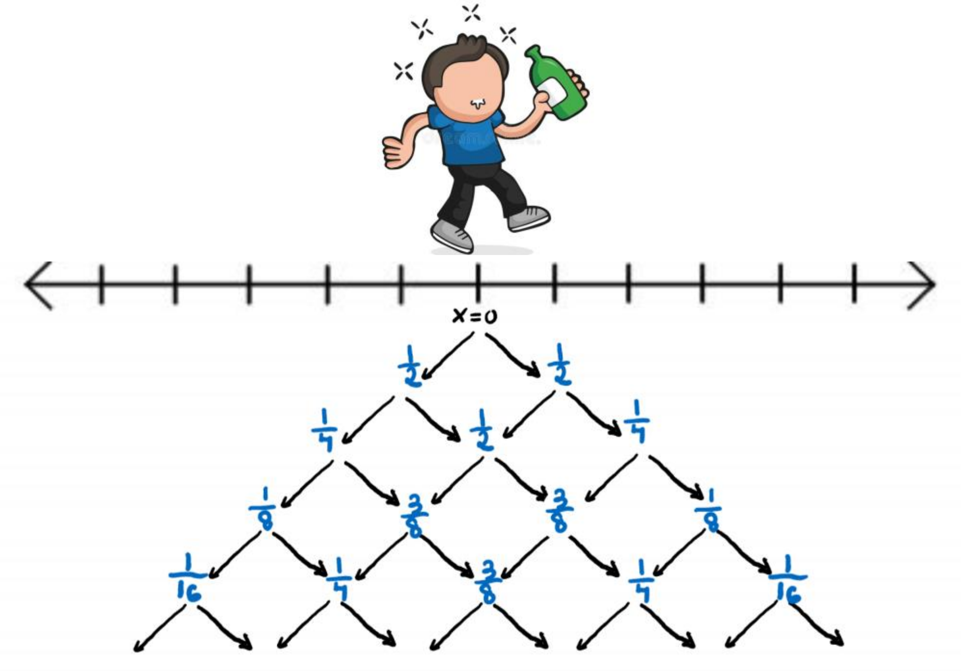
Two Particle Quantum Walk

What does drunkenness look like in the Quantum Realm?

Random Walk

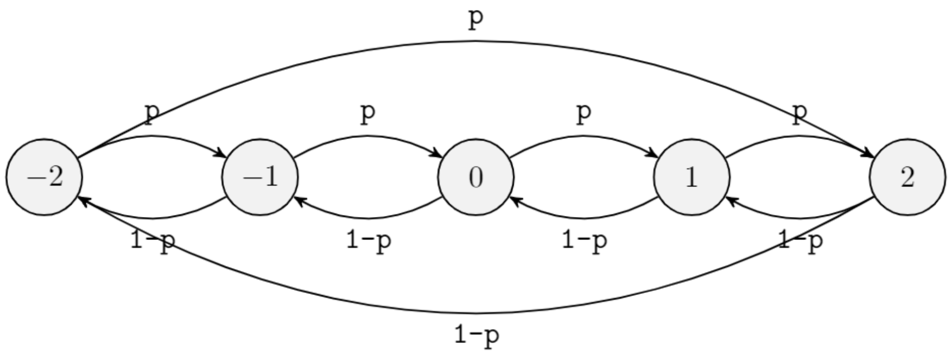


A drunkard's walk—where the choice of whether to step to the right or the left is made randomly by the toss of a coin

Random walk is a stochastic process, that describes a path that consists of a succession of random steps on some mathematical space.

Random Walk as a Markov Chain

- Random walk on a N vertex graph is given by an $N \times N$ matrix - T where, T_{ij} represents the probability of making a transition from i to j and $\sum_{j=1}^{N} T_{ij} = 1$
- The state space for a $1D$ Random Walk: $\{0, \pm 1, \pm 2\}$ with periodic boundary conditions



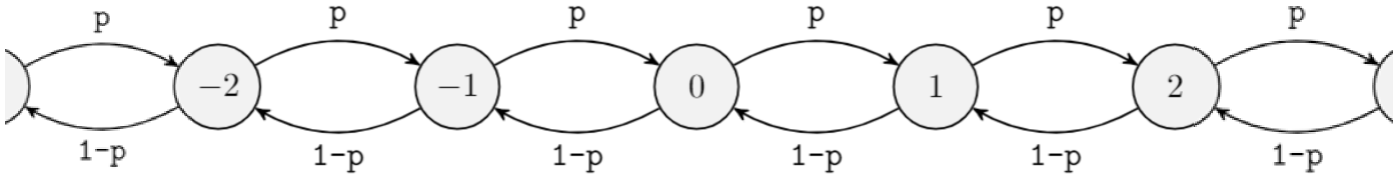
$$T = \begin{bmatrix} 0 & p & 0 & 0 & 1-p \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ p & 0 & 0 & 1-p & 0 \end{bmatrix} = T^{(L)} + T^{(R)}$$

$$T^{(R)} = p \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad T^{(L)} = (1-p) \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

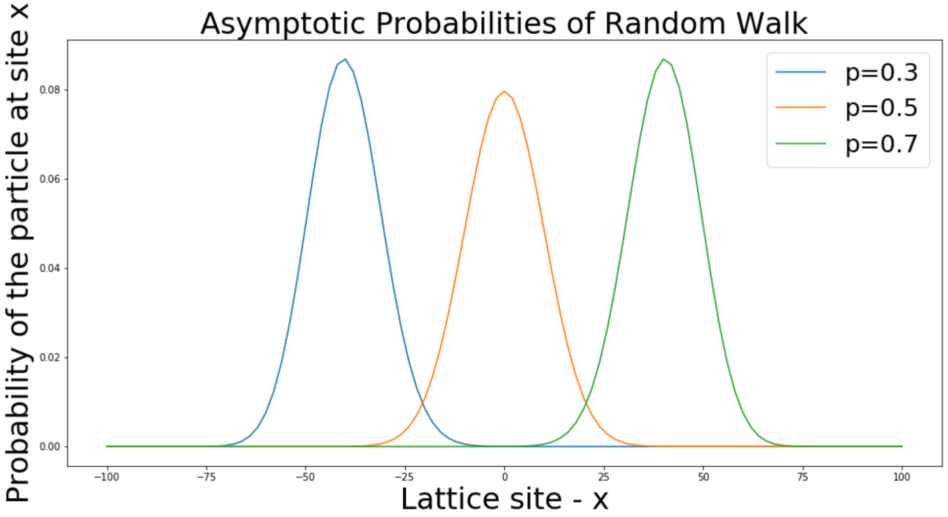
- $\mu(t+1) = \mu(t)T$, where $\mu \in \mathbb{R}^N$
- T can be used to obtain the asymptotic probability distribution π
- Here $\pi = \{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\}$, when $p = \frac{1}{2}$, where $\pi \in \mathbb{R}^N$

Random Walk on infinite 1D chain

- For an infinite 1D chain :
- The state space for infinite $1D$ Random Walk: $\{0, \pm 1, \pm 2, \pm 3 \ldots\}$

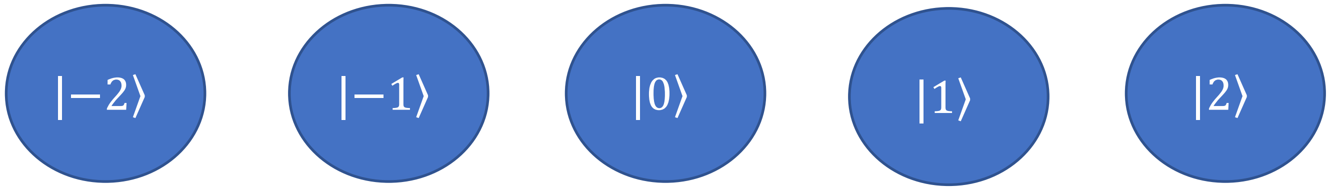


- Asymptotic probability distribution is Gaussian with the position of the peak determined by the bias of the coin.



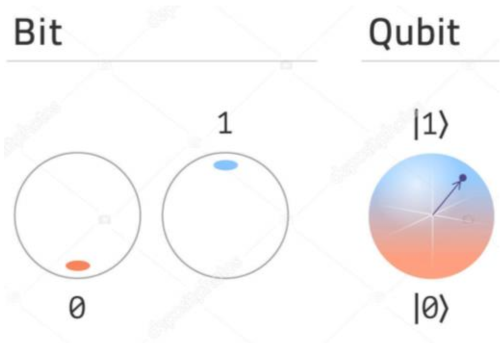
Some Postulates of Quantum Mechanics

- Hilbert space is a vector space of quantum states (represented by kets-$|\psi\rangle$) that is complete.
- Orthonormal basis states representing the position of the particle: $\{|i\rangle\}$



- Orthonormality of the basis quantum states $\langle i|j\rangle = \delta_{ij}$
- Uniquely quantum - Superposition of states is also a state in the Hilbert space.

$\{|0\rangle, |1\rangle\}$



$\{\alpha|0\rangle + \beta|1\rangle\}$

Some Postulates of Quantum Mechanics

- Observable of a physical system is described by an operator that acts on the kets.

$$\hat{p}_x|\psi\rangle = -i\hbar\frac{\partial}{\partial x}|\psi\rangle$$

- The only possible result of the measurement of an observable *A* is one of the eigenvalues of the corresponding operator

$$A|\psi_a\rangle = a|\psi_a\rangle$$

Some Postulates of Quantum Mechanics

- Immediately after the measurement of an observable A has yielded a value a_n, the state of the system is the normalized eigenstate $|\psi_{a_n}\rangle$.

- The time evolution of a quantum system is given by some unitary operator *U* that preserves the normalization of the associated ket.

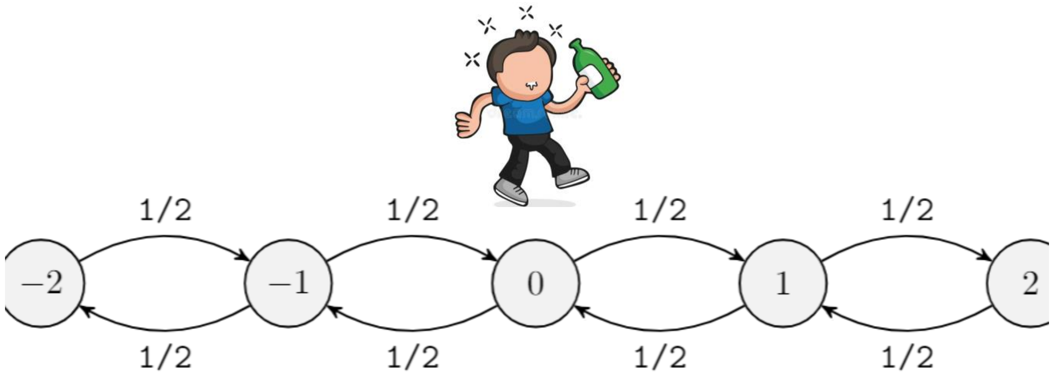
$$|\psi(t)\rangle = \mathrm{U}(\mathrm{t}, \mathrm{t}_0)|\psi(t_0)\rangle$$

i.e,

$$\sum_i |\langle i|\psi(t_0)\rangle|^2 = 1 = \sum_i |\langle i|\psi(t)\rangle|^2$$

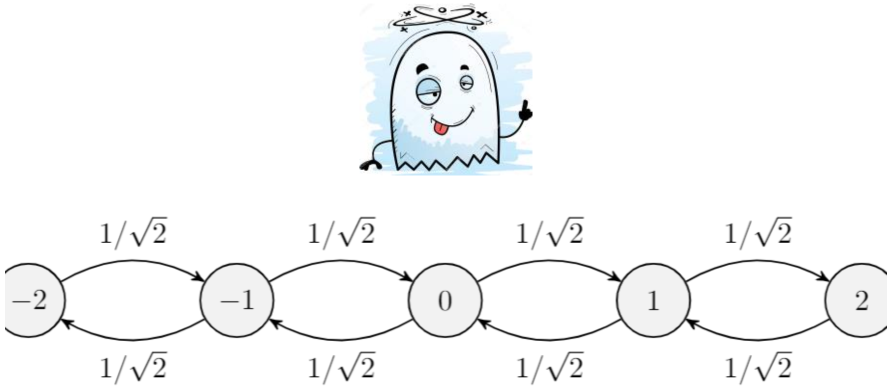
Quantizing the Random Walk

Classical Random Walk



Matrix elements of T between Markov States

Quantum Walk



, Matrix elements between basis quantum states

- Probability distribution after one transition

$\mu(t+1) = \mu(t)T$, where $\mu \in \mathbb{R}^N$

- Such that $\sum_i \mu_i(t) = 1 = \sum_i \mu_i(t+1)$

- Probability distribution after one transition

$|\psi(t+1)\rangle = U\psi(t)$, where $\psi \in \mathbb{C}^N$

- Such that ,

$\sum_i |\langle i|\psi(t)\rangle|^2 = 1 = \sum_i |\langle i|\psi(t+1)\rangle|^2$

Quantizing the Random Walk-Unitary Operator

Coin operator-Direction movement

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

$$U = (I \otimes H) \cdot S$$

Shift operator that moves the object

S= $|R\rangle\langle R| \otimes \sum_j |j+1\rangle\langle j|$

$$+|L\rangle\langle L| \otimes \sum_j |j\rangle\langle j-1|$$

$S = S^R + S^L$

$$U = (I \otimes H)S^R + (I \otimes H)S^L$$

$$T^{(R)} = p\begin{bmatrix}0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$T^{(L)} = (1-p)\begin{bmatrix}0 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\end{bmatrix}$$

Mapping the steps of a Quantum Walk

- If the object starts in the state

$$\psi(0) = |0R\rangle$$

- First step, $\psi(1) = U\psi(0)$,

$$\psi(1) = \frac{1}{\sqrt{2}}|-1L\rangle + \frac{1}{\sqrt{2}}|1R\rangle$$

Applying H $\longrightarrow \frac{1}{\sqrt{2}}(|0R\rangle + |0L\rangle)$

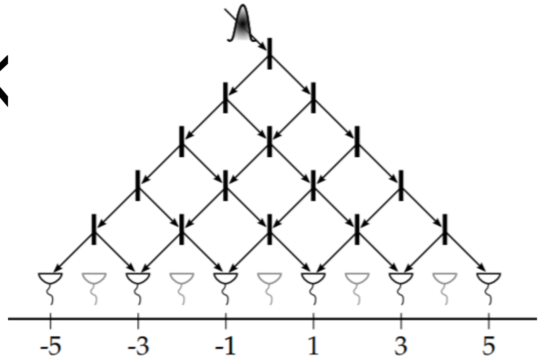
Applying S $\longrightarrow \frac{1}{\sqrt{2}}(|1R\rangle + |-1L\rangle)$

- Second step, $\psi(2) = U\psi(1)$,

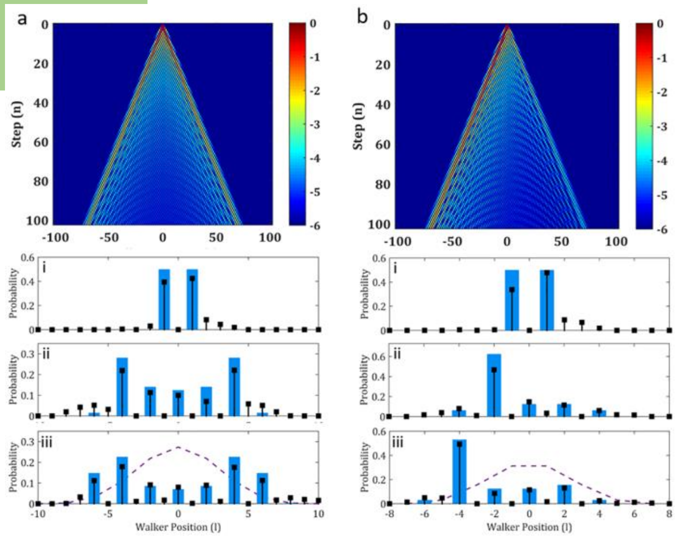
$$\psi(2) = \frac{1}{2}(|-2L\rangle + |0L\rangle + |0R\rangle - |2R\rangle)$$

- Third step, $\psi(3) = U\psi(2)$,

$$\psi(3) = \frac{1}{\sqrt{8}}(|-3L\rangle + |-1R\rangle + 2|1R\rangle + |1L\rangle + |3R\rangle)$$



The optical Galton board implementation of quantum walks on the line. The thick lines represent beamsplitters.

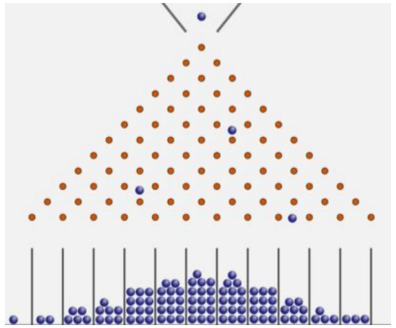


Random walk vs Quantum Walk

Random Walk

- The probability of being at position *i* after *T* steps of the classical random walk on the line starting in 0.

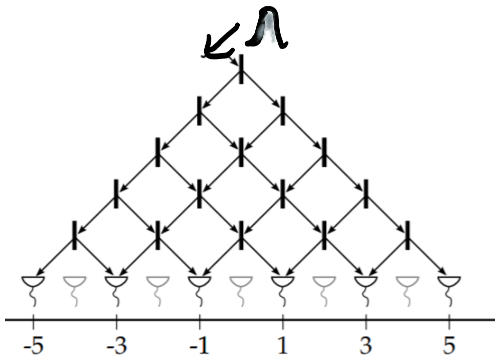
T \ i	-5	-4	-3	-2	-1	0	1	2	3	4	5
0						1					
1					$\frac{1}{2}$		$\frac{1}{2}$				
2				$\frac{1}{4}$		$\frac{1}{2}$		$\frac{1}{4}$			
3			$\frac{1}{8}$		$\frac{3}{8}$		$\frac{3}{8}$		$\frac{1}{8}$		
4		$\frac{1}{16}$		$\frac{1}{4}$		$\frac{3}{8}$		$\frac{1}{4}$		$\frac{1}{16}$	
5	$\frac{1}{32}$		$\frac{5}{32}$		$\frac{5}{16}$		$\frac{5}{16}$		$\frac{5}{32}$		$\frac{1}{32}$



Quantum Walk

- The probability of being found at position *i* after *T* steps of the quantum random walk on the line, with the initial state

T \ i	−5	−4	−3	−2	−1	0	1	2	3	4	5
0						1					
1					$\frac{1}{2}$		$\frac{1}{2}$				
2				$\frac{1}{4}$		$\frac{1}{2}$		$\frac{1}{4}$			
3			$\frac{1}{8}$		$\frac{5}{8}$		$\frac{1}{8}$		$\frac{1}{8}$		
4		$\frac{1}{16}$		$\frac{5}{8}$		$\frac{1}{8}$		$\frac{1}{8}$		$\frac{1}{16}$	
5	$\frac{1}{32}$		$\frac{17}{32}$		$\frac{1}{8}$		$\frac{1}{8}$		$\frac{5}{32}$		$\frac{1}{32}$

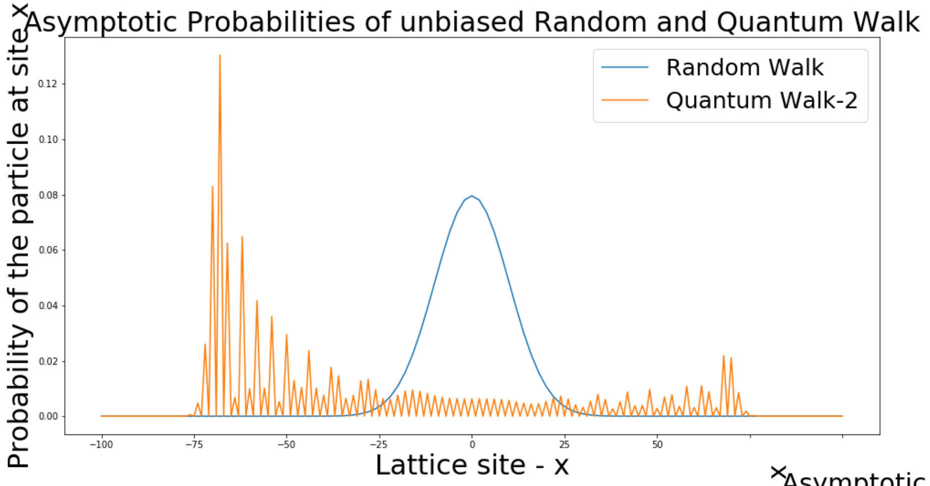


Quantum Walk

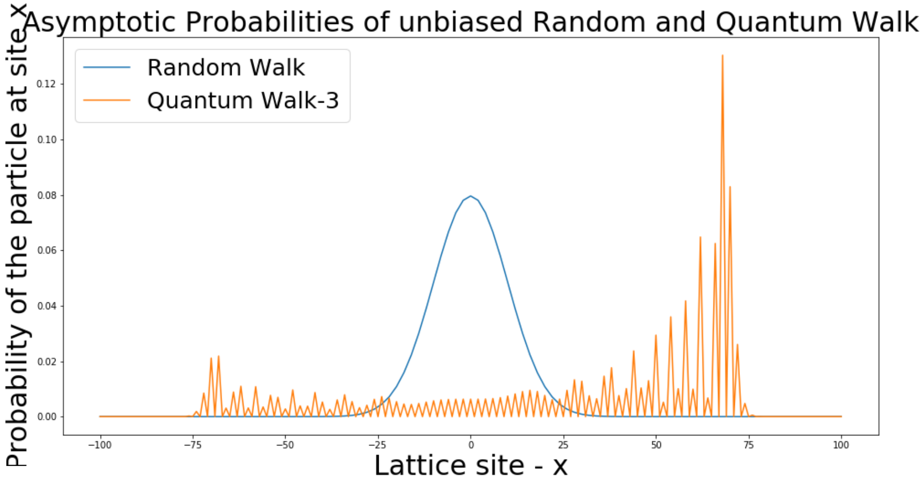
- Asymptotic probability distribution of single particle quantum walk
- Eigenstates of the Unitary operator - $U = (I \otimes H) \cdot S$
- Eigenvalues of a Unitary operator are of the form $U|\psi_\lambda\rangle = e^{i\lambda}|\psi_\lambda\rangle$
- The unitary operator has multiple eigenvalues with absolute value 1, therefore there are multiple asymptotic probability distributions

Quantum Walk

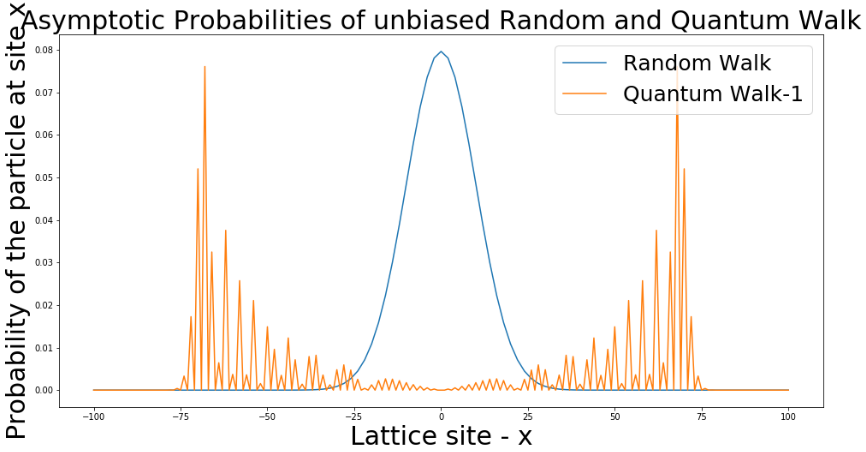
- Direction of the initial state determines which asymptotic state is attained.



$$\psi(0) = |0L\rangle$$



$$\psi(0) = |0R\rangle$$

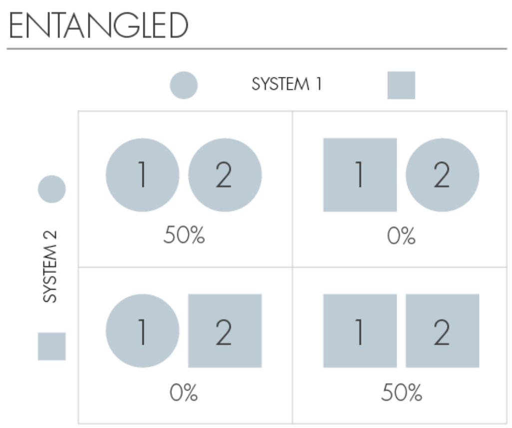


$$\psi(0) = \frac{1}{\sqrt{2}}(|0R\rangle + i|0L\rangle)$$

Two-particle Quantum Walk – non-interacting

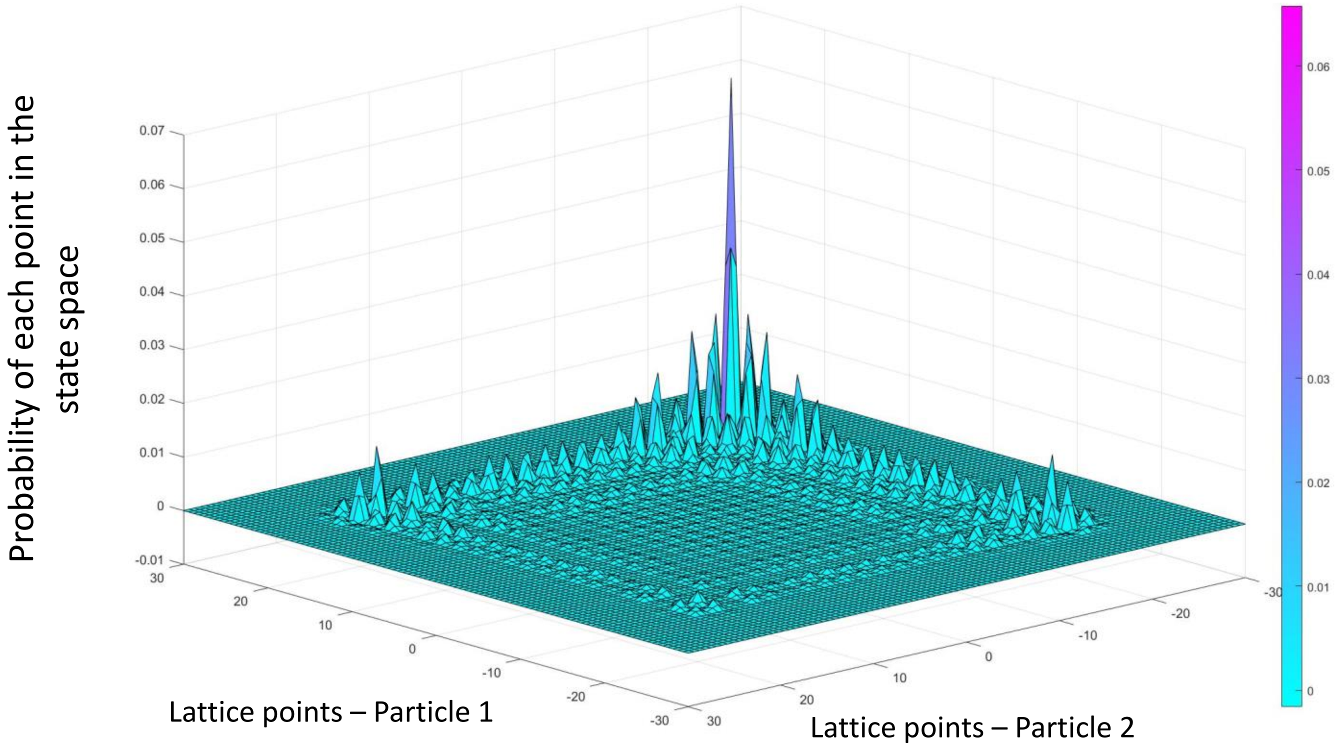
- The unitary operator for two non-interacting Quantum walkers is
- $U = U_1 \otimes U_2$, $\qquad U_1 = (I \otimes H_1) \cdot S_1, U_2 = (I \otimes H_2) \cdot S_2$

- Two-particle quantum walk allows us to study the effects of indistinguishability and entanglement(another uniquely quantum phenomena (allegedly)).



Two-particle Quantum Walk – non-interacting

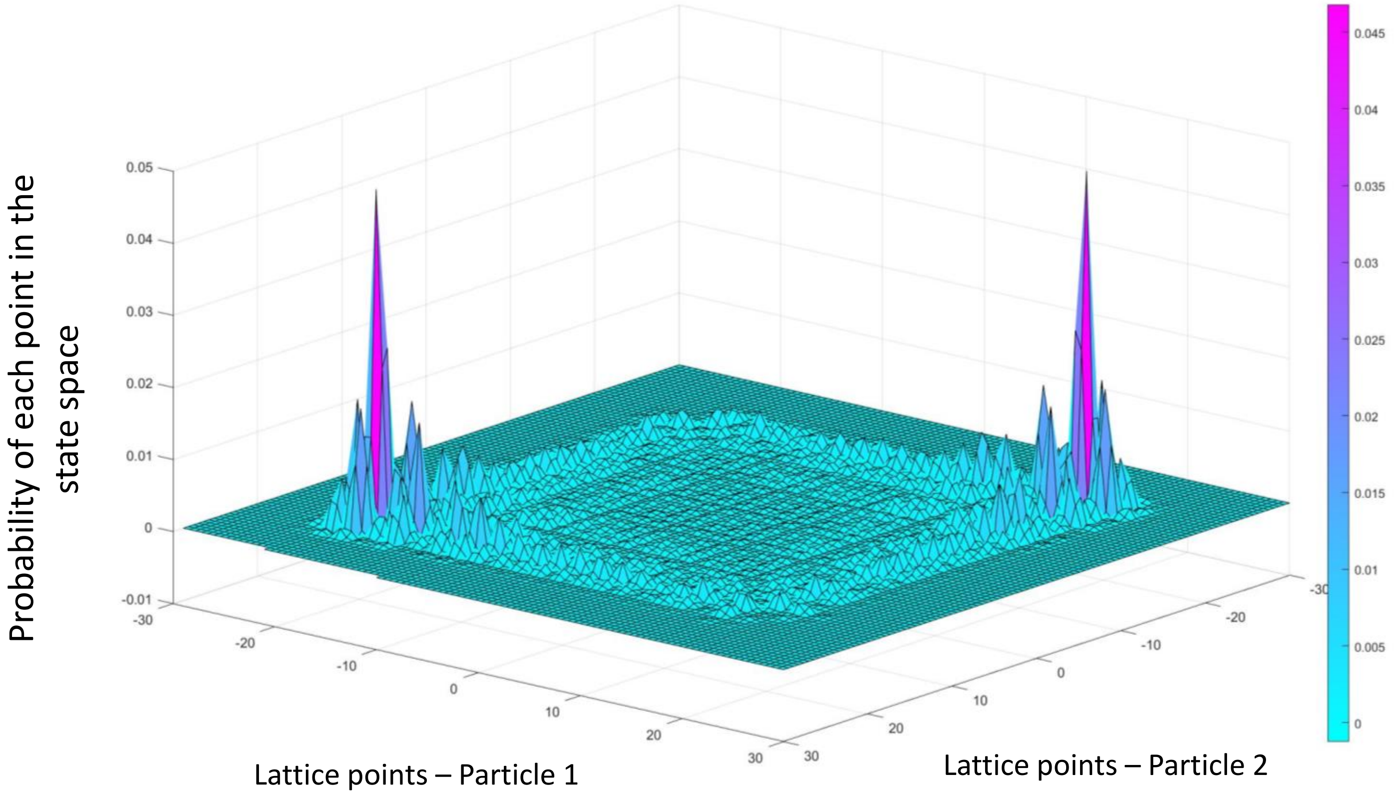
- Numerically obtained probability distribution for two non-interacting Quantum walkers starting in pure state $\psi(0) = |0L\rangle_1 |0R\rangle_2$.



The asymptotic probability distribution is product of the single particle probability distribution with one walker drifting to left and the other to right

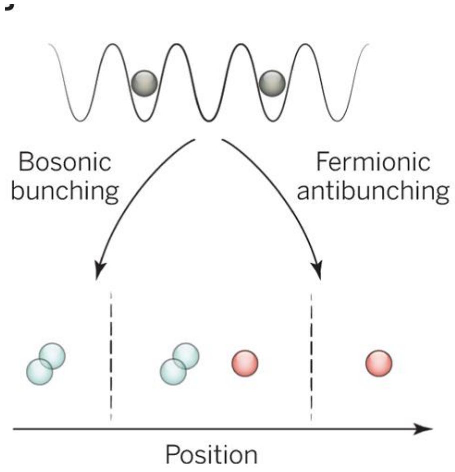
Two-particle Quantum Walk

- Numerically obtained probability distribution for two non-interacting quantum walker starting in the entangled antisymmetric fermionic state



$$\psi(0) = \frac{1}{\sqrt{2}}(|0L\rangle_1|0R\rangle_2 - |0R\rangle_1|0L\rangle_2)$$

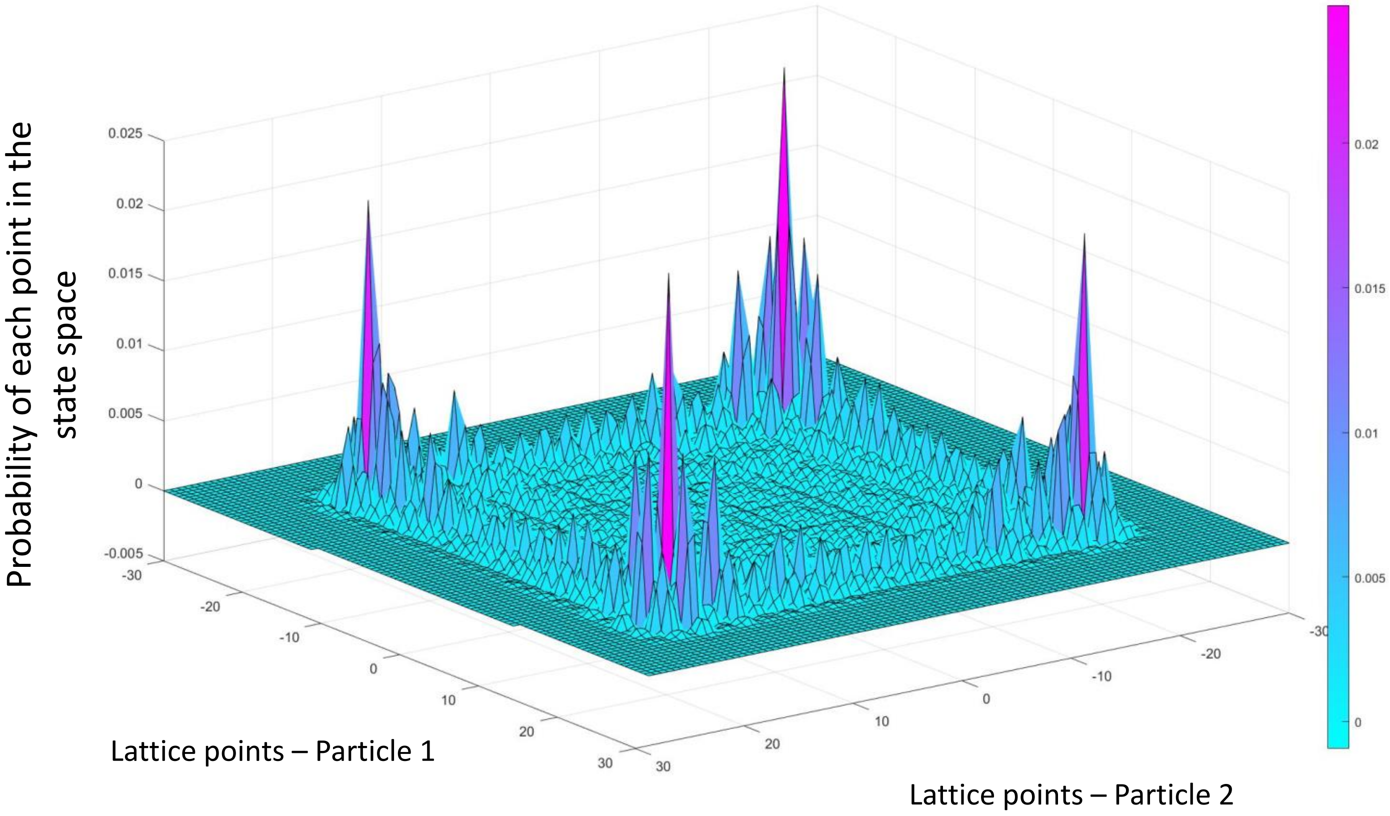
The asymptotic probability distribution is not a simple product of the single particle probability distribution. We see that the fermions move away from each other.



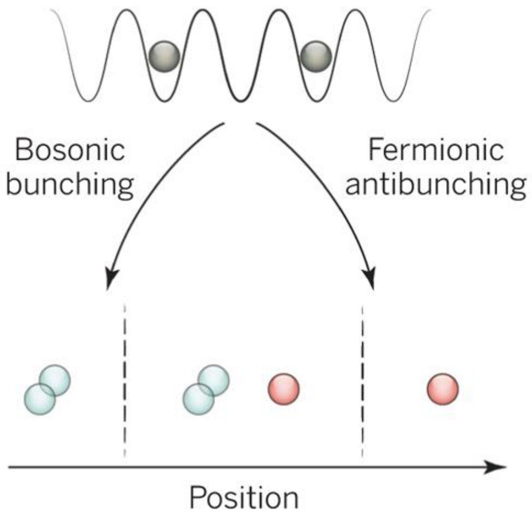
Two-particle Quantum Walk

- Numerically obtained probability distribution for two non-interacting quantum walker starting in the entangled symmetric boson state

$$\psi(0) = \frac{1}{\sqrt{2}}(|0L\rangle_1|0R\rangle_2 + |0R\rangle_1|0L\rangle_2)$$



The asymptotic probability distribution is not a simple product of the single particle probability distribution. We see that the bosons move towards each other.



Future work

- Quantum Markov Chain construction of the quantum walk.
- Compare the hitting time for classical and quantum random walkers on a graph using this construction
- Effect of the coin operator on the hitting time.
- Compare and calculate correlations and entropy rate of multiparticle quantum walks especially examining the role of entanglement.
- Entropy rate when the two coins H_1 and H_2 are entangled

Thank You

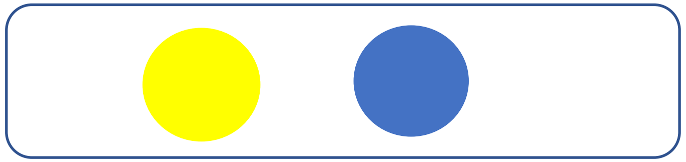
Mixed state vs Superposition of States

Mixed State

- Density Matrix of a mixed state.

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Superposition of states

- Density matrix of a superposed state.

$$\rho = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)$$

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Two Particle-Random Walk

- The state space for a $2D$ Random Walk: $\{[i,j]\}\ \ i,j \in [-N,N]$

$$T_1 = T_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- Transition matrix for a 2D Random walk.

$$T = T_1 \otimes T_2$$

- T is then used to obtain the asymptotic probability distribution π where $\pi \in \mathbb{R}^{N\times N}$

Quantum Walk

- Asymptotic probability distribution of single particle quantum walk
- Eigenstates of the Unitary operator

$$U = (I \otimes H) \cdot S$$

- Eigenvalues of a Unitary operator are of the form $U|\psi_\lambda\rangle = e^{i\lambda}|\psi_\lambda\rangle$
- Multiple eigenstates as there are multiple eigenvalues with absolute value 1

Coin operator has two eigenstates whose eigenvalues have absolute value 1.

- Diagonalizing S involves transforming to the momentum basis.
- Shift operator has two eigenstates whose eigenvalues have absolute value 1.