Critical Information: The Behavior of Information Measures Near Critically in a 2D Spin Lattice

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Abstract

Spin lattices are one of the most thoroughly studied interacting systems in condensed matter physics due to the emergence of long range interactions near criticality. The Ising model is one example of a spin lattice that, although simple in form, yields great insight into critical phenomena. Although the Ising model has been thoroughly studied, it has primarily been through the view of thermodynamics. Here we investigate some information theoretic measures such as the Shannon Entropy and the mutual information, of a 2D nearest neighbor Ising spin lattice near criticality and show that the mutual information is an order parameter that fully captures the temperature dependent behavior of this system.

1 Introduction

The Ising model is a classical spin lattice described by the Hamiltonian

$$H = \sum_{ij} J_{ij}\sigma_i\sigma_j + \sum_i h_i\sigma_i$$

where σ_i is the spin at site i, J_{ij} is the coupling between sites i and j and h is an external magnetic field. One quantity of great interest for this system is the correlation function

$$\langle\sigma_i\sigma_{i+r}\rangle = \frac{1}{\mathbf{Z}}\sum_i e^{-\beta H_i}\sigma_i\sigma_{i+r}$$

The correlation function is a measure of the connectedness of two points in a system at given distance and defines one of the fundamental length scales of a system, the correlation length, ζ. ζ can be viewed as the length over which local information is shared throughout the system. Taking this view it then seems reasonable to argue that the divergence of ζ at the critical point should also be visible in the mutual information shared between two points. The mutual information of two random variables, RV, is defined by

$$\begin{aligned} I(X;Y) &= \sum_x \sum_y p(x,y) \mathrm{Log}_2\left(\frac{\mathrm{p(x,y)}}{\mathrm{p(x)p(y)}}\right) \\ &= H(X) - H(X|Y) \end{aligned}$$

where $H(X)$ is the Shannon entropy defined by

$$H(X) = -\sum_x p(x)\mathrm{Log}_2(\mathrm{p(x)})$$

At criticality the mutual information should show behavior similar to thermodynamic order parameters, potentially demonstrating an information theoretic perspective for critical behavior.

2 Methods

The system used in this investigation is a 2D Ising spin lattice with no external field and ferromagnetic nearest neighbor coupling. Under these conditions the Hamiltonian reduces to

$$H = -J \sum_{<ij>} \sigma_i \sigma_j$$

where the brackets denote summing over nearest neighbors. To model the dynamics of this system the Metropolis–Hastings algorithm was implemented on a 50X50 square lattice. The Metropolis-Hastings algorithm uses the following criteria to update the system.

1. At $\mathrm{site}_{\mathrm{ij}}$ propose a spin flip and compute the change in energy, $\Delta\epsilon$.
2. If $\Delta\epsilon <= 0$: keep the flip and update the system.
3. If $\Delta\epsilon > 0$: Draw a random number $x \in (0, 1]$ and compare to the relative probability of the flip $e^{-\beta\Delta\epsilon}$, where $\beta = \frac{1}{k_B T}$.
4. If $x < e^{-\beta\Delta\epsilon}$: perform the flip and update the system. else: continue

The initial spin configuration was selected at random and 100 rounds of relaxation were performed to eliminate transient behavior. After the initial relaxation the temperature was incrementally lowered through the critical temperature. At each temperature the on site probability, $p(\sigma_i)$, and two point correlations at a distance r, $p(\sigma_i|\sigma_{i+r})$, over a range of distances. From these quantities the on site entropy

$$H(\sigma_i) = -\sum_{\pm} p(\sigma_i)\mathrm{Log}_2(\mathrm{p}(\sigma_\mathrm{i}))$$

The conditional entropy

$$H(\sigma_i|\sigma_{i+r}) = -\sum_{i,i+r} p(\sigma_i, \sigma_{i+r})\mathrm{Log}_2(\mathrm{p}(\sigma_\mathrm{i}|\sigma_\mathrm{i+r}))$$

and the mutual information

$$I(\sigma_i;\sigma_{i+r}) = H(\sigma_i) - H(\sigma_i|\sigma_{i+r})$$

3 Results

The exploration of this system clearly showed a connection between thermodynamic properties and information properties. The on site entropy showed results similar to earlier work[1], see Fig.1, demonstrating a transition from the maximum value, consistent with a uniform distribution of spins, to minimum of 0, consistent with a single spin value, upon moving through the critical temperature. The divergence of the correlation of length is a key indicator of a phase

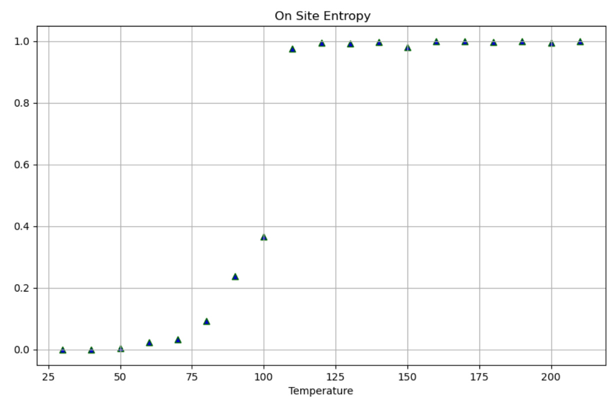

Figure 1: The on site entropy of the Ising spin lattice as a function of temperature. Note the transition at $T_C = 105.3$K

transition. Fig.2 shows the divergence of the correlation length at the critical temperature by the entropy rate as well as the entropy both dropping to zero at $T_C = 52.5$K.

The most interesting result came from the mutual information. The first set of runs used a $T_C = 52.5$K, see Fig.3. In these runs the values of the mutual information as well as the rate converge at T_c. Additional runs were carried out using $T_C = 105.3$K. Several runs were preformed to ensure consistency and each produced a similar result. Fig.4 shows the mutual information as a function of temperature. I remain convinced that the strange behavior is an error in the simulation but note the sharp dip and discontinuity in the derivative at T_c.

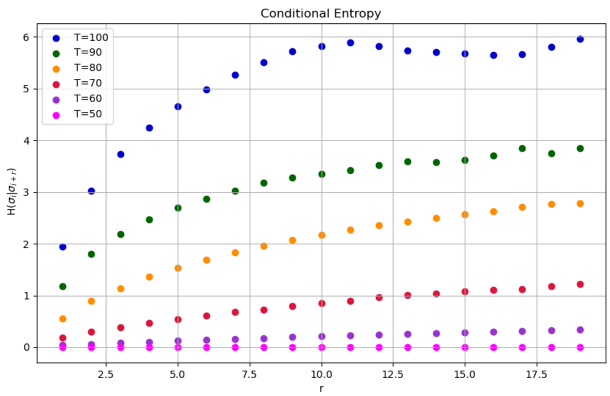

Figure 2: The conditional entropy as a function of distance over a range of temperatures. Note the decrease in not only the max value but also the rate of entropy increase as the system approaches the critical temperature.

4 Discussion

The cusp in the derivative of the mutual information remains a complete mystery to me; however, I am quite unknowledgeable in this area so it is probably something simple. Other than the cusp, the temperature dependent behavior of mutual information strongly demonstrates behavior typical of an order parameter. If the cusp is just an error then the mutual information shows the system undergoing a second order transition at the critical temperature.

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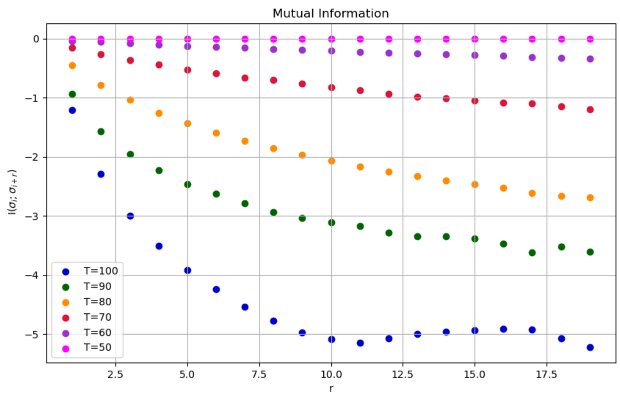

Figure 3: The mutual information as a function of distance over a range of temperatures. Note the convergence of the rate of change as the system approaches the critical temperature.

References

[1] Vikram S. Vijayaraghavan, Ryan G. James, and James P. Crutchfield. Anatomy of a spin: The information-theoretic structure of classical spin systems. *Entropy*, 19(5), 2017.

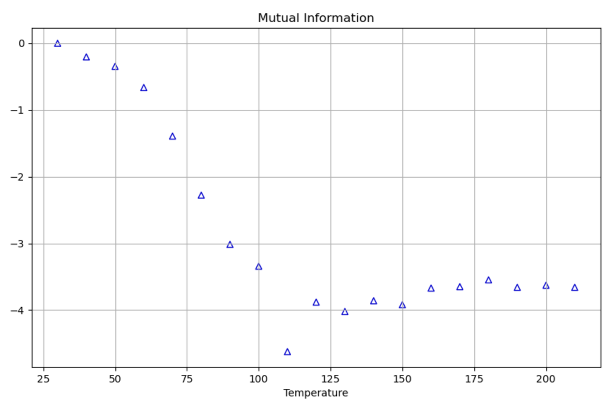

Figure 4: The mutual information as a function of temperature at r = 10.