Everything but the Kitchen Sync:
Synchronization and Desynchronization in Nonunifilar Hidden Markov Models

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Abstract

I explore how an observer synchronizes to and desynchronizes from a nonunifiliar hidden Markov model, and argue that it is natural to describe the synchronization behavior of these models within an active, dynamical framework. Very small steps are taken toward the construction of such a framework by reviewing tools that seem well-suited for studying synchronization in this way. Additionally, I define some possible indicators of a model's dynamical behavior, and describe an interesting way of computing them.

Contents

1 Introduction

"How does an observer synchronize to a process?" is a question that has been answered many times, in many ways [1–6]. A less frequently posed question is "How does an observer synchronize to a *nonunifilar model* of a process?", and a question that has, to my knowledge, never been asked in the literature is "How does an observer *desynchronize* from a *nonunifilar model* of a process?". When modeling a process, a unifilar machine is generally preferred for a number of reasons. However, there are cases when nonunifilar models are preferable, or are thrust upon us [3, 7], and in those cases it would be nice to have a complete understanding of their synchronization behavior. Reference [3] contains what I have found to be the most complete discussion of synchronization to nonunifilar models, where tools based on asymptotic forms and convergence rates are employed in the analysis. I would argue, however, that these kinds of tools are not the most natural way to study nonunifilar models. Synchronization is an inherently turbulent process for nonunifilar models; an observer is continually flung between various levels of uncertainty as she repeatedly synchronizes, desynchronizes, and resynchronizes with the model. Perhaps a more dynamical synchronization analysis would be more natural for these types of models?

The general goal of this report is to advocate for the development of a general *dynamical synchronization theory*. Obviously the development of such a theory is a tall order, so instead we focus on reviewing synchronization, discussing the effects of desynchronization, and introducing a few tools that seem uniquely useful for studying synchronization to nonunifilar models. In particular, in Subsection 3.4 we introduce the possibility machine from Ref. [5] and find that it is particularly useful, both as a tool for qualitative understanding and for quantitative computations.

2 Background

The situation depicted in Fig. 1 is the focus of this report. A black box produces symbols according to some stationary stochastic process. An observer receives the symbols generated by the black box, but cannot see its internal structure. As a result of some prior analysis of the process, the observer has obtained an accurate model of the process[1]. Since the observer cannot see inside the black box, she would like to use her model as a stand-in for the inner workings of the box. Now treating this model as if it were actually the generator producing the symbols, the observer attempts to determine the present state of the model, given the symbols she is seeing.

The models we will be studying are *hidden Markov models* (HMMs).

Definition 1. *A hidden Markov model* $M = \left(\boldsymbol{\mathcal{S}}, \mathcal{A}, \left\{T^{(x)}\right\}_{x\in\mathcal{A}}\right)$ *is defined by*

[1]In this report, we will only be concerned with models that are "accurate" in the sense that they exactly reproduce the word distributions of the observed process. We are only interested in valid *presentations* of the process.

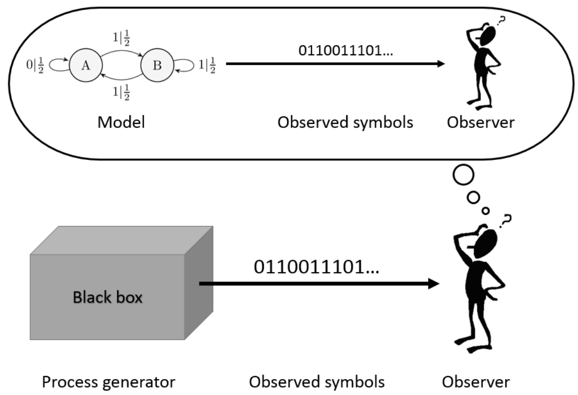

Figure 1: The "synchronization channel": An observer receiving symbols envisions that the symbols are produced by a model of the process generator. The observer attempts to ascertain the current state of the model, given the symbols she has observed.

- *a set of model states* $\boldsymbol{\mathcal{S}}$,
- *an alphabet of symbols* $\mathcal{A}$,
- *a set of symbol-labeled transition matrices* $\left\{T^{(x)}\right\}_{x\in\mathcal{A}}$, *where* $T^{(x)}_{ij}$ *is the probability of transitioning from state* $i \in \boldsymbol{\mathcal{S}}$ *to state* $j \in \boldsymbol{\mathcal{S}}$ *and generating the symbol* $x \in \mathcal{A}$.

In addition to this mathematical specification, we graphically represent HMMs as directed graphs with the edge labeling syntax

$$s \xrightarrow{x|T^{(x)}_{ss'}} s' \,.$$

We write the random variables for the model state and observed symbol at time t as $\mathcal{S}_t$ and X_t, respectively. The state and symbol time ordering during a transition is

$$\mathcal{S}_t \xrightarrow{X_t} \mathcal{S}_{t+1} \,.$$

To denote sequences of variables, we use the notation $X_{a:b} = X_a X_{a+1} \ldots X_{b-1}$.

We use *belief distribution* vectors to express the observer's uncertainty about the current internal state of the model at a particular time. The belief distribution vectors are elements of a vector space spanned by the standard basis $\{\langle \mathrm{e}_s | : s \in \boldsymbol{\mathcal{S}}\}$, where the components of $\langle \mathrm{e}_s|$ are zero, except for the component corresponding to state s, which equals 1: $\langle \mathrm{e}_s| = \begin{bmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{bmatrix}$. We define a general belief distribution via these basis vectors.

Definition 2. *Having observed the word* $x_{0:t}$, *the* belief distribution *at time* $t > 0$ *is*

$$\langle \mu_\alpha(x_{0:t})| = \sum_{s\in\boldsymbol{\mathcal{S}}} \Pr(\mathcal{S}_t = s \mid X_{0:t} = x_{0:t}, \mathcal{S}_0 \sim \alpha) \, \langle \mathrm{e}_s| \ ,$$

where α indicates the initial belief distribution.

Typically when discussing synchronization, we choose the initial belief distribution to be the stationary, asymptotic state distribution of the model $\langle\pi|$, so we set $\alpha = \pi$ for now. We will discuss other initial distributions when when we come to resynchronization and desynchronization.

Each time the observer sees a new symbol, her uncertainty about the current model state changes, and she must update her belief distribution to reflect this change. The distribution evolves in the following way:

$$\langle\mu_\pi(x_{0:t+1})| = \langle\mu_\pi(x_{0:t}x_t)| = \frac{\langle\mu_\pi(x_{0:t})|\, T^{(x_t)}}{\langle\mu_\pi(x_{0:t})|\, T^{(x_t)}\,|\mathbf{1}\rangle}\,, \tag{1}$$

where $|\mathbf{1}\rangle = \sum_{s\in\boldsymbol{\mathcal{S}}} |\mathrm{e}_s\rangle$. Part of the general time-evolution behavior of the distribution is directly determined by the *unifilarity* of the model.

Definition 3. *A* unifilar *HMM is one where knowledge of the current state and observed symbol uniquely determines the next state:*

$$H[\mathcal{S}_{t+1} \mid X_t = x, \mathcal{S}_t = s] = 0 \quad \textit{for all} \quad x \in \mathcal{A}, s \in \boldsymbol{\mathcal{S}}\,.$$

When an observer of a unifilar model becomes certain about the current state of the model, she remains certain about the current state forever, no matter how many additional symbols are observed. In general, however, HMMs need not satisfy the unifilarity condition. Those that do not are called *nonunifilar*.

Definition 4. *A* nonunifilar *HMM is one where knowledge of the current state and observed symbol is insufficient to uniquely determine the next state:*

$$H[\mathcal{S}_{t+1} \mid X_t = x, \mathcal{S}_t = s] > 0 \quad \textit{for at least one pair of} \quad x \in \mathcal{A} \quad \textit{and} \quad s \in \boldsymbol{\mathcal{S}}\,.$$

We call the state s in that symbol-state pair a nonunifilar state*.*

If an observer becomes certain about the current state of a nonunifilar model, she will not remain certain forever because encountering a nonunifilar state during time evolution can inject uncertainty into her belief distribution. As a result, the time evolution of the observer's belief distribution is much more dynamic in nonunifilar models: the distribution continually fluctuates between certainty and varying levels of uncertainty as more and more symbols are observed[2].

[2]For many nonuifilar models, it is impossible for an observer to ever reach certainty about the current state [8–10]. In this report, we will only consider nonuifilar models that are possible to synchronize to.

3 Synchronization

An observer starting off uncertain about the state of the model and attempting to determine the current state, like the scenario depicted in Fig. 1, is called *synchronization*.

Definition 5. *An observer is* synchronized *with the model at time t if she knows, with certainty, the current state of the model at time t:*

$$H[\mu_\pi(x_{0:t})] = 0\,.$$

The complementary mode of the observer's knowledge is when she is uncertain about the the current state of the model.

Definition 6. *An observer is* desynchronized *from the model at time t if she is uncertain about the current state of the model at time t:*

$$H[\mu_\pi(x_{0:t})] > 0\,.$$

During synchronization, the observer goes from being desynchronized to being synchronized.

3.1 The power automaton

To aid in our study of synchronization, we introduce the *power automaton* (PA) [6].

Definition 7. *The* power automaton $\mathrm{PA}(M) = \left(\boldsymbol{\mathcal{P}}, \mathcal{A}, \left\{L^{(x)}\right\}_{x\in\mathcal{A}}\right)$ *of a model* $M = \left(\boldsymbol{\mathcal{S}}, \mathcal{A}, \left\{T^{(x)}\right\}_{x\in\mathcal{A}}\right)$ *is defined by*

- *a set of states $\boldsymbol{\mathcal{P}}$ equal to the power set $\wp(\boldsymbol{\mathcal{S}})$,*
- *the alphabet $\mathcal{A}$ of M,*
- *a set of symbol-labeled Boolean transition matrices $\left\{L^{(x)}\right\}_{x\in\mathcal{A}}$, where $L^{(x)}_{ij} = \mathsf{True}$ if a transition from state $i \in \boldsymbol{\mathcal{P}}$ to state $j \in \boldsymbol{\mathcal{P}}$ exists, and $L^{(x)}_{ij} = \mathsf{False}$ otherwise.*

A state of the PA indicates the model states that correspond to the nonzero components in the observer's belief distribution, so it indicates what model states the observer thinks the model could be in. However, the PA completely does away with probabilities, so PA states simply indicate the presence or absence of probability in the components of the observer's belief distribution. Transitions between PA states reflect the fact that the observer's idea of possible current model states fluctuates over time; at some time she may believe that the model could be in any of the model states, and at another time she may believe that the model could only be in some subset of the states.

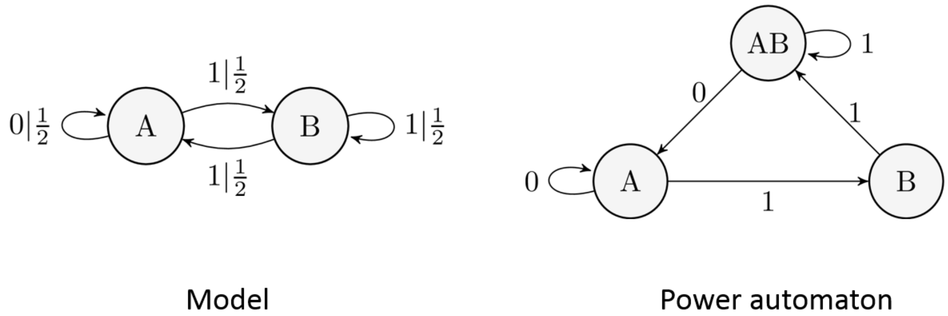

Figure 2: No Lonely Ones model and its associated power automaton.

The PA for the No Lonely Ones (NLO) model is shown in Fig. 2. The AB state represents belief distributions that contain nonzero probability for both model states A and B, while the A and B states indicate belief distributions where the observer has synchronized with the model. Given the PA, it is quite easy to read off synchronization behavior. If the observer starts out with the asymptotic state distribution as her initial belief distribution, she will start in the AB state of the PA. Given that she starts in the AB state, observing 1's will keep her in a state of desynchronization, but observing a 0 will cause her to synchronize to state A. Once synchronized in state A, she will remain synchronized indefinitely if she observes only 0's. However, observing a 1 sends her to the nonunifilar state B. She will then observe a second 1 and will end up desynchronized back in AB. This example nicely illustrates the utility of power automata when studying synchronization. Given a word generated by the model, the PA allows us to easily track the time evolution of the set of possible model states. We were already able to track the time evolution using the iterated matrix multiplication of Eq. (1), but it is unnecessary to compute the exact belief distribution each time if all one cares about is which states are possible. By eschewing the associated probabilities, the power automaton neatly captures the topological essentials of the synchronization dynamics.

3.2 Synchronizing words

In addition to aiding our qualitative understanding of synchronization, the power automaton is a structure from which we can determine sets of *synchronizing words* [6].

Definition 8. *A word w is* synchronizing *if the observer knows the current state of the model after seeing w:*

$$H[\mu_\pi(w)] = 0\,.$$

The set of length-ℓ synchronizing words is

$$W_{\text{sync}}(\ell) = \left\{ w \in \mathcal{A}^\ell : H[\mu_\pi(w)] = 0 \right\}.$$

Note that this definition depends only on the observer's uncertainty in the instant after seeing the last symbol of the word; it does not depend on what happened during the observation of the prefixes of the word. For example, 01100 is a valid synchronizing word for the NLO model in Fig. 2 even though the 011 prefix causes synchronization and then desynchronization.

In order to be more detailed with our synchronization analysis, we need to examine more closely how exactly a word causes an observer to synchronize. To illustrate why such a distinction might be necessary, consider the synchronizing words 111110 and 000000 of the NLO model. The words are of equal length, and they are both synchronizing, but the synchronization behaviors caused by the words are very different. During most of the length of 111110 the observer is in the AB state of the PA and is desynchronized from the model, only to synchronize on the very last symbol. For 000000 however, the observer synchronized on the first symbol and spends almost all of the word length synchronized to the A state. The fact that two words with such different behaviors are both equally described as "synchronizing" motivates the following definition:

Definition 9. *A word $x_{0:t}$ is* actively synchronizing *if it is synchronizing and its prefix $x_{0:t-1}$ is not synchronizing:*

$$H[\mu_\pi(x_{0:t})] = 0 \quad \textit{and} \quad H[\mu_\pi(x_{0:t-1})] > 0\,.$$

The set of length-ℓ actively synchronizing words is

$$\begin{aligned} W^{\mathrm{A}}_{\mathrm{sync}}(\ell) &= \{x_{0:\ell} \in W_{\mathrm{sync}}(\ell) : x_{0:\ell-1} \notin W_{\mathrm{sync}}(\ell-1)\} \\ &= \{x_{0:\ell} \in W_{\mathrm{sync}}(\ell) : H[\mu_\pi(x_{0:\ell-1})] > 0\}\,. \end{aligned}$$

An actively synchronizing word is one that causes synchronization upon the observation of its last symbol. In the NLO example given above, 111110 is an actively synchronizing word, while 000000 is merely synchronizing.

Another property we might wish to categorize words by is their *history.* A word's history is described by the effects of its prefixes. For example, in the NLO model, 0110 is an actively synchronizing word, but its prefix 0 is also actively synchronizing. In some situations, it may be necessary to only consider synchronizing words with no instances of synchronization in their history. For that reason, we define a third and final synchronizing word set.

Definition 10. *A word $x_{0:t}$ is a* minimal synchronizing word *if it is actively synchronizing and none of its prefixes are synchronizing:*

$$H[\mu_\pi(x_{0:t})] = 0 \quad \textit{and} \quad H[\mu_\pi(x_{0:t'})] > 0 \quad \textit{for all} \quad t' \in [0, t-1]\,.$$

The set of length-ℓ minimal synchronizing words is

$$\begin{aligned} W^{\mathrm{M}}_{\mathrm{sync}}(\ell) &= \left\{x_{0:\ell} \in W^{\mathrm{A}}_{\mathrm{sync}}(\ell) : x_{0:t} \notin W_{\mathrm{sync}}(t) \text{ for all } t \in [0, \ell-1]\right\} \\ &= \left\{x_{0:\ell} \in W^{\mathrm{A}}_{\mathrm{sync}}(\ell) : H[\mu_\pi(x_{0:t})] > 0 \text{ for all } t \in [0, \ell-1]\right\}. \end{aligned}$$

For the NLO model, the complete set of minimal synchronizing words is

$$W^{\mathrm{M}}_{\mathrm{sync}} = \bigcup_{\ell=1}^{\infty} W^{\mathrm{M}}_{\mathrm{sync}}(\ell) = \{(1)^n 0 : n \geq 0\} \, .$$

The set of minimal synchronizing words can be used to define the notions of average synchronization time and synchronization order [6], but these quantities will not be discussed in this report. Additionally, an algorithm to compute $W^{\mathrm{M}}_{\mathrm{sync}}(\ell)$ from the power automaton is given in Ref. [6]. That algorithm can be modified in order to compute $W_{\mathrm{sync}}(\ell)$ and $W^{\mathrm{A}}_{\mathrm{sync}}(\ell)$ as well.

3.3 Synchronization status and risk

For each of the synchronizing word sets introduced in the previous subsection, we can define an associated probability. Intuitively, each of these probabilities seems relevant to the description of the overall synchronization dynamics by virtue of its definition alone. However, I have not been able to formalize my idea of "synchronization dynamics", so I simply present the definitions below, without comment, to illustrate the kind of quantities I think may be important in this dynamical view of synchronization. Despite not entirely understanding the roles of these probabilities, I am able to show that they can be computed in a very interesting way. This is outlined in Subsection 3.5. Additionally, plots of these quantities, and their desynchronization counterparts, for the NLO model are in Section 5.

Definition 11. *The* synchronization status *is the probability that the observer is synchronized at time t:*

$$\begin{aligned}
\Upsilon_{\mathrm{sync}}(t) &= \Pr(H[\mu_\pi(X_{0:t})] = 0) \\
&= \sum_{x_{0:t} \in W_{\mathrm{sync}}(t)} \Pr(X_{0:t} = x_{0:t} \mid \mathcal{S}_0 \sim \pi) \\
&= \sum_{x_{0:t} \in W_{\mathrm{sync}}(t)} \langle \pi | \left(\prod_{i=0}^{t-1} T^{(x_i)} \right) | \mathbf{1} \rangle \, .
\end{aligned}$$

Definition 12. *The* absolute synchronization risk *is the probability that the observer synchronizes at time t:*

$$\begin{aligned}
\Upsilon^{\mathrm{A}}_{\mathrm{sync}}(t) &= \Pr(H[\mu_\pi(X_{0:t})] = 0, H[\mu_\pi(X_{0:t-1})] > 0) \\
&= \sum_{x_{0:t} \in W^{\mathrm{A}}_{\mathrm{sync}}(t)} \Pr(X_{0:t} = x_{0:t} \mid \mathcal{S}_0 \sim \pi) \\
&= \sum_{x_{0:t} \in W^{\mathrm{A}}_{\mathrm{sync}}(t)} \langle \pi | \left(\prod_{i=0}^{t-1} T^{(x_i)} \right) | \mathbf{1} \rangle \, .
\end{aligned}$$

Definition 13. *The* singular synchronization risk *is the probability that the observer synchronizes for the first time at time t:*

$$\begin{aligned}\Upsilon^{\mathrm{M}}_{\mathrm{sync}}(t) &= \Pr(H[\mu_\pi(X_{0:t})] = 0, H[\mu_\pi(X_{0:t-1})] > 0, \ldots, H[\mu_\pi(X_{0:1})] > 0) \\ &= \sum_{x_{0:t} \in W^{\mathrm{M}}_{\mathrm{sync}}(t)} \Pr(X_{0:t} = x_{0:t} \mid \mathcal{S}_0 \sim \pi) \\ &= \sum_{x_{0:t} \in W^{\mathrm{M}}_{\mathrm{sync}}(t)} \langle \pi | \left(\prod_{i=0}^{t-1} T^{(x_i)} \right) |\mathbf{1}\rangle \ .\end{aligned}$$

3.4 The possibility machine

In an attempt to gain a better understanding of the synchronization behavior of some nonunifilar model, we would like to somehow reincorporate probabilities into the power automaton representation, while still retaining the benefits we gained from the utility and simplicity of the PA. A natural first choice is to turn to the mixed state presentation (MSP) of the model. The MSP can be considered to be the canonical form of this "probabilistic power automaton" we are looking for since the states of the MSP are belief distributions. One complication with the MSP is the fact that mixed state presentations of nonunifilar models generally have an infinite number of states. While there has been considerable recent development in the treatment of these MSPs [8–10], it may be beneficial to see how far we can get with finite representations before we start wrestling with uncountably infinite mixed states.

Our "probabilistic power automaton" takes the form of the *possibility machine* (PM) [5]:

Definition 14. *The* possibility machine $\mathrm{PM}(M) = \left(\boldsymbol{\mathcal{J}}, \mathcal{A}, \left\{Q^{(x)}\right\}_{x\in\mathcal{A}}\right)$ *of a model* $M = \left(\boldsymbol{\mathcal{S}}, \mathcal{A}, \left\{T^{(x)}\right\}_{x\in\mathcal{A}}\right)$ *is defined by*

- *a set of states $\boldsymbol{\mathcal{J}} = \{(s,p) : s \in p, p \in \boldsymbol{\mathcal{P}}\}$, where $\boldsymbol{\mathcal{P}}$ is the set of states of the power automaton of M,*
- *the alphabet $\mathcal{A}$ of M,*
- *a set of symbol-labeled transition matrices $\left\{Q^{(x)}\right\}_{x\in\mathcal{A}}$.*

The elements of the transition matrices are given by

$$Q^{(x)}_{(s,p),(s',p')} = T^{(x)}_{ss'} \left[L^{(x)}_{pp'} \right] ,$$

where $L^{(x)}$ is a symbol-labeled transition matrix of the power automaton, and $[\cdot]$ is the Iverson bracket[3]*.*

[3] Recall that $L^{(x)}$ is a Boolean matrix.

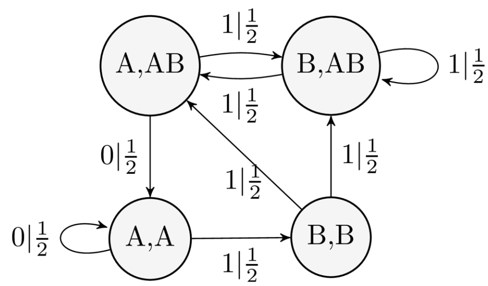

Figure 3: The possibility machine for the No Lonely Ones model in Fig. 2.

The possibility machine for the NLO model is shown in Fig. 3. The PM can be thought of as a sort of "product" of the model and its power automaton. Its states can be interpreted as simultaneously indicating the true current state of the model, and what model states the observer believes are currently possible. Therefore, the PM displays synchronization very transparently: the observer is synchronized if and only if the possibility machine's current state is (s, s) for some $s \in \boldsymbol{\mathcal{S}}$. Otherwise, the observer is desynchronized. We partition $\boldsymbol{\mathcal{J}}$ into two subsets: a set of synchronized states, and a set of desynchronized states.

Definition 15. *A possibility machine state is a* synchronized state *if it is a member of the set*

$$\mathbb{S} = \{(s, s) : s \in \boldsymbol{\mathcal{S}}\} .$$

Definition 16. *A possibility machine state is a* desynchronized state *if it is a member of the set*

$$\mathbb{D} = \boldsymbol{\mathcal{J}} - \mathbb{S} .$$

Additionally, we single out the "maximally desynchronized states" of the PM. We call them *initial states* because the PM starts in a distribution over those states at the beginning of a synchronization process, similar to how the observer's belief distribution starts in the the asymptotic state distribution.

Definition 17. *A possibility machine state is an* initial state *if it is a member of the set*

$$\mathbb{I} = \{(s, \boldsymbol{\mathcal{S}}) : s \in \boldsymbol{\mathcal{S}}\} .$$

3.5 Wordless computation of $\Upsilon_{\text{sync}}(t)$, $\Upsilon^{\text{A}}_{\text{sync}}(t)$, and $\Upsilon^{\text{M}}_{\text{sync}}(t)$

In Subsection 3.3, we defined the synchronization status $\Upsilon_{\text{sync}}(t)$ and risks $\Upsilon^{\text{A}}_{\text{sync}}(t)$ and $\Upsilon^{\text{M}}_{\text{sync}}(t)$ in terms of sets of synchronizing words. It may be surprising then to learn that the status and risks can be calculated directly from the PM, without needing to construct the synchronizing word sets.

Let $\{\langle\tilde{\mathrm{e}}_j|\}_{j\in\boldsymbol{\mathcal{J}}}$ be the standard basis representing the states of the PM. Define the initial state distribution over the PM states:

$$\langle\tilde{\pi}| = \sum_{(s,\boldsymbol{\mathcal{S}})\in\mathbb{I}} \langle\pi|\mathrm{e}_s\rangle \left\langle\tilde{\mathrm{e}}_{(s,\boldsymbol{\mathcal{S}})}\right| ,$$

where $\langle\pi|$ is the asymptotic state distribution of the model, and $|\mathrm{e}_s\rangle$ is a member of the model's standard basis. In addition to its definiton in terms of the synchronizing word set, the synchronization status $\Upsilon_{\mathrm{sync}}(t)$ is more generally defined as the probability that the observer is synchronized at time t. That is the same as the probability that the PM is in a synchronized state at time t. This probability is easily computed from the PM:

$$\Upsilon_{\mathrm{sync}}(t) = \sum_{j\in\mathbb{S}} \Pr(\mathcal{J}_t = j \mid \mathcal{J}_0 \sim \tilde{\pi}) = \sum_{j\in\mathbb{S}} \langle\tilde{\pi}|\, Q^t \,|\tilde{\mathrm{e}}_j\rangle ,$$

where $Q = \sum_{x\in\mathcal{A}} Q^{(x)}$.

Similarly, the absolute synchronization risk $\Upsilon^{\mathrm{A}}_{\mathrm{sync}}(t)$ can be computed directly from the PM. The absolute synchronization risk is the probability that the observer synchronizes at time t. Synchronization is very easy to detect on the PM. The observer synchronizes exactly when the PM transitions from a state in $\mathbb{D}$ to a state in $\mathbb{S}$. Therefore the absolute risk is just the total probability that the PM undergoes a synchronizing transition at time t:

$$\begin{aligned}
\Upsilon^{\mathrm{A}}_{\mathrm{sync}}(t) &= \sum_{j\in\mathbb{S}} \sum_{j'\in\mathbb{D}} \Pr\big(\mathcal{J}_t = j, \mathcal{J}_{t-1} = j' \mid \mathcal{J}_0 \sim \tilde{\pi}\big) \\
&= \sum_{j\in\mathbb{S}} \sum_{j'\in\mathbb{D}} \Pr\big(\mathcal{J}_t = j \mid \mathcal{J}_{t-1} = j', \mathcal{J}_0 \sim \tilde{\pi}\big) \Pr\big(\mathcal{J}_{t-1} = j' \mid \mathcal{J}_0 \sim \tilde{\pi}\big) \\
&= \sum_{j\in\mathbb{S}} \sum_{j'\in\mathbb{D}} \langle\tilde{\pi}|\, Q^{t-1} \left|\tilde{\mathrm{e}}_{j'}\right\rangle \left\langle\tilde{\mathrm{e}}_{j'}\right|Q|\tilde{\mathrm{e}}_j\rangle \\
&= \sum_{j\in\mathbb{S}} \langle\tilde{\pi}|Q^{t-1} P_{\mathbb{D}} Q|\tilde{\mathrm{e}}_j\rangle ,
\end{aligned}$$

where $P_{\mathbb{D}}$ is the projection operator over the desynchronized states:

$$P_{\mathbb{D}} = \sum_{j\in\mathbb{D}} |\tilde{\mathrm{e}}_j\rangle \langle\tilde{\mathrm{e}}_j| .$$

Perhaps most surprisingly, $\Upsilon^{\mathrm{M}}_{\mathrm{sync}}(t)$ can also be calculated without the set of minimal synchronizing words. The singular risk is equal to the probability that the observer synchronizes for the first time at time t. On the PM, that corresponds to the sum of probabilities of words with paths that start at the initial distribution, end at a synchronized state, and only ever pass through desynchronized states:

$$
\begin{aligned}
\Upsilon^{\mathrm{M}}_{\mathrm{sync}}(t) &= \sum_{j_0\in\mathbb{S}}\sum_{j_1\in\mathbb{D}}\cdots\sum_{j_{t-1}\in\mathbb{D}} \Pr(\mathcal{J}_t = j_0, \mathcal{J}_{t-1} = j_1, \ldots, \mathcal{J}_1 = j_{t-1} \mid \mathcal{J}_0 \sim \tilde{\pi}) \\
&= \sum_{j_0\in\mathbb{S}}\sum_{j_1\in\mathbb{D}}\cdots\sum_{j_{t-1}\in\mathbb{D}} \Pr(\mathcal{J}_t = j_0 \mid \mathcal{J}_{t-1} = j_1)\Pr(\mathcal{J}_{t-1} = j_1 \mid \mathcal{J}_{t-2} = j_2)\ldots \\
&\qquad\qquad\qquad\qquad \times \Pr(\mathcal{J}_2 = j_{t-2} \mid \mathcal{J}_1 = j_{t-1})\Pr(\mathcal{J}_1 = j_{t-1} \mid \mathcal{J}_0 \sim \tilde{\pi}) \\
&= \sum_{j_0\in\mathbb{S}}\sum_{j_1\in\mathbb{D}}\cdots\sum_{j_{t-1}\in\mathbb{D}} \left\langle \tilde{\pi} \middle| Q \middle| \tilde{\mathrm{e}}_{j_{t-1}} \right\rangle \left\langle \tilde{\mathrm{e}}_{j_{t-1}} \middle| Q \middle| \tilde{\mathrm{e}}_{j_{t-2}} \right\rangle \ldots \langle \tilde{\mathrm{e}}_{j_2} | Q | \tilde{\mathrm{e}}_{j_1} \rangle \; \langle \tilde{\mathrm{e}}_{j_1} | Q | \tilde{\mathrm{e}}_{j_0} \rangle \\
&= \sum_{j_0\in\mathbb{S}} \langle \tilde{\pi} | Q P_{\mathbb{D}} Q P_{\mathbb{D}} \ldots P_{\mathbb{D}} Q P_{\mathbb{D}} Q | \tilde{\mathrm{e}}_{j_0} \rangle \\
&= \sum_{j\in\mathbb{S}} \langle \tilde{\pi} | Q (P_{\mathbb{D}} Q)^{t-1} | \tilde{\mathrm{e}}_j \rangle \ .
\end{aligned}
$$

So we see that all three probabilities can be computed from just the PM, without any knowledge of the sets of synchronizing words. I do not know enough about the analysis of algorithms to be able to speak confidently about the efficiency of these formulas, but it seems like they would be more computationally efficient at large values of t compared to computing these probabilities from the synchronizing word sets.

4 Beyond synchronization

So far we have only discussed a very specific type of synchronization: synchronization given the observer's initial belief distribution is the asymptotic state distribution of the model. We have been assuming this because we have been considering the situation where the observer starts observing symbols at $t = 0$ and has no knowledge about any symbols prior to $t = 0$. In that case, the observer's best choice for the initial belief distribution is the asymptotic state distribution because we have also been assuming that the process generator is operating in its steady-state.

4.1 Resynchronization

Since we are looking at nonunifilar models from which an observer can desynchronize, the type of synchronization described above is no longer the only kind that can occur. For example, consider the Random Double One model in Fig. 4. If an observer's initial belief distribution is the asymptotic state distribution, the PA starts at ABC at $t = 0$. If the observer then sees the word 01, she is synchronized after the 0 and then she is desynchronized after the 1. Additionally, we notice that the ABC state is a transient state, so the observer will never return to it. So how do we describe *re*synchronization after seeing the word 01?

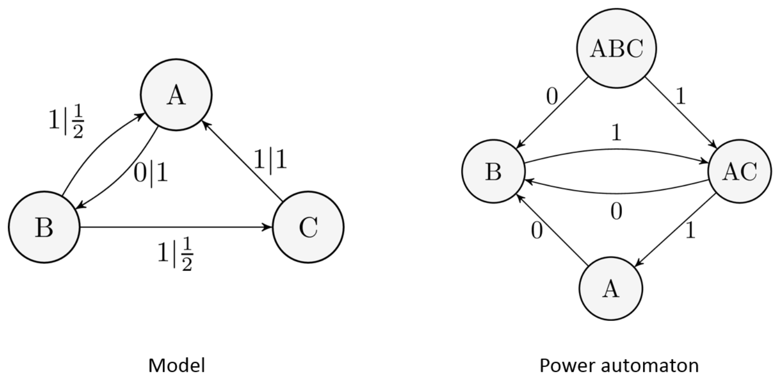

Figure 4: Random Double One model and its power automaton

Fortunately, very little about the structure of the formalism we've developed so far changes. The most significant change is the replacement of π in each equation with the new initial belief distribution

$$\langle\gamma| \equiv \langle\mu_\pi(01)| = \begin{bmatrix}1/2 & 0 & 1/2\end{bmatrix}.$$

Additionally, many quantities we have defined are functions of the initial belief distribution, but much of our notation does not make this explicit. For example, the synchronization status should really be written like $\Upsilon_{\text{sync}}(\pi, t)$ since π appears in Def. 11. This is obviously not just a notational distinction; if, for example, our observer wanted to compute the synchronization status for her synchronization from $\langle\gamma|$, she would have to compute $\Upsilon_{\text{sync}}(\gamma, t)$, where $\Upsilon_{\text{sync}}(\gamma, t) \neq \Upsilon_{\text{sync}}(\pi, t)$. The time indexing should also be shifted to accommodate this new viewpoint. If a word w is observed after the observer sees 01, the resulting belief distribution would be $\langle\mu_\pi(01w)|$ in the old notation and $\langle\mu_\gamma(w)|$ in the new notation. We shift $t = 0$ to match the observer's new location.

Besides these minor mathematical details, there is really nothing conceptually different about "synchronization" versus "resynchronization after desynchronization". Therefore, we will generally refer to either process as "synchronization" without discrimination.

4.2 Desynchronization

Synchronization and desynchronization are opposite processes. In synchronization, the uncertainty in the current model state is initially positive, then becomes zero after observing a synchronizing word. In desynchronization, the uncertainty in the current state is zero initially, then becomes positive after observing a desynchronizing word. In fact, synchronization and desynchronization are essentially exactly opposite such that understanding one is basically enough to understand the other. That is not to say that the implications or effects of synchronization and desynchronization are the same, only that they are mechanically very similar. As a result, essentially all of the quantities we defined for synchronization have well-defined desynchronization counterparts. However, translating from synchronization to desynchronization quantities is not as easy as it was for resynchronization due to the fact that their starting and ending points differ.

In general, desynchronization quantities, like resynchronization quantities, are inherently state-dependent. In actuality, most of the quantities we have defined depend explicitly on the observer's initial belief distribution, we have just been ignoring this since we have been assuming that the initial distribution is $\langle\pi|$. However, state dependence is perhaps a more significant property of desynchronization than it is for resynchronization, the reason being that the way an observer can desynchronize is highly-dependent on what model state the observer is initially synchronized to. For example, consider the NLO model in Fig. 2. If the observer is initially synchronized to A, it is possible that the observer only sees 0s and never desynchronizes. However, if the observer is initially synchronized to B, desynchronization is guaranteed to occur upon observation of the next symbol. Different model states can have wildly different desynchronization behaviors, so desynchronization is really fundamentally state-dependent. It is for this reason that I have stuck with probabilities and avoided using many information-theoretic in this report. Calculating something like $H[\mathcal{S}_t|X_{0:t}]$ mixes contributions from all the states together, and that might smear out some detail if we are trying to studying desynchronization. Despite only focusing on synchronization quantities for the bulk of the report, I have tried to make the quantities useful for both synchronization and desynchronization and easy to convert between the two, hence the absence of quantities like $H[\mathcal{S}_t|X_{0:t}]$. This is point that I believe might be important in a full-fledged dynamical synchronization theory. Quantities of interest should be agnostic about whether the observer is currently synchronizing or desynchronizing, so that the behavior can be tracked over any length of time. I know that the status and risk parameters I defined violate this rule, but you have to start somewhere.

So how do you convert a synchronization quantity into a desynchronization quantity? The first two steps are identical to the procedure for resynchronization, but the last one is unique for desynchronization:

1. Change the initial belief distribution.

2. Re-index time so that $t = 0$ coincides with the beginning of the desynchronization window.

3. Make sure to change any synchronization quantity in the expression into a desynchronization quantity, and vice versa.

For example, the equation for calculating the singular synchronization risk from the possibility machine is

$$\Upsilon^{\mathrm{M}}_{\mathrm{sync}}(t) = \sum_{j\in\mathbb{S}} \langle\tilde{\pi}|Q(P_{\mathbb{D}}Q)^{t-1}|\tilde{\mathrm{e}}_j\rangle \ ,$$

and its counterpart, the singular desynchronization risk, is

$$\Upsilon^{\mathrm{M}}_{\mathrm{desync}}(s,t) = \sum_{j\in\mathbb{D}} \left\langle \tilde{\mathrm{e}}_{(s,s)} \middle| Q(P_{\mathbb{S}}Q)^{t-1} \middle| \tilde{\mathrm{e}}_j \right\rangle .$$

5 No Lonely Ones plots

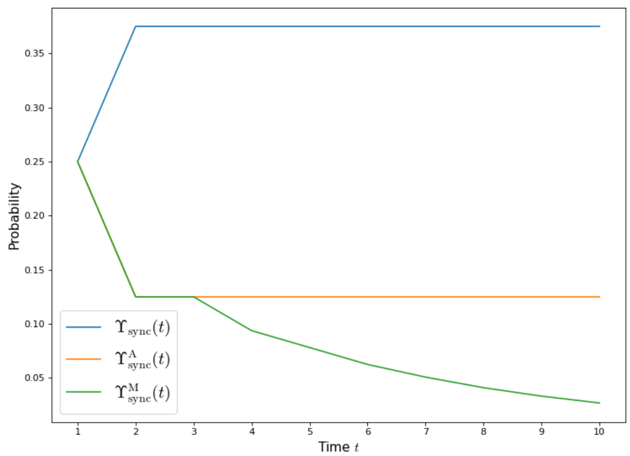

Figure 5: Synchronization status and risks for the No Lonely Ones model.

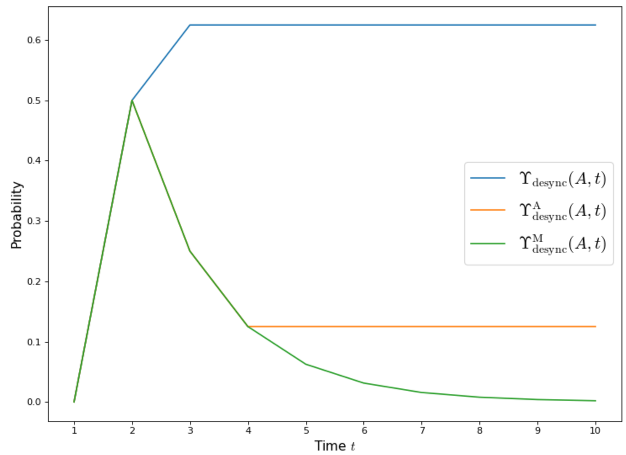

Figure 6: Desynchronization status and risks for the No Lonely Ones model. Initial state is A.

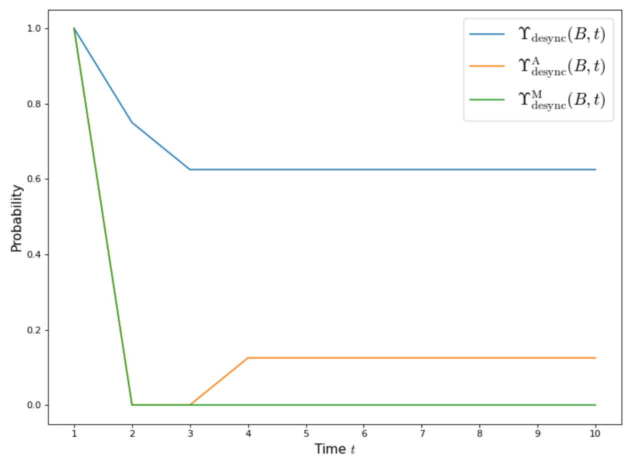

Figure 7: Desynchronization status and risks for the No Lonely Ones model. Initial state is B.

6 Conclusion

We began with a brief review of synchronization, and found the power automaton to be a very useful tool for constructing sets of synchronizing words, as well as for building a qualitative understanding of the active, volatile nature of synchronization to nonunifilar models. Following that, the synchronization status and risks were introduced as potentially useful quantities to probe dynamic synchronization behavior. The possibility machine of Ref. [5] proved to be a very natural and illustrative environment in which to study the dynamics, and it allowed us to obtain simple expressions for the synchronization status and risks. We also discussed the relative equivalence of synchronization, resynchronization, and desynchronization, establishing that an appropriate set of tools allows us to analyze all three processes from a unified viewpoint.

Clearly substantial progress must be made in these studies before they begin to form something resembling a general dynamical synchronization theory, but I hope my argument for why such a theory should be developed is at least somewhat convincing. Considerable progress has been made in describing the dynamics of unsynchronizable models [8–10], and I am hoping that some of the tools developed for that will help with describing synchronizable models as well. Additionally, the possibility machine construction appears to be uniquely useful for understanding synchronization dynamics, and for computing useful quantities. Further investigations into its utility for this topic should be performed.

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