Desynchronization in Nonunifilar Hidden Markov Models

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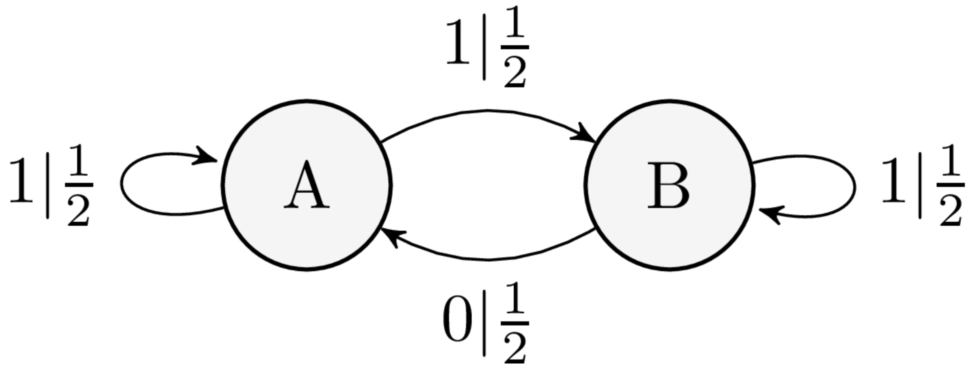
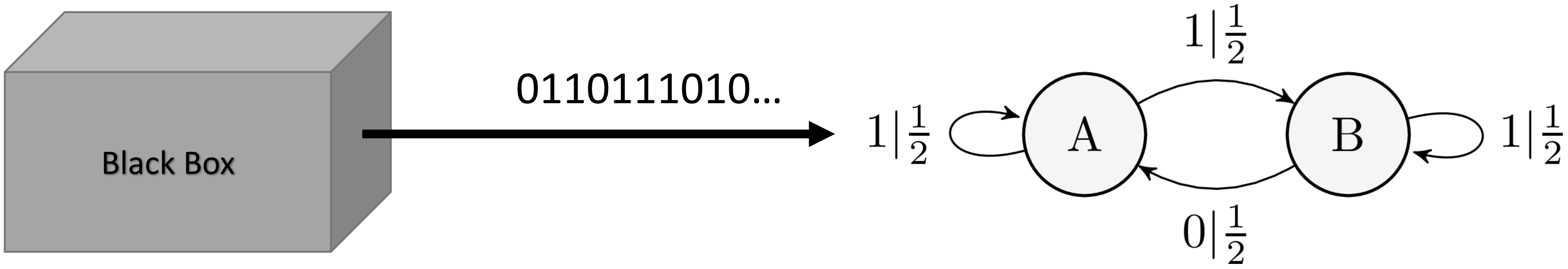


General Setup



Process generator

Observed symbols

Model
(presentation of process)

- Before any symbols are observed, the current model state is unknown (asymptotic state distribution)
- Each time a symbol is observed, update the state distribution

Notation & Conventions

Model states (not causal): S_t, s_t

Observed symbols: X_t, x_t

Sequences: $X_{a:b} = X_a X_{a+1} X_{a+2} \dots X_{b-1}$

Time ordering: $S_t \xrightarrow{X_t} S_{t+1}$

Notation & Conventions

Synced state distribution ("pure"): $\langle e_i| = \begin{bmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{bmatrix}$

General state distribution ("mixed"): $\langle \mu(x_{0:t})|$

$$\langle \mu(x_{0:t})|e_i\rangle = \Pr(S_t = i | X_{0:t} = x_{0:t})$$

Sync vs. Desync

Synchronization

- Initially unsynced: $\langle \mu(\lambda) | = \langle \pi |$
  $H[\mu(\lambda)] > 0$
- At some L, $H[\mu(x_{0:L})] = 0$
- Unifilar: $H[\mu(x_{0:t})] = 0 \quad \forall t \geq L$

Desynchronization

- Initially synced: $\langle \mu(x_{0:L}) | = \langle e_i |$
  $H[\mu(x_{0:L})] = 0$
- At some $K > L$, $H[\mu(x_{0:K})] > 0$
- Unifilar: Never happens

Interested in models that can sync and desync → Nonunifilar*

Desync Properties: Desyncing Words

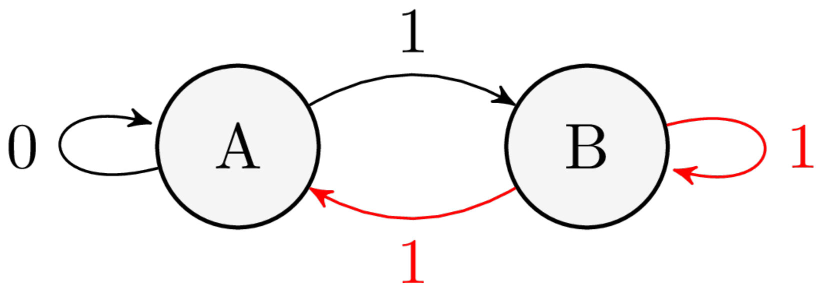
Set of minimal desynchronizing words for state s: $\mathcal{L}_{\mathrm{desync}}(s)$

For $w \in \mathcal{L}_{\mathrm{desync}}(s)$:

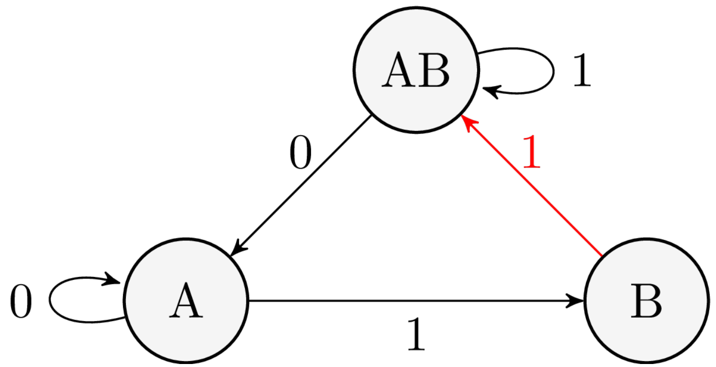
- $H[\mu(w)|S_0 = s] > 0$
- No prefix of w is also a desynchronizing word

Desync Properties: Desyncing Words

Model



Power automaton



$$\mathcal{L}_{\text{desync}}(A) = \{0^n 11 | n \in \mathbb{N}_0\}$$

$$\mathcal{L}_{\text{desync}}(B) = \{1\}$$

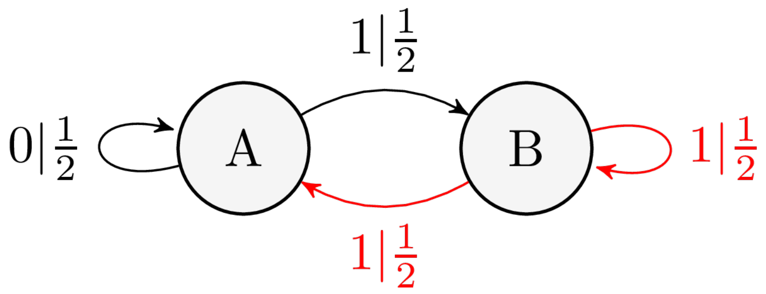
Desync Properties: Desync Risk

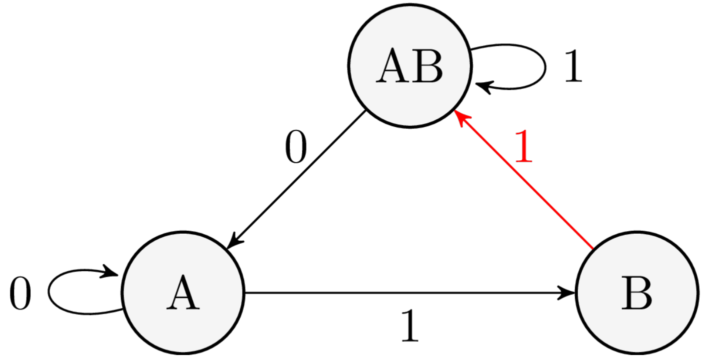
Desynchronization risk at time t: $\Upsilon(s,t)$

$$\Upsilon(s,t) = \sum_{w \in \mathcal{L}_{\mathrm{desync}}(s,t)} \Pr(X_{0:t} = w | S_0 = s)$$

$$\mathcal{L}_{\mathrm{desync}}(s,t) = \{ w \in \mathcal{L}_{\mathrm{desync}}(s) \,|\, |w| = t \}$$

Desync Properties: Desync Risk





$$\mathcal{L}_{\text{desync}}(A) = \{0^n 11 | n \in \mathbb{N}_0\}$$

$$\Upsilon(A, 0) = 0$$

$$\Upsilon(A, 1) = 0$$

$$\Upsilon(A, t) = \langle e_A | \left(T^{(0)}\right)^{t-2} \left(T^{(1)}\right)^2 |1\rangle = 2^{1-t} \quad \text{for} \quad t \geq 2$$

Desync Properties: Desync Risk

Since desynchronization is only ever caused by nonunifilar transitions, can calculate risk without $\mathcal{L}_{\text{desync}}(s, t)$

$$\begin{aligned}
\Upsilon(s,t) &= \sum_{w \in \mathcal{L}_{\text{desync}}(s,t)} \Pr(X_{0:t} = w | S_0 = s) \\
&\stackrel{?}{=} \sum_{s' \in N} \sum_{x \in \mathcal{L}_{\text{desync}}(s',1)} \Pr(S_{t-1} = s', X_{t-1} = x | S_0 = s) \\
&\stackrel{?}{=} \sum_{s' \in N} \sum_{x \in \mathcal{L}_{\text{desync}}(s',1)} \sum_{s''} T^{(x)}_{s's''} \left(T^{t-1}\right)_{ss'}
\end{aligned}$$

where N is the set of nonunifilar states

Open Questions

- Other relevant quantities?
  - Desynchronization order
  - Average desynchronization time
- Applications?
  - Quantum
  - Information engines
- Resynchronization (sync after desync)?
  - Synchronization dynamics?

Thank you!

Questions?