal{L}_{\text{sync}}$ . This is, trivially, the longest word in  $\mathcal{L}_{\text{sync}}$ . With this, computing the synchronization order reduces to:

#### Algorithm 2.

1. If  $\mathcal{L}_{\text{sync}}$  is infinite, return  $\infty$ .
2. Enumerate each word in  $\mathcal{L}_{\text{sync}}$  and return the length of the longest word.

The test in the first step can be done simply by running a loop-detection algorithm on DFA  $\mathcal{T}$ . If there is a loop, then  $\mathcal{L}_{\text{sync}}$  is infinite.

### 3. Markov Order

Due to Thm. 2, a process's Markov order can be computed by finding the synchronization order of the process's  $\epsilon$ -machine. If one does not have the  $\epsilon$ -machine for a process, but rather some other unifilar presentation, it is still possible in some cases to obtain the Markov order through the synchronization order. That is, the algorithms for  $k_{\mathcal{S}}$  and  $k_{\chi}$  provide probes into the presentation's length scales. It can be the case that  $R$  is accessible to those probes, if  $k_{\chi} < k_{\mathcal{S}}$ , but it is only guaranteed to be accessible in the case of  $\epsilon$ -machines. Note, there exist techniques for constructing the  $\epsilon$ -machine from any presentation [16].

### 4. Cryptic Order

In the following algorithm  $\mathcal{T}$  refers to the power automaton of the machine  $\mathcal{M}$ .  $\mathcal{T}$ 's states— $p$ ,  $q$ , and  $r$ —are elements of the power set of the states of  $\mathcal{M}$ . By the *predecessors* of a state  $q$  along edge  $\xrightarrow{p} xq$  we refer to the set  $p' = \{m | (\xrightarrow{m} xn) \in \mathcal{M} \text{ and } m \in p \text{ and } n \in q\}$ . These are the states  $m \in p$  that actually transition to a state  $n \in q$  on symbol  $x$ . By *subset construction* below we refer to the standard NFA-to-DFA conversion algorithm [9].

#### Algorithm 3.

1. Construct the power automaton  $\mathcal{T} = 2^{\mathcal{M}}$  via subset construction.
2. Push each edge  $\xrightarrow{p} xq$  in  $\mathcal{T}$  to a queue.
3. While queue is not empty:
  - (a) Pop edge  $\xrightarrow{p} xq$  in the queue.
  - (b) If edge is in processed list:
    - i. Restart loop, popping the next edge from the queue.
  - (c) Find the predecessors  $p'$  of  $q$  along  $\xrightarrow{p} xq$ .
  - (d) If  $p' \neq p$ :
    - i. Remove edge  $\xrightarrow{p} xq$  from  $\mathcal{T}$ .
  - (e) If  $|p'| > 1$ :
    - i. Perform subset construction on  $p'$  (implicitly, this adds the edge  $\xrightarrow{p'} xq$  to  $\mathcal{T}$ ).
    - ii. Push each edge created in the prior step into the queue.
    - iii. For each  $\xrightarrow{r} yp$  in  $\mathcal{T}$ :
      - A. Add edge  $\xrightarrow{r} yp'$  to  $\mathcal{T}$ .
      - B. Add edge  $\xrightarrow{r} yp'$  to the queue.
    - iv. Add  $\xrightarrow{p} xq$  and  $\xrightarrow{p'} xq$  to the processed list.

The result is an automaton  $\mathcal{T}'$ . The longest path in  $\mathcal{T}'$  through transient states ending in a recurrent state is the cryptic order. Skipping previously processed edges is important since for some topologies the algorithm can enter a cycle where it will remove and then later add the same edge, ad infinitum.

There are three simple additions to this algorithm that result in a sizable decrease in running time. The first is to store the edges to be processed in a priority queue, such that an edge  $\overset{p}{\rightarrow} xq$  is popped before an edge  $\overset{r}{\rightarrow} ys$  if  $|q| < |s|$ , or if  $|q| = |s|$ , then  $|p| < |r|$ . The second optimization is to trim dangling states after each pass through the outer loop. A dangling state is a state  $p$  such that there is no path from  $p$  to the recurrent states. The last method for improving speed is to not add edges between recurrent states to the queue in step 2.

This algorithm for computing the cryptic order only holds for unifilar presentations.

## Appendix D: Statistical Measures of Synchronization

### 1. The Synchronization Distribution

Taking a slightly more general view than the synchronization order, we consider statistical properties of synchronization, rather than just the absolute length at which an ensemble will all be synchronized. In this vein, we define a distribution that gives the probability for a word to first synchronize at length  $\ell$ .

**Definition 4.** *The synchronization distribution  $S$  gives the probability of synchronizing to a presentation at length  $\ell$ :*

$$S(\ell) \equiv \sum_{w \in \mathcal{L}_{\text{sync}}} \Pr(w) \delta(|w| - \ell) . \quad (\text{D1})$$

where  $\delta$  is the Kronecker delta function.

**Remark.**  $S$  is normalized:  $\sum_{\ell=0}^{\infty} S(\ell) = 1$ .

We now draw out two particular quantities from this distribution—quantities that have observable meaning for a presentation.

**Definition 5.** *The synchronization time  $\tau$  [5] is the average number of observations needed to synchronize to a presentation:*

$$\tau \equiv \sum_{w \in \mathcal{L}_{\text{sync}}} |w| \Pr(w) \quad (\text{D2})$$

$$= \mathbb{E}_{\ell}[S(\ell)] , \quad (\text{D3})$$

where the second equality shows that  $\tau$  is also equal to the expectation value of the synchronization distribution.

The synchronization time is useful for understanding how long it takes *on average* to synchronize to a model. This is in contrast to the Markov order which is the minimal longest-synchronization-time across all presentations of a process.

**Definition 6.** *The synchronization entropy  $H_{\text{sync}}$  is the uncertainty in the synchronization distribution:*

$$H_{\text{sync}} \equiv H[S(\ell)] . \quad (\text{D4})$$

**Remark.** *Note that this is quite distinct from the synchronization information  $\mathbf{S}$  of Ref. [14]:*

$$\mathbf{S} = \sum_{\ell=1}^{\infty} H[S_{\ell}|X_{0:\ell}] .$$

The synchronization entropy, in contrast, measures the flatness of the synchronization distribution. And, since the synchronization distribution decays exponentially with length, the fatter the tail, the higher the uncertainty in synchronization.

### 2. The Synchronization Distribution

There are two methods to compute the synchronization distribution. The first requires an  $\epsilon$ -machine with finite recurrent and transient components. The second, only a finite recurrent component. We present the former case first.

**Algorithm 4.**

1. Perform an ordered breadth-first traversal of the finite  $\epsilon$ -machine.
2. While traversing, keep track of the word induced by the path and the product of the probabilities along that path.
3. When a recurrent node is reached, stop that particular thread of the traversal.
4. Sum the probabilities of all words with the same length.

This algorithm produces each minimal synchronizing word and its probability in lexicographic order. Then words of each length can be grouped and their probabilities summed to get the synchronization distribution.

The second algorithm is used when an  $\epsilon$ -machine with finite transient structure is not available:

**Algorithm 5.**



1. Produce all the minimal synchronizing words from the DFA given in Algorithm 1.

2. For each minimal synchronizing word, compute its probability using the recurrent  $\epsilon$ -machine [14].

3. Sum the probabilities of all words with the same length.

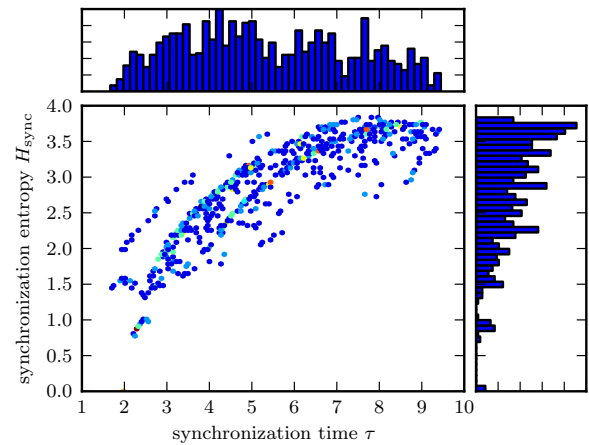


FIG. 16. Distribution of synchronization time  $\tau$  and synchronization entropy  $H_{\text{sync}}$  for all 1,388 four-state, binary-alphabet, exactly-synchronizing  $\epsilon$ -machines with uniform outgoing transition probabilities. Individual histograms for each property are shown above and to the right.

Once the distribution is computed using one of the above algorithms, it is trivial to compute the Shannon entropy and mean of the distribution to get the synchronization entropy and synchronization time, respectively.

### 3. Results

Finally, we survey the distribution of synchronization times  $\tau$  and synchronization entropies  $H_{\text{sync}}$  for all 1,388 four-state, binary-alphabet, exact  $\epsilon$ -machines with uniform outgoing transition probabilities [18]. See Fig. 16. It is interesting to note that there is structure in the distribution in the form of veils. However, the veils are not the entirety of the distribution, there are many machines that fall elsewhere.