FLUCTUATIONS AND THE ONSET OF CHAOS

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Received 3 April 1980

We consider the role of fluctuations on the onset and characteristics of chaotic behavior associated with period doubling subharmonic bifurcations. By studying the problem of forced dissipative motion of an anharmonic oscillator we show that the effect of noise is to produce a bifurcation gap in the set of available states. We discuss the possible experimental observation of this gap in many systems which display turbulent behavior.

It has been recently shown that the deterministic motion of a particle in a one-dimensional anharmonic potential, in the presence of damping and a periodic driving force, can become chaotic [1]. This behavior, which appears after an infinite sequence of subharmonic bifurcations as the driving frequency is lowered, is characterized by the existence of a strange attractor in phase space and broad band noise in the power spectral density. Furthermore, it was predicted that under suitable conditions such turbulent behavior may be found in strongly anharmonic solids [2]. Since condensed matter is characterized by many-body interactions, one may ask about the effects that random fluctuating forces have on both the nature of the chaotic regime and the sequence of states that lead to it. This problem is also of relevance to the behavior of stressed fluids, where it has been suggested that strange attractors play an essential role in the onset of the turbulent regime [3]. Although there are experimental results supporting this conjecture [4–6], other investigations have emphasized the possible role of thermodynamic fluctuations directly determining the chaotic behavior [7].

With these questions in mind, we study the role of fluctuations on the onset and characteristics of chaotic behavior associated with period doubling subharmonic bifurcations. We do so by solving the problem of forced dissipative motion in an anharmonic potential with the aid of an analog computer and a white-noise generator. As we show, although the structure of the strange attractor is very stable even under the influence of large fluctuating forces, their effect on the set of available states is to produce a symmetric gap in the deterministic bifurcation sequence. The magnitude of this bifurcation gap is shown to increase with noise level. By keeping the driving frequency fixed we are also able to determine that increasing the random fluctuations induces further bifurcations, thereby lowering the threshold value for the onset of chaos. Finally, the universality of these results is tested by observing the effect of random errors on a one-dimensional map, and suggestions are made concerning the possible role of temperature in experiments that study the onset of turbulence.

Consider a particle of mass m, moving in a one-dimensional potential $V = a\eta^2/2 - b\eta^4/4$, with η the displacement from equilibrium and a and b positive constants. If the particle is acted upon by a periodic force of frequency ω_d and amplitude F, and a fluctuating force $f(t)$, with its coupling to all other degrees of freedom represented by a damping coefficient γ, its equation of motion in dimensionless units reads

$$\frac{d^2\psi}{dt^2} + \alpha\frac{d\psi}{dt} + \psi - 4\psi^3 = \Gamma\cos\left(\frac{\omega_d}{\omega_0}\right)t + f(t) \qquad (1)$$

with $\psi = \eta/2\eta_0$, the particle displacement normalized to the distance between maxima in the potential (η_0

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$= (a/b)^{1/2}$), $\alpha = \gamma/(ma)^{1/2}$, $\Gamma = Fb^{1/2}/2a^{3/2}$, $\omega_0 = (a/m)^{1/2}$ and $f(t)$ a random fluctuating force such that

$$\langle f(t)\rangle = 0 \tag{2a}$$

and

$$\langle f(0)f(t)\rangle = 2A\,\delta(t) \tag{2b}$$

with A a constant proportional to the noise temperature of the system.

The range of solutions of eq. (1), in the case where $f(t) = 0$ (the deterministic limit) has been investigated earlier [1]. For values of Γ and ω_d such that the particle can go over the potential maxima, as the driving frequency is lowered, a set of bifurcations takes place in which orbits in phase space acquire periods of 2^n times the driving period, T_d. At a threshold frequency ω_{th}, a chaotic regime sets in, characterized by a strange attractor with "periodic" bands. Within this chaotic regime, as the frequency is decreased even further, another set of bifurcations takes place whereby 2^m bands of the attractor successively merge in a mirror sequence of the 2^n periodic sequence that one finds for $\omega \to \omega_{th}^+$. The final chaotic state corresponds to a single band strange attractor, beyond which there occurs an irreversible jump into a periodic regime of lower amplitude.

In order to study the effects of random fluctuations on the solutions we have just described, we solved eq. (1) using an analog computer in conjunction with a white-noise generator having a constant power spectral density over a dynamical range two orders of magnitude larger than that of the computer. Time series and power spectral densities were then obtained for different values of Γ, A and ω_d. While we found that the folding structure of the strange attractor is very stable under the effect of random forces, the bifurcation sequence that is obtained in the presence of noise differs from the one encountered in the deterministic limit.

Our results can be best summarized in the phase diagram of fig. 1, where we plot the observed set of bifurcations (or limiting set) as a function of the noise level, N, normalized to the rms amplitude of the driving term, Γ. The vertical axis denotes the possible states of the system, labeled by their periodicity $P = 2^n$, which is defined as the observed period normalized to the driving period, T_d. As can be seen, with increasing noise level a symmetric bifurcation gap appears, depleting states both in the chaotic and periodic phases. This set of inaccessible states is characterized by the fact that the longest periodicity which is observed before a strange attractor appears is a decreasing function of N, with the maximum number of bands which appear in the strange attractor behaving in exactly the same fashion. This gap extends over a large range of noise levels (up to $N = 1.5$), beyond which the motion either becomes unstable (i.e., $|\psi| \to \infty$; lower dashed line) or an amplitude jump takes place from the chaotic regime to a limit cycle of period 1 (upper dashed line).

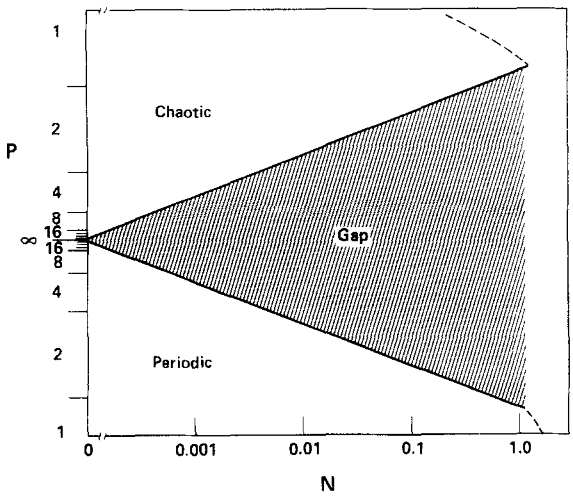

Fig. 1. The set of available states of a forced dissipative anharmonic oscillator as a function of the noise level. The vertical axis denotes the periodicity of a given state with $P = T/T_d$. The noise level is given by $N = A/\Gamma_{rms}$. The shaded area corresponds to inaccessible states.

We can illustrate this behavior by looking at the power spectral densities, $S(\omega)$ at fixed values of the driving frequency while increasing N. Fig. 2 shows such a sequence for $\omega_d/\omega_0 = 0.6339$, $\Gamma = 0.1175$, and $\alpha = 0.4$. Fig. 2(a) corresponds to $S(\omega)$ near the deterministic limit which, for the parameter values used, displays a limit cycle of period four. As N is increased, a transition takes place into a chaotic regime characterized by broad band noise with subharmonic content of periodicity $P = 4$ (fig. 2(b)) [‡1]. As the noise is increased even further, a new bifurcation occurs from which a new

[‡1] We should mention that the Poincare map corresponding to this state clearly shows a four-band strange attractor with a single fold.

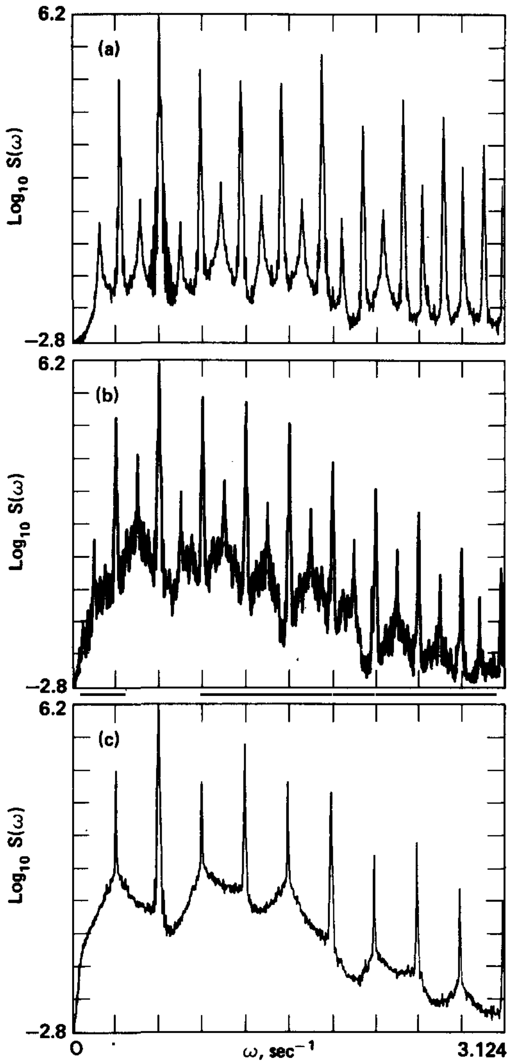

Fig. 2. Power spectral densities at increasing values of the effective noise temperature, for $\Gamma = 0.1175$, $\alpha = 0.4$, and $\omega_d = 0.6339\ \omega_0$. Fig. 2(a): $N = 10^{-4}$. Fig. 2(b): $N = 0.005$. Fig. 2(c): $N = 0.357$.

chaotic state with $P = 2$ emerges. Physically, this sequence reflects the fact that a larger effective noise temperature (and hence a larger fluctuating force) makes the particle gain enough energy so as to sample increasing nonlinearities of the potential, with a resulting motion which in the absence of noise could only occur for longer driving periods ‡2.

A different set of states appears if the noise level is kept fixed while changing the driving frequency. In this case the observed states of the system correspond to vertical transitions in the phase diagram of fig. 1, with the threshold value of the driving period, $T_{\rm th}^n$, at which one can no longer observe periodicities $P \geqslant 2^n$, behaving like

$$T_{\rm th}^n = T_{\rm th}^\infty(1 - N_{\rm n}^\gamma) \qquad (3)$$

for $0 \leqslant N_{\rm n} \leqslant 1$, with $N_{\rm n}$ the corresponding noise level, $T_{\rm th}^\infty$ the value of the driving period for which the deterministic equation undergoes a transition into the chaotic state, and γ a constant which we determined to be $\gamma = \sim 1$ for $P \geqslant 2$ ‡3.

In order to test the universality of the bifurcation gap we have just described, we have also studied the bifurcation structure of the one-dimensional map described by

$$x_{L+1} = \lambda x_L(1 - x_L) + n_L(0, \sigma^2) \qquad (4)$$

where $0 \leqslant x_L \leqslant 1$, $0 \leqslant \lambda \leqslant 4$, and n_L is a gaussian random number of zero mean and standard deviation σ. For $n_L = 0$, eq. (4) displays a set of 2^n periodic states universal to all single hump maps [8–9], with a chaotic regime characterized by 2^m bands that merge pairwise with increasing λ [10]. For $n_L \neq 0$ and a given value of σ, the effect of random errors on the stability of the limiting set is to produce a bifurcation gap analogous to the one shown in fig. 1.

The above results are of relevance to experimental studies of turbulence in condensed matter, for they show that temperature plays an important role in the observed behavior of systems belonging to this same universality class. In particular, Belyaev et al. [11], Libchaber and Maurer [12] and Gollub et al. [13] have reported that under certain conditions the transition to turbulence is preceded and followed by different finite sets of 2^n subharmonic bifurcations. It would therefore be interesting to see if temperature changes or external sources of noise in the fluids can either reduce or increase the set of observed frequencies, thus providing for a test of these ideas. In the case of solids such as superionic conductors, the expo-

‡2 In the regime of subharmonic bifurcation the dependence of response amplitude on driving frequency is almost linear.

‡3 Using the scaling relation $(T_{\rm th} - T_n)/(T_{\rm th} - T_{n+1}) = \delta$ [1] this implies that the threshold noise level scales like $N_n/N_{n+1} = \delta$, with $\delta = 4.669201609$ ....

nential dependence on temperature of their large diffusion coefficients might provide for an easily tunable system with which to study the existence of bifurcation gaps. Last, but not least, these studies can serve as useful calibrations on the relative noise temperature of digital and analog simulations.

In concluding we would like to emphasize the wide applicability of the effects that we have reported. Beyond the experimental studies of turbulence, there exist other systems which belong to the same universality class as the anharmonic oscillator and one-dimensional maps. These systems range from the ordinary differential equations studied by Lorenz [10], Robbins [14], and Rossler [15] to partial differential equations describing chemical instabilities [16]. Since period doubling subharmonic bifurcation is a universal feature of all these models, our results provide a quantitative measure of the effect of noise on their non-linear solutions.

The authors wish to thank D. Farmer, N. Packard, and R. Shaw for helpful discussions and the use of their simulation system.

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