Searching the minimal Quantum ϵ - Machine

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Overview

The presentation is organized as follows:

- ϵ Machine and formalism of Quantum ϵ Machine
- Examples of Quantum ϵ Machine
- Discussion and future scope of work

ϵ Machine

- Causal states: s_i, $i = 1, \cdots, n$
- Output alphabet: $\mathcal{A}$
- Transition matrices: T^a_{ij}, $a \in \mathcal{A}$
- Property : Unifilarity

Formalism of Quantum ϵ Machine

- Causal states: $|\psi_i\rangle$, $i = 1, \cdots, n$ (Normalized)
- Operators: $\mathcal{T}^a$, $a \in \mathcal{A}$
- Unifilarity: $\mathcal{T}^a|\psi_i\rangle = U_{ij}^a|\psi_j\rangle \sim |\psi_j\rangle$

Probability: Suppose we are currently in state ψ_i, the probability of observing 01 is given by:

$$\text{Prob} = \sum_j |\langle\psi_i|\mathcal{T}^0\mathcal{T}^1|e_j\rangle|^2 \tag{1}$$

where e_j are basis vectors. We can rewrite the probability as:

$$\text{Tr}(|\psi_i\rangle\langle\psi_i|\mathcal{T}^0\mathcal{T}^1(\mathcal{T}^1)^\dagger(\mathcal{T}^0)^\dagger) \tag{2}$$

In general if we are in ρ the probability of observing $a_1 \cdots a_n$ is:

$$\text{Tr}(\rho\mathcal{T}^{a_1}\cdots\mathcal{T}^{a_n}(\mathcal{T}^{a_n})^\dagger\cdots(\mathcal{T}^{a_1})^\dagger) \tag{3}$$

Properties of Operators and States

- For any state ρ, the total probability of observing all outputs is 1:

$$1 = \sum_a \mathrm{Tr}(\rho \mathcal{T}^a (\mathcal{T}^a)^\dagger) \tag{4}$$

This is true for any states so we must ask

$$\sum_a \mathcal{T}^a (\mathcal{T}^a)^\dagger = \mathbf{1} \tag{5}$$

Properties of Operators and States

- The transition probabilities are encoded in this way:

$$\text{Prob(i} \to \text{a, j)} = \text{Tr}(|i\rangle\langle i|\mathcal{T}^a(\mathcal{T}^a)^\dagger) = |U_{ij}^a|^2 = \Pr(j, a|i) \qquad (6)$$

  In general we have:

$$\mathcal{T}^a|i\rangle = \sqrt{\Pr(j, a|i)}\exp(i\theta_{ij}^a)|j\rangle \qquad (7)$$

- The inner product of two states are given by:

$$\begin{aligned} \langle i|j\rangle &= \langle i|\mathbf{1}|j\rangle = \langle i|\sum_a \mathcal{T}^a(\mathcal{T}^a)^\dagger|j\rangle \qquad (8) \\ &= \sum_a \sqrt{\Pr(l, a|j)\Pr(k, a|i)}(i\theta_{jl}^a - i\theta_{ik}^a)\langle k|l\rangle \end{aligned}$$

Properties of Operators and States

- For each transition $i \to j$ with output a we have a phase parameter θ_{ij}^{a}
- We want to find out the parameters that minimize the von Neumann entropy

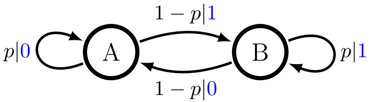


Figure 1: Perturbed Coin

$$\mathcal{T}^0 \, |A\rangle = \sqrt{p} \; \exp(i\theta_1) \; |A\rangle \qquad \mathcal{T}^0 \, |B\rangle = \sqrt{1-p} \; \exp(i\theta_2) \; |A\rangle \qquad (9)$$
$$\mathcal{T}^1 \, |B\rangle = \sqrt{p} \; \exp(i\theta_3) \; |A\rangle \qquad \mathcal{T}^1 \, |A\rangle = \sqrt{1-p} \; \exp(i\theta_4) \; |B\rangle$$

$$\begin{aligned} \langle B|A\rangle &= \sqrt{p}\sqrt{1-p}\Big(\exp(i\theta_1 - i\theta_2) + \exp(i\theta_4 - i\theta_3) \Big) \\ &= \sqrt{p}\sqrt{1-p} \; r \exp(i\theta) \end{aligned} \qquad (10)$$

where $r \in [0, 2]$ and $\theta \in [0, 2\pi]$.

Perturbed Coin II

The state we prepare is:

$$\rho = \frac{1}{2}|A\rangle\langle A| + \frac{1}{2}|B\rangle\langle B| \tag{11}$$

The eigenvalues are:

$$\lambda_1 = \frac{1}{2}\left(1 - r\sqrt{p - p^2}\right), \lambda_2 = \frac{1}{2}\left(1 + r\sqrt{p - p^2}\right) \tag{12}$$

When $r = 2$ von Neumann entropy obtains its minimum. They are the same as q-machine:

$$\begin{aligned} |B\rangle &= \sqrt{1-p}|e_0\rangle + \sqrt{p}|e_1\rangle \\ |A\rangle &= \sqrt{p}|e_0\rangle + \sqrt{1-p}|e_1\rangle \end{aligned} \tag{13}$$

where $|e_0\rangle$ and $|e_1\rangle$ are two orthogonal states.

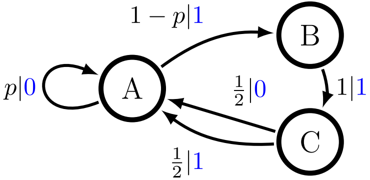


Figure 2: Nemo process

$$
\begin{aligned}
\mathcal{T}^0 \, |A\rangle &= \sqrt{p}\ \exp(i\theta_1)\ |A\rangle \\
\mathcal{T}^0 \, |B\rangle &= 0 \\
\mathcal{T}^0 \, |C\rangle &= \sqrt{1/2}\exp(i\theta_2)|A\rangle \\
\mathcal{T}^1 \, |A\rangle &= \sqrt{1-p}\ \exp(i\theta_3)\ |B\rangle \\
\mathcal{T}^1 \, |B\rangle &= \exp(i\theta_4)\ |C\rangle \\
\mathcal{T}^1 \, |C\rangle &= \sqrt{1/2}\exp(i\theta_5)|A\rangle
\end{aligned} \tag{14}
$$

Nemo Process II

$$\langle A|B\rangle = \frac{\sqrt{p(1-p)}}{1+p}\exp(i\theta_1 - \theta_2 - \theta_3 + \theta_5) \quad (15)$$

$$\langle B|C\rangle = \frac{\sqrt{p}}{1+p}\exp(i\theta_1 - \theta_2 - \theta_4 + \theta_5)$$

$$\langle A|C\rangle = \frac{\sqrt{2p}}{1+p}\exp(i\theta_1 - \theta_2)$$

We can choose phases such that $\langle A|B\rangle$ and $\langle B|C\rangle$ are real numbers:

$$\langle A|B\rangle = \frac{\sqrt{p(1-p)}}{1+p} \quad (16)$$

$$\langle B|C\rangle = \frac{\sqrt{p}}{1+p}$$

$$\langle A|C\rangle = \frac{\sqrt{2p}}{1+p}\exp(i\theta)$$

Nemo Process III

The mixed state we prepare is:

$$\rho = \frac{1}{3-2p}|A\rangle\langle A| + \frac{1-p}{3-2p}|B\rangle\langle B| + \frac{1-p}{3-2p}|C\rangle\langle C| \tag{17}$$

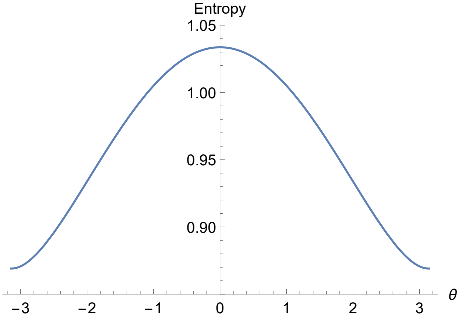
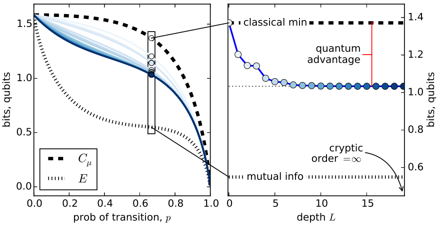


Figure 3: Entropy-θ for p=0.666 Nemo Process

Nemo Process IV



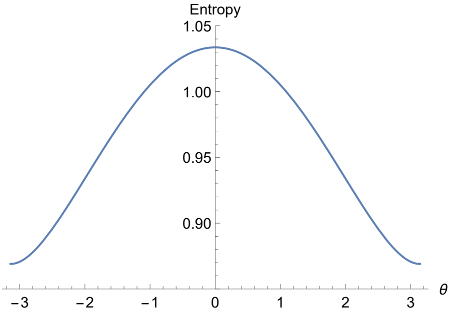


Figure 4: Entropy for p=0.666 Nemo Process. (Mahoney, Aghamohammadi, Crutchfield. 2016)

Nemo Process V

- For Nemo process we obtain the minimum entropy at

$$\begin{aligned}\langle A|B\rangle &= \frac{\sqrt{p(1-p)}}{1+p} \\ \langle B|C\rangle &= \frac{\sqrt{p}}{1+p} \\ \langle A|C\rangle &= -\frac{\sqrt{2p}}{1+p}\end{aligned} \tag{18}$$

- When p=$\frac{\sqrt[3]{9+\sqrt{87}}}{6^{2/3}} - \frac{1}{\sqrt[3]{6\left(9+\sqrt{87}\right)}} \sim 0.589755$, we can use only two states to simulate this process.

3 States MBW process

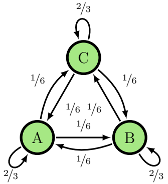


Figure 5: 3 States MBW process

- The inner products are given by:

$$\begin{aligned} \langle A|B\rangle &= r_{AB} \\ \langle B|C\rangle &= r_{BC} \\ \langle A|C\rangle &= r_{AC}\exp(i\theta) \end{aligned} \tag{19}$$

where $r_{AB}, r_{BC}, r_{AC} \leq \frac{5}{6}$ and
$1 + 2\, r_{AB}\, r_{BC}\, r_{AC} \cos(\theta) \geq r_{AB}^2 + r_{BC}^2 + r_{AC}^2$.

- The mixed state we prepare is:

$$\rho = \frac{1}{3}|A\rangle\langle A| + \frac{1}{3}|B\rangle\langle B| + \frac{1}{3}|C\rangle\langle C| \tag{20}$$

- The minimum are given at:

$$\begin{aligned} \langle A|B\rangle &= \frac{5}{6} \\ \langle B|C\rangle &= \frac{5}{6} \\ \langle A|C\rangle &= \frac{5}{6}\exp(i\theta) \end{aligned} \tag{21}$$

  where $\cos(\theta) = \frac{117}{125}$

- Three eigenvalues are $\{0, \frac{1}{18}\left(9-4\sqrt{3}\right), \frac{1}{18}\left(9+4\sqrt{3}\right)\}$.
- von Neumann entropy $S = 0.515109$ bits. For 3D q-machine($r_{AB}, r_{BC}, r_{AC} = \frac{5}{6}, \theta = 0$) $S_{q3} = 0.61$ bits and for 2D q-machine ($r_{AB}, r_{BC}, r_{AC} = \frac{1}{2}, \theta = \pi$) $S_{q2} = 1$ bits.(Loomis, Crutchfield 2018)

Future

- For larger $\epsilon-$Machine, how to find out the parameters that minimize von Neumann entroy
- By using mixed states to represent causal states can we do better?

THANK YOU!