Postscript to *The Attractor-Basin Portrait of a Cellular Automaton*

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Abstract

The definition of regular attractor, originally presented in *The Attractor-Basin Portrait of a Cellular Automaton*,[1] is reexamined and improved.

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Definitions of Regular Attractor‡

The original definition of regular attractor appeared in ref. [1], and is reproduced as definition (1) below. Since that time, further reflection has shown the desirability of improvements on the original. One proposed modification follows as definition (2).

It bears repeating that the notion of attractor defined here is of an attracting *language*—i.e., a pattern. The intuitive notion of a regular attractor is of a regular language A (such as, for example, a domain) that is stable under small perturbations—i.e., states in some vicinity $V(A)$ eventually fall back into A.

First we quote the original definition, as follows.

Definition (1). An invariant language A is a **fixed-point regular attractor** if it is a regular language and there exists a regular vicinity $V(A)$ such that

i. All states in V remain in V forever: $\mathbf{\Phi}^t(V) \subseteq V,\ \forall\ t \geq 0$; and
ii. All states in V, except possibly for a set R of zero size, i.e., $S(R|V) = 0$, eventually fall back into A. Thus for almost all $\mathbf{s} \in V$, there exists a $t \geq 0$ such that $\Phi^t(\mathbf{s}) \in A$.

On further examination, it became evident that definition (1) used an unnecessarily restrictive notion of stability. For example, although the Hamming vicinity $V_H(A)$ provides the most compelling notion of "small" perturbation, it need not satisfy condition (i), even in cases where perturbed states eventually fall back into A. The iterates $\mathbf{\Phi}^t(V_H(A))$ need not remain in $V_H(A)$, but instead may evolve into some larger vicinity that might or might not be regular. This is not unlike the situation in continuous-state dynamical systems, in which a small perturbation of magnitude δ away from a fixed point $\mathbf{x}_0$ remains within some larger region of size $\varepsilon \geq \delta$ about $\mathbf{x}_0$.[3]

We therefore relax condition (i) slightly to arrive at an alternative definition:

Definition (2). An invariant language A is a **fixed-point regular attractor** if it is a regular language and there exist regular vicinities $V_1(A)$ and $V_2(A)$ such that

i. $V_1(A) \subseteq V_2(A)$;
ii. $V_2(A)$ is a trapping region: $\mathbf{\Phi}(V_2) \subseteq V_2$; and
iii. All states in $V_1(A)$, except possibly for a set R such that $S(R|V_1) = 0$, eventually collapse down onto A. Thus for almost all $\mathbf{s} \in V_1$, there exists a $t \geq 0$ such that $\Phi^t(\mathbf{s}) \in A$.

Conditions (i) and (ii) imply that iterates of $V_1(A)$ remain in $V_2(A)$ for all time: $\mathbf{\Phi}^t(V_1) \subseteq V_2,\ \forall\, t \geq 0$. This is the direct analog of Lyapunov stability, with V_1 being the set of states

‡The material in this article originally appeared in ref. [2]. See either of refs. [1, 2] for notation and definitions of terms used here.

perturbed away from the invariant set A, and V_2 serving as the larger vicinity containing all the iterates of V_1. Condition (iii) is then the analog of asymptotic stability, with the difference that a set of zero size is allowed to remain outside the attractor.

Discussion

Using definition (1), ref. [1] investigated the properties of elementary CA 18, arguing in particular for the existence of two regular attractors, denoted by Λ^0 and $\Lambda^{0,0}$. Since that time, an error has been found in the analysis. As we will see below, this error is of minor technical significance and does not affect the overall argument.

It was incorrectly reported in ref. [1] that CA configurations in the Hamming vicinity $V_H(\Lambda^0)$ of the attractor Λ^0 remain in $V_H(\Lambda^0)$ for all time. The correct statement is that configurations perturbed into $V_H(\Lambda^0)$ quickly leak out into the larger b_2-subbasin vicinity of Λ^0, $V_{b_2}(\Lambda^0)$. The same problem is found with $\Lambda^{0,0}$.

With the new definition of regular attractor, this oversight is corrected. For the case in which A is the domain language Λ^0, we identify the Hamming vicinity $V_H(\Lambda^0)$ and the subbasin vicinity $V_{b_2}(\Lambda^0)$ with V_1 and V_2 in definition (2). Then it is straightforward to show that requirements (i) and (ii) are satisfied. This situation is exactly analogous for the two-domain case, $\Lambda^{0,0} = A$, for which we take the subbasin vicinity $V_{b_3}(\Lambda^{0,0})$ as V_2 and the Hamming vicinity $V_H(\Lambda^{0,0})$ as V_1. The remaining steps in the demonstration of attraction (i.e., requirement (iii) for $\Lambda^0 = A$ and $\Lambda^{0,0} = A$) are essentially unchanged from those in ref. [1]. Further investigation of the stability of Λ^0 and $\Lambda^{0,0}$ can be found in ref. [4]. In the end, the correction of the error quoted above does not change the attractor-basin portrait of ECA 18.

Even definition (2) must be regarded as imperfect, however. Although it captures the notion of stability to the extent allowed by the discrete CA state space, the use of two nested vicinities is problematic. This is made evident by the realization that the entire extended state space $\mathcal{A}^*$ is a regular vicinity of *any* regular language. Since $\mathbf{\Phi}(\mathcal{A}^*) \subseteq \mathcal{A}^*$ for any CA rule $\mathbf{\Phi}$, conditions (i) and (ii) can always be trivially satisfied by taking $V_2 = \mathcal{A}^*$ regardless of $\mathbf{\Phi}$, A and V_1. The content of definition (2) is then effectively reduced to the single condition (iii): that almost all states in V_1 must fall back into A. In practice, we will choose as small a vicinity as possible for V_2, though again, at the root of this problem is the lack of a good definition of "small".

The definition may prove to be more useful in CA with more than one attractor (for one possible example, see ref. [5]). In such cases we could require that the two vicinities be disjoint, i.e. that two invariant languages A_0 and A_1 are *resolvable* regular attractors if there exist vicinities $V_2(A_0)$ and $V_2(A_1)$ such that $V_2(A_1) \cap V_2(A_2) = \emptyset$. Further development and testing against many more example systems are clearly necessary.

Also conspicuously missing from both definitions (1) and (2) is any requirement that an attractor be *transitive*—i.e., that it cannot be partitioned up into disjoint subsets that are

themselves attractors. Again, when the state space is discrete, it is hard to even formulate a satisfactory notion of transitivity. On a finite lattice with periodic boundaries, all asymptotic states lie on temporally periodic cycles, which (if there is more than one) are disjoint. This might suggest that only the cycles can be considered as attractors, as, in fact, they have often been defined.[6, 7, 8] Yet the number and periodicity of the cycles changes drastically with lattice size; furthermore, there is no notion of *pattern* in that view. All hope of describing the generic behavior of a given CA on an arbitrary lattice is lost, thus abandoning the very essence of the analysis of CA pattern dynamics. We regard this as unsatisfactory.

It was originally supposed that the discreteness of the state space would simplify the analysis considerably. Instead we find that it has made formulating satisfactory definitions of certain standard notions of dynamical systems theory—i.e., infinitesimal perturbation and transitivity—more difficult, especially when coupled with the absence of a preferred spatial origin.

On the other hand, in the CA which have been analyzed (such as ECA 18), this definition of regular attractor correctly identifies the system's asymptotic behavior, and provides a compact and powerful characterization of the system's qualitative dynamics. At this point, we are therefore left with the realization that definition (2), though incomplete, is demonstrably useful—and that it can only become more so if a fully appropriate way of completing it is found.

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