Complexity:
Order contra Chaos

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Abstract

A concise commentary on observing and modeling complexity within the framework of dynamical systems, information, and computation theories. Deterministic chaos, randomness, order, predictability, uncertainty, and complexity are contrasted. The descriptive abstraction of state is seen to be a natural extension of the symbolization necessary for language and of the number concept. Philosophical consequences of deterministic chaos are noted.

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Chaos and Complexity

Chaos is the deterministic production of behaviour that is unpredictable over long times. Although there are a number of ways to express its defining properties, a simple example will serve to introduce the key considerations in deterministic chaos: the breakdown of predictability, observation of a complex process, and the mathematical effort required to forecast. These have their analogues in the dynamical systems theory of chaos, information theory of measurement, and computation theory of modeling.

The weather is often considered a prime example of unpredictable behaviour. In fact, it is quite predictable. Over the period of one minute (say), one can surely predict it. With a glance out the nearest window to note the sky's disposition, one can immediately report back a forecast. To predict over one hour, one would search to the horizon, noting more of the sky's prevailing condition. Only then, and not without pause to consider how that might change during the hour, would one offer a tentative prediction. If asked to forecast two weeks in advance one would probably not even attempt the task since the necessary amount of information and the time to assimilate it would be overwhelming. Despite the long term unpredictability, a meteorologist can write down the equations of motion for the forces controlling the weather dynamics in each case. In this sense, the weather's behaviour is symbolically specified in its entirety. How does unpredictability arise in such a situation?

The short answer is that the governing natural laws, even though expressible in a compact symbolic form, can implicitly prescribe arbitrarily complicated behaviour. To the extent that the natural laws are objectively understood, they are written as *equations of motion*. These are a procedure that, given a sufficient measurement of a system's configuration, specifies how to compute future behaviour. Often articulated in the language of differential calculus, the equations of motion codify the interplay of the components of a system's configuration. They are, in fact, incremental rules, i.e. an *algorithm*, that determine the configuration at the next moment in terms of the one immediately preceding. Forecasting, though, requires knowing the behaviour for any future time.

The belief that this could be done and the assumption that it was easy to do so was most succinctly expressed by Laplace more that two centuries ago:

> The present state of the system of nature is evidently a consequence of what it was in the preceding moment, and if we conceive of an intelligence which at a given instant comprehends all the relations of the entities of the universe, it could state the respective positions, motions, and general affects of all these entities at any time in the past or future.

While Newton's and Leibniz's invention of the differential calculus gave a new language with which to model natural phenomena, its direct implementation as a procedural description has only recently become feasible. Before this time, when sequential, compounded computation could only be performed by hand, even the simplest prediction problems demanded arduous and typically impractical effort. Thus, mathematical techniques were developed to *invert* the equations of motion. In the limited settings for which this could be carried out, viz. *linear* equations, the analytic methods yielded *closed-form solutions* which short-cut the direct incremental computation of future behaviour. The main characteristic of linear equations is that given two solutions a third may be found as their sum.

A vast array of phenomena do not share this property. Despite this limitation, closed-form solution has been the dominant criterion for understanding dynamical behaviour since the time of Newton. Its range of applicability has ceased increasing. The types of phenomena now demanding scientific attention, such as the weather and even substantially smaller systems, are explicitly *non-linear* and do not, even in principle, allow for closed-form solution. That there was a fundamental limit to finding closed-form solutions was appreciated by Poincaré at the turn of this century. Although he despaired of this, he was also the initiator of the alternative approach to describing complex behaviour, *qualitative dynamics*, which later became *dynamical systems theory*.

Dynamical Systems Theory

A central abstraction in dynamical systems theory is that the instantaneous configuration of a process is represented as a point, or *state*, in a space of states. The *dimension* of the state space is the number of numbers required to specify uniquely the system's configuration at each instant. With this, the temporal evolution of the process becomes the motion from state to state along an *orbit* or *trajectory* in the state space.

For a simple clock pendulum the state space is the two-dimensional plane. A state here consists of two numbers: one denoting the position, the other the velocity. The state space of a fluid in a closed box is the collection of all velocity fields: the space of all possible instantaneous changes in fluid particle positions. If every particle moves independently, the dimension of the equivalent dynamical system is exceedingly large: proportional to the number of particles. Despite the difficulty in picturing this representation directly, the temporal evolution of the fluid is abstractly associated with a trajectory in this high-dimensional state space. In the fortunate case when there is strong coherence between components of a large system or when the system itself has only a few significant components, the trajectory can be visualized in a much lower dimensional space.

If a temporal sequence of configurations is observed to be stable under perturbations and is approximately recurrent, then the trajectory is said to lie on an *attractor* in the state space. The attractor concept is a generalization of the classical notion of equilibrium. One of the main contributions of (dissipative) dynamical systems theory is the categorization of all long-term behaviour into three attractor classes. A *fixed point* attractor is a single, isolated state toward which all neighbouring states evolve. A *limit cycle* is a sequence of states that are repetitively visited. These attractors describe predictable behaviour: two orbits starting from nearby states on such an attractor stay close as they evolve. Unpredictable behaviour, for which the latter property is not true, is described by *chaotic attractors*. In a crude approximation, these are often defined negatively as attractors that are neither fixed points, limit cycles, nor products of limit cycles.

There are several complementary descriptions of the basic properties of chaotic attractors. Analytically, they consist of highly convoluted orbits. An infinite number of unstable limit cycles and an infinite number of aperiodic orbits can be embedded in a chaotic attractor. Topologically, chaotic attractors often display self-similar, or *fractal*, structure. Geometrically, although globally stable to perturbations off the attractor, they exhibit average local instability. Orbits starting at close initial states on a chaotic attractor separate exponentially fast. Physically, this local instability amplifies microscopic fluctuations to affect macroscopic scales. Although the resulting macroscopic behaviour may be predictable over sufficiently short times, to an observer it is unpredictable over long times. Even in the absence microscopic fluctuations, forecasting typical chaotic orbits requires maximal computational effort on the part of an observer who knows the governing equations of motion. The size of the minimal computer program to predict grows with the length of the forecast.

Aside from attractor classification, another significant contribution of dynamical systems theory is a geometric picture of *transients*: how states off an attractor relax onto it. An attractor's *basin of attraction* is the set of all initial states that evolve onto it. There can be multiple basins, so that radically different behaviour may be seen depending on the initial configuration. The complete catalogue of attractors and their basins for a given dynamical system is called its *attractor-basin portrait*.

Dynamical systems theory is also the study of how attractors and basin structures change with the variation of external control parameters. A *bifurcation* occurs if, with the smooth variation of a control, the attractor-basin portrait changes qualitatively.

Sources of Randomness

To summarize, dynamical systems theory has identified three sources of unpredictability or effective randomness.

1. *Sensitive dependence on initial condition*: To which attractor does the system go? The borders between basins can be highly convoluted, so that completely different attractors can be seen with very small changes in initial condition.
2. *Deterministic chaos*: This is unpredictability of long-term behaviour due to local instability on the attractor.

3. *Sensitive dependence on control parameter*: The attractor-basin portrait can be arbitrarily sensitive to changes in control parameters.

Poincaré expressed an appreciation that such sensitivities could arise in systems governed by known laws as follows

> But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation *approximately*. If that enabled us to predict the succeeding situation with the *same approximation*, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon.

The remark closes with an implicit operational definition of randomness as a phenomenon which appears fortuitous due to ignorance. This and similar notions of uncertainty play an important role in probabilistic descriptions of unpredictable behaviour.

In chaotic systems uncertainty and approximation are rapidly amplified. This precludes not only the long-term prediction of their behaviour, but also the closed-form solution of their equations of motion. Reminiscent of quantum theory, the first difficulty necessitates, even in the classical setting of dynamical systems, a complete accounting of the measurement process. The second requires a computational theory of inferring models from measurements.

Information and Measurement

An observation of a natural process entails measurement of its state. The act of measurement is a codification of the physical configuration. But how much do observations tell one about the process? Information theory measures the amount of information in an observation as the negative logarithm of its probability. Information itself is never rigorously defined; it is only quantified. The most concise attempt, however, is due to Bateson: information is a difference that makes a difference. This expresses the origin of information in the unanticipation of an event and also its essential relativity.

The average information contained in isolated measurements is called the *dimension* of the underlying process: the minimum amount of information necessary to uniquely identify a configuration. In a complementary way the *dynamical entropy* quantifies how much can be predicted about the next measurement given that one knows the entire history up to that point. It measures the average temporal rate of information loss once a measurement is made. If a process is chaotic a new measurement must be made after a short time since the information about its previous state is lost. From the observer's viewpoint, the dynamical entropy is the rate at which a process produces new information.

Information theory does not give a direct indication of a process's underlying geometric structure, since it is a probabilistic description of the behaviour. The geometry of the underlying attractor can be recovered, however, even from a single component time series produced by a multi-dimensional process. Reconstruction methods produce an equivalent state space representation from a time series of observations. They provide a direct connection between experimental data and the geometric tools of dynamical systems theory.

Complexity and Modeling

The minimal computation required to forecast and to model observed behaviour are two measures of its complexity. They are especially important for deterministic chaos. Since there are no closed-form solutions for chaotic orbits, there are no algorithmic short-cuts enabling one to avoid direct incremental computation of future states from the equations of motion. Laplace parenthetically acknowledged the importance of computation complexity for exact prediction:

> But ignorance of the different causes involved in the production of events, as well as their complexity, taken together with the imperfection of analysis, prevents our reaching the same certainty about the vast majority of phenomena.

The complexity associated with forecasting dynamical systems, introduced by Chaitin and Kolmogorov as a computational measure of randomness, is equivalent to the dynamical entropy. A repetitive process is easy to predict, since there are only a few measurement sequences to anticipate. An ideal random process is difficult to predict due to the diversity of sequences. The repetitive process produces little or no information; the random process produces a maximal amount.

The complexity of modeling, however, is complementary to such “randomness” measures. The model of the repetitive process is simple: listing the basic pattern again and again is all that is required. The random process is also quite simple, but from a statistical viewpoint. For a random process, one’s model is simply to guess at successive measurements. Both repetitive and random processes have low modeling complexity. A complex process is an amalgam of both deterministic and random computations.

Modeling complexity is maximized in processes that are at the border between order and chaos. This is a concise summary of the information processing capabilities of dynamical systems. It is particularly germane to processes at phase transitions, such as the transition between ice (order) and water (chaos). For adaptive and evolving systems, such as found in biology, the notion of modeling complexity captures the necessary interplay between innovation and utility of function. Innovation allows an organism to adapt to a changing environment. Ordered behaviour and structure are necessary as a foundation for further evolution and in order to take advantage of regularity.

Methodology

Deterministic chaos has found its particular niche in the taxonomy of complex dynamical behaviour. Indeed, research has advanced to an “engineering” phase in which chaos is designed to control, eliminate, or enhance unpredictability.

Deterministic chaos forced a change in scientific methodology away from the emphasis on closed-form representations for single orbits. One result is that the Baconian notion of inexorable progress in the refinement of scientific theories via experimentation is not strictly valid, since a model prediction will eventually differ from observed behaviour. The error in this prediction can be as large as the attractor itself. The refinement of a "theory" for a single chaotic orbit cannot be improved beyond that irreducible and large error. One response is to use probabilistic descriptions of the apparent random behaviour. This ignores the tremendous structure in deterministic behaviour, such as the short-term predictability and the shape of a chaotic attractor. Qualitative dynamics is a geometric approach intermediate between exact solution and probabilistic methods.

Chaos, though, is only a shadow of forms of complicated behaviour still to be perceived. What will last, then, is not so much the phenomenon of deterministic chaos, but rather the methodology, *experimental mathematics*, that has been developed to explore it. The goal there is to circumvent the analytical and expressive deficiencies of closed-form solutions in order to directly explore the complexity of analytic models. Digital computers have facilitated much of its development by providing access to vast amounts of numerical computation. The basic methodology draws on the geometric representations from dynamical systems theory, the quantitative probabilistic descriptions of information theory, and the structural analysis of complexity developed in computation theory.

Name, Number, State

The primary concept on which dynamical systems theory and its applications rest is the notion of *state*. From a scientific-historical perspective, the very recent use of the state concept is seen as only the most recent example of a series of improved descriptive abstractions. These are modes of symbolic representation that facilitate modeling the perceived world. The first in the series might be taken to be the development of language, or more basically, the naming of objects in the perceptual environment. The second was the number concept which gave a refined precision in differentiating named objects.

In the development of descriptive abstractions, new modes do not replace existing ones, but instead are built out of them in a procedural hierarchy:

number is an ordering property of sets of named objects. Each mode sets the substrate for a level of modeling and so the complexity at that level depends on that of the lower levels. Although number is an essential aspect of the measurement process, the state abstraction builds on it and introduces a *geometrization* of procedure. Through it time and, especially, the evolution of behaviour become objects for description.

Philosophy of Chaos

The discovery of deterministic chaos and the success of dynamical systems theory belie a re-invigoration of mechanism. Unlike the determinant and lifeless mechanism of a century ago, mechanical systems are now seen as sources of effective randomness, surprise, and innovation. At one and the same time subjectivity enters in an essential way into descriptions of complex behaviour.

The detailed structural theory of chaos and the vast array of non-linear systems exhibiting it make it clear that *randomness* is an ideal only approximable by physical processes. It is characteristic of scientific progress that original concepts give way to a refined understanding. Major advancements in scientific knowledge often exact a toll in discovery of new limitations of the explanatory reach of the existing world view. With the development of geometric descriptions of complexity, there comes the appreciation of the fundamental limits on their predictability.

Many problems in dynamical systems theory derive from the essential tension between local determinism and global indeterminism. The equations of motion specify local space-time rules and so determine the evolution from an initial state entirely. Nonetheless, the long-term and large-scale structures responsible for the observed properties cannot be directly inferred from them. When observed with any finite accuracy, chaotic processes exhibit a preferred direction of time, even though the microscopic equations do not. Via local instability microscopic determinism leads to macroscopic irreversibility.

The behaviour of non-linear systems cannot be understood solely in terms of their constituents' behaviour. Indeed, it is the very interaction of the constituents that produces complex behaviour. The reductionist methodology fails entirely to capture the structures that arise specifically due to interaction.

From the mechanistic viewpoint, the physical brain is the substrate supporting the mind. There is now the stronger dynamic interpretation: the mind is the dynamics of the brain. Chaos sheds no light on the literal interpretation of the problem of a mind expressing free will. The classical universe described by dynamical systems theory is deterministic. The existence of deterministic chaos does, however, expand the discussion of free will to include the notion of stability of the physical substrate. A system can appear to have *effective free will* in the sense that no one, especially not the system itself, can decide whether its macroscopic behaviour is completely determined, let alone fully predict it. If deterministic chaos were found via experimental investigation to be an essential and common behavioural mode of the physical brain, then one could reasonably conclude that individuals express effective free will.

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