Drazin Inverse per Spectral perSpective

with Applications to Complex Systems

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Joint work with James P. Crutchfield

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Discussion based on:

J. P. Crutchfield, C. J. Ellison and P. M. Riechers.
"Exact complexity: the spectral decomposition of intrinsic computation." arXiv:1309.3792 [cond-mat.stat-mech].

and

P. M. Riechers and J. P. Crutchfield.
"Spectral decomposition of structural complexity: the meromorphic functional calculus of nondiagonable dynamics." (In preparation.)

How to Drazin:

Invert what's invertible;
leave the rest alone.

Many complex stochastic processes of interest can be modeled by a hidden Markov model (HMM).

Any HMM will have:

- some set of states $\boldsymbol{\mathcal{S}}$,
- an alphabet $\mathcal{A}$ of observables,
- a set of output-labeled transition matrices $T^{\mathcal{A}} = \left\{ T^{(x)} : T^{(x)}_{i,j} = \Pr(\mathcal{S}_t = \sigma^j | \mathcal{S}_{t-1} = \sigma^i) \right\}_{x \in \mathcal{A}}$ constituting the row-stochastic state-to-state transition matrix $T = \sum_{x \in \mathcal{A}} T^{(x)}$.

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In principle, can calculate everything about a process directly from HMM, but *different questions require different HMM representations.*

Question about process reduces to function of appropriate transition dynamic.

The Zero Eigenvalue, (di)Graphically

$$\Lambda_A = \left\{ \lambda \in \mathbb{C} : \det(A - \lambda I) = 0 \right\}$$

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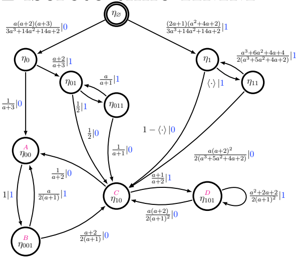
Discrete time HMM



Time evolution: T^n

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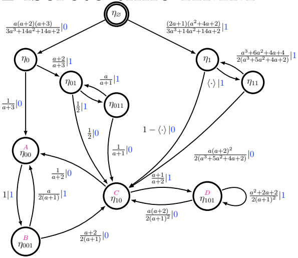
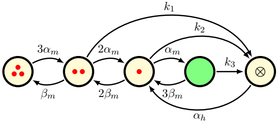


Time evolution: T^n

Continuous time HMM



Time evolution: e^{tG}

An Operator and its Spectrum

Spectrum

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Resolvent

The *resolvent* of A, $\mathcal{R}(z; A) \equiv (zI - A)^{-1}$, where z is a continuous complex variable, thus contains all of the spectral information about A (and more).

A finite square matrix and its eigenvalues

- If an operator A can be represented as a finite square matrix, then its spectrum is just the set of A's *eigenvalues*:

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- Algebraic multiplicity a_λ
- Geometric multiplicity g_λ
- The *index* ν_λ of the eigenvalue λ is the size of the largest Jordan block associated with λ.

Definition

Projection Operator

The *projection operator* of A associated with the eigenvalue λ is:

$$\begin{aligned} A_\lambda &\equiv \frac{1}{2\pi i} \oint_{C_\lambda} \mathcal{R}(z; A)\, dz \\ &= \operatorname{Res}\left[(zI - A)^{-1}, z \to \lambda\right] \end{aligned}$$

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If A is diagonalizable, then the projection operator can be simply expressed as:

$$A_\lambda = \prod_{\zeta \in \Lambda_A \setminus \{\lambda\}} \frac{A - \zeta I}{\lambda - \zeta} .$$

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If $a_\lambda = 1$, then the projection operator can be simply expressed as:

$$A_\lambda = \frac{1}{\langle \boldsymbol{\lambda} | \boldsymbol{\lambda} \rangle} \left| \boldsymbol{\lambda} \right\rangle \left\langle \boldsymbol{\lambda} \right| \ ,$$

where $\langle \boldsymbol{\lambda}|$ is the left eigenvector of A associated with λ and $|\boldsymbol{\lambda}\rangle$ is the right eigenvector of A associated with λ. (Note: $\langle \boldsymbol{\lambda}| \neq |\boldsymbol{\lambda}\rangle^\dagger$!)

Some General Properties of Projection Operators

- $\{A_\lambda\}$ is a mutually orthogonal set:

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- The projection operators are a resolution of the identity:

$$I = \sum_{\lambda \in \Lambda_A} A_\lambda$$

Choosing a functional calculus

Taylor Series

Inspired by
$f(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\xi)}{n!} (a - \xi)^n$.

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$$f(A) = \frac{1}{2\pi i} \oint_{C_{\Lambda_A}} f(z)(zI - A)^{-1} \, dz$$

Extends $f(A)$ beyond Taylor domain.

The Resolvent Resolved: Partial Fraction Decomposition of the Resolvent:

$$
\begin{aligned}
\mathcal{R}(z;A) &= (zI - A)^{-1} \\
&= \frac{\mathcal{C}^\top}{\det(zI - A)} \\
&= \frac{\mathcal{C}^\top}{\prod_{\lambda \in \Lambda_A} (z-\lambda)^{a_\lambda}} \\
&= \sum_{\lambda \in \Lambda_A} \sum_{m=0}^{a_\lambda - 1} \frac{1}{(z-\lambda)^{m+1}} A_{\lambda,m} \\
&= \sum_{\lambda \in \Lambda_A} \sum_{m=0}^{\nu_\lambda - 1} \frac{1}{(z-\lambda)^{m+1}} A_\lambda \big(A - \lambda I\big)^m
\end{aligned}
$$

for $z \notin \Lambda_A$, where $\mathcal{C}$ is the matrix of cofactors of $zI - A$; A_λ is the *projection operator* associated with λ.

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$$= \sum_{\lambda \in \Lambda_A} \sum_{m=0}^{\nu_\lambda - 1} A_\lambda (A - \lambda I)^m \left(\frac{1}{2\pi i} \oint_{C_\lambda} \frac{f(z)}{(z-\lambda)^{m+1}} \, dz \right) .$$

Extends $f(A)$ beyond both Taylor and holomorphic domain.

Meromorphic functional calculus

Meromorphic functional calculus

$$f(A) = \sum_{\lambda \in \Lambda_A} \sum_{m=0}^{\nu_\lambda - 1} A_\lambda (A - \lambda I)^m \left(\frac{1}{2\pi i} \oint_{C_\lambda} \frac{f(z)}{(z-\lambda)^{m+1}} \, dz \right) ,$$

- Poles and zeros of $f(z)$ can interact with poles of resolvent.
- Extends $f(A)$ beyond holomorphic domain.

Powers of Matrices

$$A^L = \sum_{\lambda \in \Lambda_A \setminus \{0\}} \sum_{m=0}^{\nu_\lambda - 1} \binom{L}{m} \lambda^{L-m} A_\lambda (A - \lambda I)^m$$
$$+ [0 \in \Lambda_A] \sum_{m=0}^{\nu_0 - 1} \delta_{L,m} A_0 A^m$$

for any $L \in \mathbb{C}$, where $\binom{L}{m}$ is the generalized binomial coefficient:

$$\binom{L}{m} = \frac{1}{m!} \prod_{n=1}^{m} (L - n + 1)$$

with $\binom{L}{0} = 1$.

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for any $L \in \mathbb{C}$, $\binom{L}{m} = \frac{1}{m!} \prod_{n=1}^{m} (L - n + 1)$ with $\binom{L}{0} = 1$.

E.g.:
$\binom{0}{m} = 0$ unless $m = 0$, so:

$$A^0 = \sum_{\lambda \in \Lambda_A} A_\lambda = I \ .$$

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E.g.:
$\binom{1}{m} = 0$ unless $m = 0$ or $m = 1$, so:

$$\begin{aligned} A^1 &= \sum_{\lambda \in \Lambda_A \setminus \{0\}} \sum_{m=0}^{1} \lambda^{L-m} A_\lambda (A - \lambda I)^m + [0 \in \Lambda_A] \, A_0 A \\ &= \sum_{\lambda \in \Lambda_A} \lambda A_\lambda + \sum_{\lambda \in \Lambda_A} A_\lambda (A - \lambda I) \qquad . \end{aligned}$$

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$$= \sum_{\lambda \in \Lambda_A} \lambda A_\lambda + \sum_{\lambda \in \Lambda_A} A_\lambda (A - \lambda I) = D + N = A \; .$$

where D is diagonable, N is nilpotent, and $[D, N] = 0$.

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$\binom{-|L|}{m} = (-1)^m \binom{|L|+m-1}{m}$ and $\binom{m}{m} = 1$, so:

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where $A^{\mathcal{D}}$ is the *Drazin inverse* of A.

Drazin Inverse, Spectrally

$$A^{\mathcal{D}} = \sum_{\lambda \in \Lambda_A \setminus \{0\}} \sum_{m=0}^{\nu_\lambda - 1} (-1)^m \lambda^{-1-m} A_\lambda (A - \lambda I)^m \ .$$

When A is diagonable, this reduces to:

$$A^{\mathcal{D}} = \sum_{\lambda \in \Lambda_A \setminus \{0\}} \lambda^{-1} A_\lambda \qquad \text{if } \nu_\lambda = 1 \text{ for all } \lambda \in \Lambda_A \ .$$

Drazin inverse, Axiomatically

A^{-1} from the meromorphic functional calculus satisfies all of the properties of the unique Drazin inverse as it is axiomatically defined for matrices with index ν_0:

$$\begin{aligned} &(1^{\nu_0}) && A^{\nu_0} A^{\mathcal{D}} A = A^{\nu_0} \\ &(2) && A^{\mathcal{D}} A A^{\mathcal{D}} = A^{\mathcal{D}} \\ &(5) && [A, A^{\mathcal{D}}] = 0 \ . \end{aligned}$$

So $A^{-1} = A^{\mathcal{D}}$,

a.k.a., the $\{1^{\nu_0}, 2, 5\}$-inverse.

Drazin inverse or Group inverse?

For $\nu_0 = 1$, $A^{\mathcal{D}} = A^{\sharp}$.
For $\nu_0 > 1$, $A^{\sharp}$ does not exist.

We are really bad at asking novel questions!

Most questions are of the form:

$$\langle \cdot | \, T^L \, | \cdot \rangle$$

or

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Different representations of a process (diff. Ts or diff. Gs) needed to answer different questions (using linear algebra).

- Every time we ask a question like the bottom row, we invoke the Drazin inverse.

Correlations and Power Spectra are directly about observables:
$\longrightarrow T$ of any representative HMM will do:

$$\gamma[\tau] = \left\langle \overline{X}_n X_{n+\tau} \right\rangle_n$$

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$\longrightarrow T$ of any representative HMM will do:

$$\begin{aligned}\gamma[\tau] &= \left\langle \overline{X}_n X_{n+\tau} \right\rangle_n \\ &= \langle \overline{\mathcal{A}} | \, T^{|\tau|-1} \, | \mathcal{A} \rangle \ .\end{aligned}$$

$$P_{\mathrm{c}}(\omega) = 2 \, \mathrm{Re} \, \langle \overline{\mathcal{A}} | \, (e^{i\omega} I - T)^{-1} \, | \mathcal{A} \rangle + \text{const} \ .$$

Information metrics require *distributions* over states of any rep. of process:
$\longrightarrow$ Use W of mixed state presentation (MSP) of any rep. of process.

E.g.,
Myopic entropy rates:

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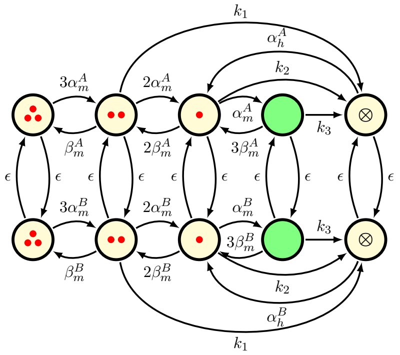
Past–future mutual info:

$$I(X_{:0}; X_{0:}) = \mathbf{E} = \sum_{L=1}^{\infty} \left[h_\mu(L) - h_\mu \right] = \langle \cdot | \, (I - W)^{\mathcal{D}} \, | \cdot \rangle$$

Measures of intrinsic computation require distributions over *causal* states:
$\longrightarrow$ Use $\mathcal{W}$ of MSP of ϵ-machine.

$\ldots$

Joint Environmental-driving–System-state Dynamic



$$Q_{\text{ex}} = \sum_{n=1}^{N} Q_{\text{ex}}[x_n; s_{n-1} \to s_n]$$

$$Q_{\mathrm{ex}} = \textstyle\sum_{n=1}^{N} Q_{\mathrm{ex}}[x_n; s_{n-1} \to s_n]$$

$$\begin{aligned}\langle Q_{\mathrm{ex}} \rangle &= \langle \boldsymbol{\mu}_0 | \Big(\sum_{n=0}^{N-1} \mathcal{T}^n \Big) Q^{(X\mathcal{S})} \, |\mathbf{1}\rangle \\ &\to \langle \boldsymbol{\mu}_0 | \Big(\int_{t=0}^{\tau} e^{t\mathcal{G}} \, dt \Big) \mathcal{Q}^{(X\mathcal{S})} \, |\mathbf{1}\rangle\end{aligned}$$

For a broad subset of HMMs, the heat matrix turns out to be:

$$\mathcal{Q}_{ij}^{(X\mathcal{S})} = \frac{\mathcal{G}_{ij}}{\beta_j} \left[\ln\left(\pi_{x^j}(s^i)\right) - \ln\left(\pi_{x^j}(s^j)\right) \right] \; .$$

$Q_{\mathrm{ex}} = \sum_{n=1}^{N} Q_{\mathrm{ex}}[x_n; s_{n-1} \to s_n]$

$$\begin{aligned}\langle Q_{\mathrm{ex}} \rangle &= \langle \boldsymbol{\mu}_0 | \Big(\sum_{n=0}^{N-1} \mathcal{T}^n \Big) Q^{(X\mathcal{S})} | \mathbf{1} \rangle \\ &\to \langle \boldsymbol{\mu}_0 | \Big(\int_{t=0}^{\tau} e^{t\mathcal{G}} \, dt \Big) \mathcal{Q}^{(X\mathcal{S})} | \mathbf{1} \rangle \\ &= \tau \, \langle \boldsymbol{\pi} | \mathcal{Q}^{(X\mathcal{S})} | \mathbf{1} \rangle + \langle \boldsymbol{\mu}_0 | \, \mathcal{G}^{\mathcal{D}} \big(e^{\tau\mathcal{G}} - I \big) \mathcal{Q}^{(X\mathcal{S})} | \mathbf{1} \rangle\end{aligned}$$

since $\int_0^{\tau} e^{t\mathcal{G}} \, dt = \tau \, | \mathbf{1} \rangle \langle \boldsymbol{\pi} | + \mathcal{G}^{\mathcal{D}} \big(e^{\tau\mathcal{G}} - I \big)$.

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Where did you come from?
Nobody knows.
But Drazin knows best.

Remove your asymptotics;
I want to see your transients.

$\mathcal{L}$ove,

$\mathcal{D}razin$