POWER SPECTRAL ANALYSIS OF A DYNAMICAL SYSTEM *

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Power spectra for chaotic transitions in three dimensions are presented for a dynamical system first proposed by Rössler. Relations between the spectra and the topology of the corresponding strange attractor are discussed.

Modern experiments in Couette and Bénard flows often use power spectral analysis as a measure of the temporal behavior of fluid motions, and the transition to turbulence [1]. At the same time there is interest in the study of simple dynamical systems which may exhibit chaotic behavior owing to the existence of a "strange attractor" [2]. The most familiar example is the highly idealized model for Bénard convection of Lorenz [3]. It is natural to study the power spectra of simple dynamical systems, in the hope of gaining insight for the interpretation of the spectra of real fluids.

There has been some work on the power spectra of strange attractors, by Ueda [4], Holmes [5] and others, but there remains a need for systematic studies of the changes in power spectra of dynamical systems as they bifurcate from one attractor topology to another. We present here results for a particular bifurcation sequence, one which occurs widely in the transition to chaotic behavior of vector fields in three dimensions.

The construction of digital computer solutions to dynamical systems followed by power spectral analysis is time-consuming. We have developed a hybrid computer system by solving the equations on an analog computer and performing power spectral analysis

on a digital computer. The analog computer is only accurate to a percent or so, but high accuracy is not needed for these largely qualitative studies.

Fig. 1 displays the transition in a system originally studied by Rössler [6]:

$$\dot{x} = -(y + z), \quad \dot{y} = x + 0.2y,$$
$$\dot{z} = 0.2 + xz - Cz, \tag{1}$$

as the parameter C is varied. Projections onto the $x-y$ plane of these equations after transients have been allowed to die out are shown. We chose this particular system for study because the "branched manifold" [7,8], enclosing its attractor has the simplest topology which will still produce a strange attractor.

To obtain these power spectra we solved the equations on an analog computer, with the natural time unit taken to be 0.01 s, sampled the solutions until 4096 points were accumulated, and used an FFT to compute the discrete spectra. For each parameter value this process was repeated 10 times and the resulting spectra averaged. Some of the results for time series obtained from $z(t)$ are shown in fig. 1. The time series derived from $x(t)$ and $y(t)$ gave similar results.

The sequence of bifurcations is schematically represented in fig. 2. The bifurcation sequence for $C < C_\infty \approx 4.20$, consisting of the successive appearance of
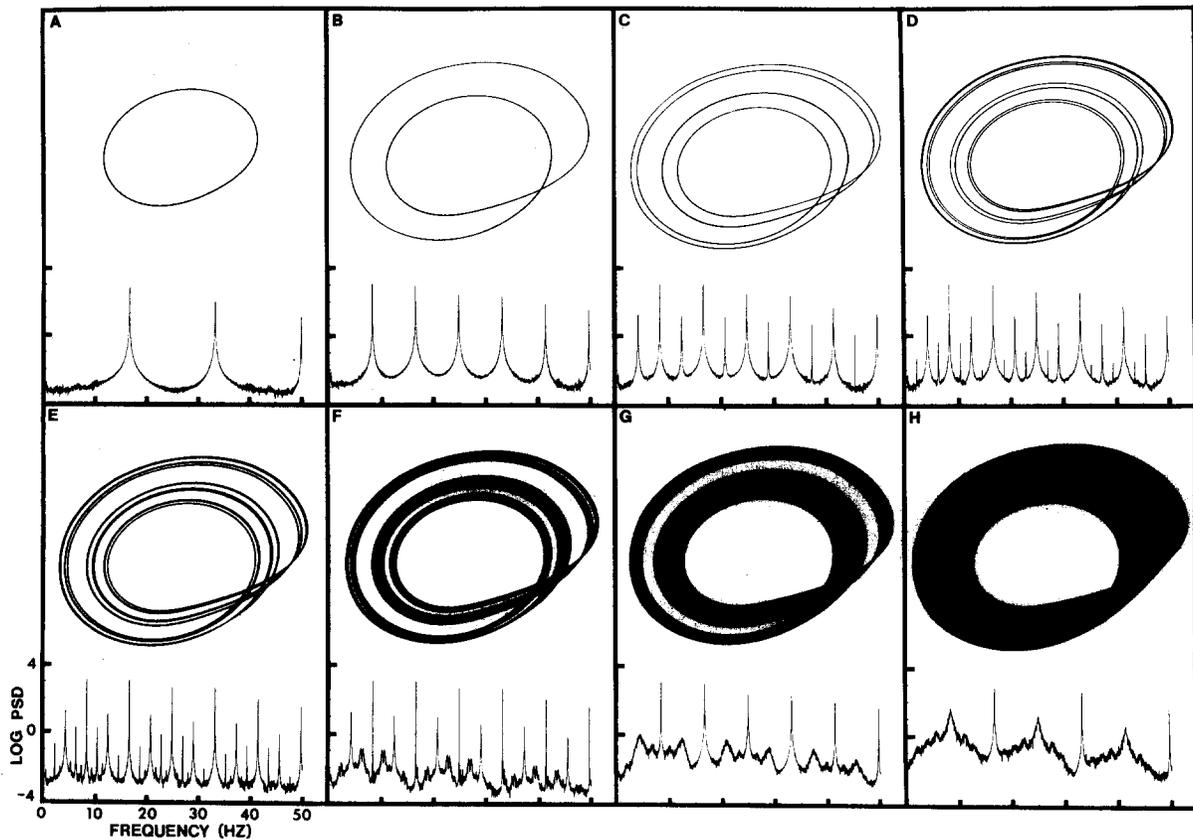
Fig. 1. Plots of the solutions of eq. (1) on the $x-y$ plane after transients have died out. Directly below each phase trajectory is the corresponding power spectral density (PSD) as a function of frequency. Details of these plots are contained in table 1.
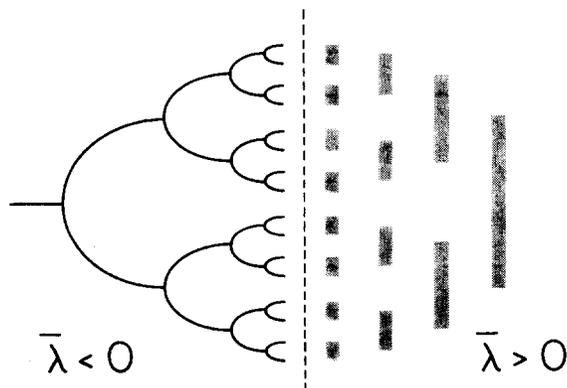
Fig. 2. The bifurcation sequence obtained from the data of fig. 1. $\bar{\lambda}$ increases from left to right. The shaded regions denote strange bands.

subharmonics, is familiar from the work of May on one-dimensional maps [9], was mentioned by Brunovsky [10], was observed by Feit in two-dimensional maps [11] and has been reported in a driven oscillator system by Coullet et al. [12]. At each step the limit cycle "unwinds", roughly doubling the period of a complete orbit. This bifurcation was known to Poincaré [13].

The doubling process reaches an accumulation point at C_∞, however, and is succeeded by a qualitatively different behavior. The largest non-zero Liapunov characteristic exponent $\bar{\lambda}$ (see fig. 3) becomes positive, reflecting the exponential divergence of trajectories [14–16]. Families of orbits remain confined to thin bands, which rejoin in a pairwise manner, as illustrated. A complete bifurcation sequence has been
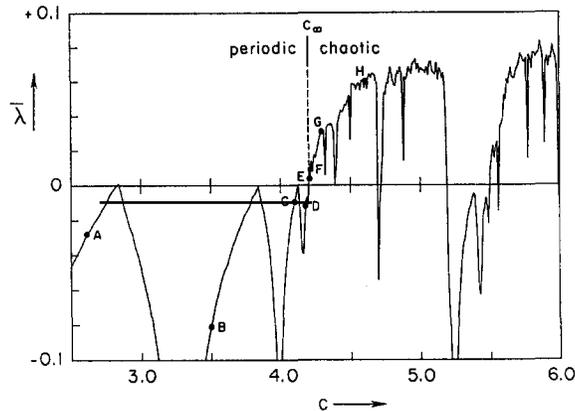
Fig. 3. Plot of the largest non-zero characteristic exponent $\bar{\lambda}$ as a function of C. Letters refer to the plots of fig. 1. A line has been drawn through around 300 calculated points.

described by Shimada and Nagashima [17] and is visible in a figure in a paper by Li and Yorke [18]. This sequence occurs quite generally in maps or flows containing simple folds. It has been observed in the Rössler system, the Lorenz system for high values of the R parameter [17], and we have observed it in the driven Van der Pol equations, the driven Duffing equations [19], and several other sets of equations, and one-dimensional maps. Periodic orbits, or other behavior, may be interspersed in the sequence, however.

Fig. 1 displays the corresponding power spectra. Each subharmonic bifurcation doubles the number of sharp frequency components, and each pairwise rejoining broadens every other sharp spike. The fundamental, at about 16 Hz, moves little during the series. Typical parameters are given in table 1.

A remarkable feature of this transition is the presence of sharp frequency components in a chaotic attractor. Even at the final stage, when the trajectories fill out the complete Rössler attractor, the spectrum retains a peak which appears to be instrumentally sharp. We believe this to be a feature of some attractors whose branched manifolds are simply connected, in the sense that all trajectories are constrained to revolve about a single hole [20]. Attractors containing fixed points, such as the familiar Lorenz, do not have this property, and do not have sharp spikes in their power spectra.

It would be convenient to have a simple measure over the power spectrum of "chaos". Several measures are capable of distinguishing sharp spikes from broad features in power spectra, but it is unclear to the writers which, if any, might have a close relation to the topology or the characteristic exponents of the underlying attractor.

One such quantity is tabulated in table 1. The number of "degrees of freedom" of a discrete spectrum of n frequencies is given by:

Table 1

Data for the bifurcation sequence shown in figs. 1 and 2. f and T refer to the natural frequency (~ 16 Hz) and period of the system. Subharmonic bifurcations occur where the curve of fig. 3 has a point of tangency with the line $\bar{\lambda} = 0$.

Fig.	C	Phase trajectory	Spectral lines	$10^3 N$
1A	2.6	period $1T$ limit cycle	f, and harmonics	1.17
1B	3.5	period $2T$ limit cycle	$\frac{1}{2}f, f$ and harmonics	2.40
1C	4.1	period $4T$ limit cycle	$\frac{1}{4}f, \frac{1}{2}f, f$ and harmonics	2.19
1D a)	4.18	period $8T$ limit cycle	$\frac{1}{8}f, \frac{1}{4}f, \frac{1}{2}f, f$ and harmonics	1.99
1E	4.21	broadening of band of period $\sim 8T$	$\frac{1}{8}f, \frac{1}{4}f, \frac{1}{2}f, f$ and harmonics (weak $\frac{1}{16}f$ present)	1.86
1F	4.23	broadening of band of period $\sim 4T$	$\frac{1}{4}f, \frac{1}{2}f, f$ and harmonics	1.84
1G	4.30	broadening of band of period $\sim 2T$	$\frac{1}{2}f, f$ and harmonics	2.73
1H	4.60	broadening of band of period $\sim T$	f and harmonics	5.97

a) The accumulation point C_∞ occurs near here. We are able to resolve a period $16T$ limit cycle and band of period $\sim 16T$, not illustrated.

$$N = \left(\sum_{i=1}^{n} P_i \right)^2 \Big/ n \sum_{i=1}^{n} P_i^2 , \qquad (2)$$

where P_i is the power at the ith frequency. For a sine wave, $N = 1/n$, for white noise, $N = 1$. One can see from table 1 that changes in N reflect bifurcations in the system, but are decisive only when the single strange band appears as in fig. 1H. Another measure can be obtained by normalizing the spectral power density and summing $P_i \log P_i$. This measure tends toward $\log n$ for a sine wave, and is equal to zero for white noise. Still another measure of the $P \log P$ type, due to Aikake [21], has been used by Yahata [22] to characterize theoretically derived power spectra relating to Couette flow. We have used the measures N and $P_i \log P_i$ in some unpublished experimental research on transitions in Couette flow. We are not convinced that they are experimentally useful measures of the power spectra, and expect further work is required on the characterization of power spectra.

The transition from a simple to a chaotic attractor occurs, in three dimensions, in one of a few distinct ways. Each of these has a characteristic signature in the power spectra domain. We have presented one in this letter, the others, as well as some higher dimensional results, will be detailed in a forthcoming paper.

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