The Quantum Path Integral of a Free Particle as a State Machine: A PHY 256 Project

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The path integral formulation of non-relativistic quantum mechanics is a construction of the time evolution of a particle through a chain of probability amplitudes, or kernels. Each kernel is evaluated across a time-slice of our time interval of interest. This approach to time evolution in quantum mechanics draws us towards a state machine viewpoint for our particle – the particle evolving under a potential can be seen as transitioning across states over each discrete time slice, outputting a time series of observables by measurement. To achieve continuous spacetime, we simply take the number of slices to infinity and the slice size to 0. This final project for PHY 256 was a preparatory investigation for a much longer, deeper study into this state machine of time-evolving quantum systems through the path integral formulation. This report presents what findings and insights the project has provided so far, with a particular focus on the free particle as the working example.

I. INTRODUCTION

"I know that I know nothing" – Socrates(?)

While the attribution of the quote is the subject of dispute, its essence has been, in many ways, central to the development of physics over the past century. The advent of quantum mechanics gave us a description of nature that, rather than presuming one can know everything about the state of a system with perfect accuracy, admits that we can only see the observables produced by a system and not its internal state itself. A quantum system is encoded not immediately by its co-ordinates in a phase space of position and momentum, but by a complex-numbered, unobservable state function, onto which measurements are made in order to extract an observable carried by that state function.

Similar manners of thought have popped up in other areas of science – information theory is fundamentally the study of uncertainties in a system or communication channel, and the study of extrapolating system behaviour via vast amounts of empirical data rather than a priori assumptions have gained frightening momentum in recent years. Computational mechanics, as a certain merger of the two, views a sophisticated system as a "computer" that prints out empirical observables, and the objective is to identify what the underlying programme of the computer is based on the observables that we have seen from it. A presentation of the system as a machine of evolving states can be proposed as the underlying process of a system of interest, and we may analyse its information properties to see whether it truly is an optimal presentation of what the process actually is. The goal, then, naturally becomes to identify what would be the minimal state machine that wastes the least amount of information about the process.

A. Motivation

Here, one may ask a question: "how would time evolution of quantum systems, itself some process to be uncovered, be presented as a state machine?" All the broad notions line up quite well: we have an unobservable state function that, by execution of some process, evolves into another state function through time, and by taking measurements throughout its evolution we can track a series of output observables. Even if we do not know exactly what point in phase space this system may actually be in, we can still find what process the system is undergoing by some presentation of its evolution based on its observables.

Out of the several formulations of quantum mechanics we have, the path integral formulation draws particular attention. The more conventional Schrödinger and Heisenberg formulations define time evolution through a unitary operator acting on the state function across continuous spacetime. This formalism of operators in continuous spacetime has been incredibly powerful, due to its openings into group theory and thus connections into symmetry properties. However, what the operator physically does to the system through time can get somewhat obscured behind the mathematical language. As such, an alternative, more intuitive approach to quantum mechanics was desired. One such approach is the path integral formulation, which replaces the unitary operator in time evolution with transitional

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probabilities from one state to another across a set time interval. Essentially, transitions across infinitesimal time intervals are chained up into a history of states across the time interval, and by summing across all possible histories, we can find information about the final state of the system.

It was thus natural for me to consider the path integral formulation as the paradigm to construct state machines out of time-evolving quantum systems. One of my own long-term research interests is evolution of quantum systems in spacetime as state machines executing underlying processes. For non-relativistic quantum systems, the path integral approach seemed the most intuitive and similar to the ϵ-machine constructions given in our PHY 256 class. One may dissect the evolution of a system from some spacetime point to another as a chain of transitions from one internal state to another, and send the number of transitions in that chain to be very large, or in theory, infinity.

B. Why Is It Interesting?

The path integral interpreted as a state machine is of particular interest mainly due to two reasons: i) recent research has encouraged study of quantum processes as state machine presentations, and ii) such research has mostly been done under the conventional operator formalisms rather than the path integral formalism, where the state-to-state transitions are, at least at first glance, made much more explicit.

1. Recent Research and Quantum Causal States

Throughout the course of PHY 256B, we have found many instances of classical binary processes and their ϵ-machines. Each transition from one causal state to another carries a definitive coinciding observable with it and transition probability. There is, however, a subtlety to this. If we consider two causal states S_j, S_k both transition into the same causal state S_l with the same observable r, the machine, in the next time step, loses memory about whether it was previously in S_j or S_k unless the machine is also counifilar. This memory loss means additional bits are required in the machine to forget its past, so $C_\mu > \mathbf{E}$ and classical ϵ-machines are, in general, irreversible. This excess amount of required bits in executing the process means that, quite ironically, a classical state machine with definite observables for each transition is not optimal to present a classical process [1].

Thus, recent research [1] [2] has introduced *quantum* ϵ-machines, where a quantum causal state is defined as $|S_j\rangle = \sum_k \sum_{r\in\Sigma} \sqrt{T_{jk}^{(r)}}\,|r\rangle\,|k\rangle$, where $|r\rangle$ and $|k\rangle$ are orthogonal basis vectors on Hilbert spaces of sizes $|\Sigma|$ and $|S|$ respectively and $T_{jk}^{(r)}$ is the transition probability with output r from S_j to S_k. This eliminates the constraint that causal states have to be mutually orthogonal to each other, allowing a nonzero overlap $\langle S_j|S_k\rangle$ for two states that lead into the same causal state with the same observable. This nonzero overlap presents additional information to an observer that reduces the statistical complexity C_q from its classical value C_μ, making it more optimal than the classical machine constructed from mutually orthogonal causal states.

2. Complex Kernels

Notice the square root on the transition probabilities for quantum casual states as defined above. This square root is due to the structure of Hilbert space in that real-valued probabilities are defined through magnitudes squared of inner products of vectors that represent the state of the system. But for a given real magnitude, there can be an infinite number of complex values that give that same magnitude, so already we can foresee a redundancy in the above representation of transition probabilities for quantum processes such as the time evolution of a subatomic particle. For transitions in a state machine for a *quantum* process, then, one may consider using not the real probabilities of transition, but the complex valued inner products between the initial and final states of a system across an interval of interest.

And that is where the path integral formalism stands out. The path integral formalism is all about calculating these inner products of initial and final states called probability amplitudes without having to invoke the state vectors of the initial and final states themselves. One of the most interesting aspects of this project is to see just how shifting from real probabilities into complex probability amplitudes may give us a state machine construction for a time-evolving quantum system even if we do not know what the system's basis vectors are, as would usually be required in other formalisms.

3. Free Particle

As a choice for a simple, illustrative example, I chose the non-relativistic 1D free particle for my study. The free particle is especially tractable due to the absence of a position-dependent potential, giving us both energy and momentum conservation throughout the entire process. The free particle is also a rudimentary example well studied in path integrals, so from a pedagogical standpoint it also made the most sense.

C. Synopsis

This report thus presents some of the very basic, early investigations I have made into the path integral formalism and how it can be used to construct state machines of a quantum time evolution. A brief background on measurements in quantum mechanics is given, as well as an exposition on quantum path integrals as the method of constructing a state machine. A state machine constructed from the path integral of the free particle is presented as a result, and perhaps most importantly, a discussion of how the presented state machine could be further analyzed and how these studies can extend further into more nontrivial systems, such as the quantum harmonic oscillator. A major difference from the classical state machines we have studied in the course is that, instead of transition probabilities, I assign transition probability amplitudes for each possible transition.

II. BACKGROUND

A. Measurements, Heisenberg and Schrödinger

The key idea in quantum mechanics is to define the state of a particle with some unobservable complex state vector $|\psi\rangle$. A measurement for a particular observable of the system is done through a measurement operator, which in general the operator acts on the state itself and may change the state vector. Only when the state remains unchanged and some real value is extracted as an eigenvalue is a measurement then made and an observable extracted.

In mechanics, we have defined a system's states through two observables: position and momentum, both parametrised by time. In classical mechanics, one can simultaneously observe both position and momentum to complete accuracy, determining a point in the phase space spanned by two values. Since in quantum mechanics measurements are achieved through corresponding operators, one may define a position operator $\hat{x}$ and momentum operator $\hat{p}$ that will extract, from their respective eigenstates observable positions x and momenta p. To define the point in phase space, all we need is a set of simultaneous eigenstates for both $\hat{x}$ and $\hat{p}$.

Here is where quantum mechanics becomes distinct from classical mechanics. Heisenberg's canonical commutator states:

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar. \tag{1}$$

If there was an eigenstate for both $\hat{x}$ and $\hat{p}$, the commutator above should be 0 when acted on that eigenstate. The commutator is a nonzero constant, however, so $\hat{x}$ and $\hat{p}$ are shown to not share simultaneous eigenstates. This is crucial to constructing state machines for quantum processes, for what this means is that when we choose a basis for the Hilbert space of the state vector $|\psi\rangle$, we must choose between either the position eigenstates or the momentum eigenstates. If we, for instance, choose the position eigenbasis for a single particle, the observables of our state machine will be the position x of the particle, but the momentum of the particle p will be hidden from us. In fact, an application of this commutator is the famous uncertainty principle:

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \tag{2}$$

The more accurate measurements of x become, the more inaccurate measurements of p become. If we wish to keep errors in measurements to a finite value, there is always some trade-off.

As for how the states actually evolve in time, this is determined by Schrödinger's equation:

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle , \tag{3}$$

where $|\psi\rangle$ is the state function of the system and $\hat{H}$ is the Hamiltonian, the operator that extracts the energy out of its eigenvectors, of a system. The Hamiltonian, as the sum of the kinetic energy and potential energy operators, is what, in principle, defines the quantum system.

Let us consider when $\hat{H}$ is independent of time. Without pondering too much on rigour, one may then see this is just a first-order differential equation with respect to time, so we find a solution for how $|\psi\rangle$ evolves:

$$|\psi(t)\rangle = \exp[-\frac{i}{\hbar}\hat{H}(t-t_0)]\,|\psi(t_0)\rangle\,, \tag{4}$$

where $|\psi(0)\rangle$ is an initial state vector at $t = t_0$, given as a boundary condition.

It is here where we introduce the *probability amplitude*, or *kernel* of some time evolution. Consider a "trivial" time evolution where the time interval is just 0, so $|\psi(t_0)\rangle$ just remains $|\psi(t_0)\rangle$. One condition of the state vectors in the Hilbert space is that the probability of a state becoming itself across no time interval is just 1 – the state remains exactly as it is. This condition is represented by $|\,\langle\psi(t_0)|\psi(t_0)\rangle\,|^2 = 1$, where $\langle\psi(t_0)|\psi(t_0)\rangle$ may be complex. If we replace the left side in the bra-ket with $|\psi(t)\rangle$, we get to define the kernel $K = \langle\psi(t)|\psi(t_0)\rangle$ – a quantity that, when its magnitude is squared, gives the probability of transition across some time interval $[t_i, t_f]$.

Note that we must choose either a position or momentum basis in order to extract an observable. Let us choose the position as our observable, so that our states are in terms of the eigenstates $|x\rangle$ of the position. Then, we may define a position basis kernel $K(x_f, t_f; x_i, t_i) = \langle x_f, t_f|x_i, t_i\rangle$ across some time interval $[t_i, t_f]$ and position interval $[x_i, x_f]$ in one dimension. It is finding this K that is the goal of the path integral formulation to describe time evolution of quantum systems.

III. THE DYNAMICAL SYSTEM: FREE PARTICLE

A time-independent 1D Hamiltonian $\hat{H}$ for a single particle is generally given as:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}), \tag{5}$$

where $V(\hat{x})$ is the potential energy of the system and m is the mass of the particle. As stated in Subsection II A, the free particle was chosen due to its simplicity and well-known result for path integrals. The free particle is simply a single particle with $V(\hat{x}) = 0$, or

$$\hat{H} = \frac{\hat{p}^2}{2m}. \tag{6}$$

Note the invariance with respect to position and time.

The equation of motion of the free particle is just Equation 6 plugged into Equation 3. As such, in the following section, I will present a derivation of $K = \langle x_f, t_f|x_i, t_i\rangle$ for general time-independent 1D Hamiltonians (Equation 5), and apply Equation 6 as a special case.

IV. METHOD

A. The Path Integral: A Sum Over Histories

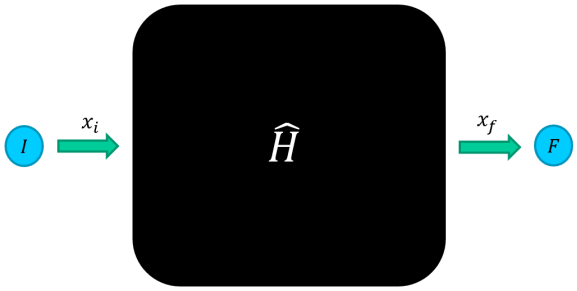

FIG. 1: A quantum time-evolution as a black box box process.

Consider now the particle is in some initial state I at $t = t_i$, with a coinciding output x_i. It undergoes time evolution through a "black box" of a Hamiltonian, and then it emerges at $t = t_f$ with output x_f and transitioning into a final state F. In classical, deterministic mechanics, even if we cannot see what is going inside the black box, there is only one unique history possible for a particle going from x_i to x_f across the time interval, determined by the classical equations of motion (Euler-Lagrange equations, principle of stationary action etc.).

This classical solution, however, is possible due to how both the position and momentum of the particle can be observed at once. In quantum mechanics, we know this is not possible, so if we see the particle at some point $x \in [x_i, x_f]$, we cannot observe what the momentum of the particle is. This uncertainty allows a distribution of nonzero kernels for all sorts of histories of the observable x, where the particle may have been speeding up at some instances or even going backwards away from x_f at some instances. Thus, the kernel is found through a sum across *all* possible histories of x, or paths, across $[t_i, t_f]$ connecting $[x_i, x_f]$, which translates to an integral in a continuum. Thus, the *path integral* is conceived.

The following derivation of the path integral is based on [3] and [4], but starts off with a background from Schrödinger's equation.

The first step is then to consider what happens along one history, or path, that connects x_i and x_f. To do so, we discretise time into N time slices of size $\delta t = (t_f - t_i)/N$. To achieve continuum, we simply send $N \to \infty$, or $\delta t \to 0$, reducing the time slices into infinitesimal time steps.

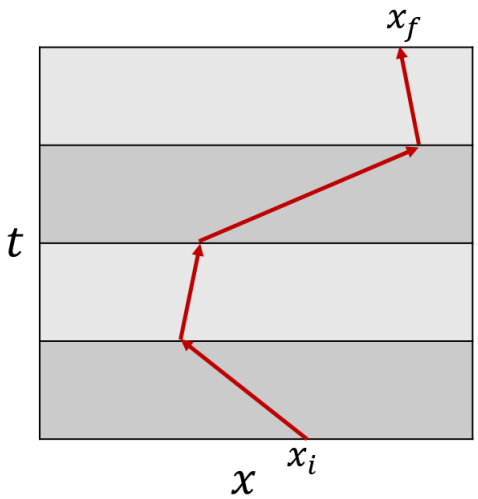

FIG. 2: Slicing of the time interval into N slices of size $\delta t = (t_f - t_i)/N$. To achieve continuity, send $N \to \infty$ and thus $\delta t \to 0$.

The time evolution operator in Equation 4 is then $\exp[-\frac{i}{\hbar}\hat{H}\delta t] = \exp[-\frac{i}{\hbar}\{\frac{\hat{p}^2}{2m}+V(\hat{x})\}\delta t]$. In general, the exponential expands out in a much more nontrivial manner, but here the infinitesimal size of δt helps us out: $\exp[-\frac{i}{\hbar}\{\frac{\hat{p}^2}{2m} + V(\hat{x})\}\delta t] = \exp[-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\delta t]\exp[-\frac{i}{\hbar}V(\hat{x})\delta t] + \mathcal{O}(\delta t^2)$, where we can drop the higher order terms.

Then, for some latticised time t_j and the particle's position observable x_j at that time, we find:

$$K(x_{j+1}, t_{j+1}; x_j, t_j) = \langle x_{j+1}| \exp[-\frac{i}{\hbar}\hat{H}\delta t] |x_j\rangle \approx \langle x_{j+1}| \exp[-\frac{i}{\hbar}\{\frac{\hat{p}^2}{2m}\delta t\}] \exp[-\frac{i}{\hbar}V(\hat{x})\delta t] |x_j\rangle . \tag{7}$$

The potential operator is in terms of just the position operator, which just extracts the position eigenvalues according to the potential $V(x)$. Thus,

$$K(x_{j+1}, t_{j+1}; x_j, t_j) = \langle x_{j+1}| \exp\left[-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\delta t\right] |x_j\rangle \exp\left[-\frac{i}{\hbar}V(x_j)\delta t\right] \tag{8}$$

Here, we use our lack of knowledge for the momentum p by summing over the momentum eigenvectors $|p\rangle$:

$$K(x_{j+1}, t_{j+1}; x_j, t_j) = \left(\int_{-\infty}^{\infty} dp \, \langle x_{j+1}| \exp[-\frac{i}{\hbar}\{\frac{\hat{p}^2}{2m}\delta t\}] |p\rangle \langle p|x_j\rangle\right) \exp[-\frac{i}{\hbar}V(x_j)\delta t]. \tag{9}$$

In position basis, $|p\rangle$ are expressed as $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp[ipx/\hbar]$. For infinitesimal time, the observable $p \approx m\frac{(x_{j+1}-x_j)}{\delta t}$, so we get our infinitesimal time kernel:

$$K(x_{j+1}, t_{j+1}; x_j, t_j) = \sqrt{\frac{m}{2\pi i\hbar\delta t}} \exp[\frac{i}{\hbar}\{\frac{m}{2}\frac{(x_{j+1}-x_j)^2}{\delta t} - V(x_j)\delta t\}]. \tag{10}$$

Notice how there are no other x dependencies in the kernel above other than x_j and x_{j+1}. Making a physical assumption of no-teleportation – continuity in the value x through time – we can thus argue that the kernel of a particular path is just going to be a chain product of its infinitesimal time kernels across N time slices – essentially, a Markov chain:

$$K_{\text{history}} = \langle x_f, t_f|x_N, t_N\rangle \langle x_N, t_N|x_{N-1}, t_{N-1}\rangle \ldots \langle x_2, t_2|x_1, t_1\rangle \langle x_1, t_1|x_i, t_i\rangle \tag{11}$$

To sum across all possible paths, we sum (integrate) across all the possible values intermediate x_j on the real line:

$$K(x_f, t_f; x_i, t_i) = \int dx_N \int dx_{N-1} \ldots \int dx_1 (\frac{m}{2\pi i\hbar\delta t})^{\frac{N+1}{2}} \exp[\frac{i}{\hbar}\delta t \sum_{j=0}^{N} \frac{m}{2}\frac{(x_{j+1}-x_j)^2}{\delta t^2} - V(x_j)] \tag{12}$$

Again, $\delta t \to 0$, so $\delta t \sum_{j=0}^{N}\{\frac{m}{2}\frac{(x_{j+1}-x_j)^2}{\delta t^2} - V(x_j)\} \approx \int_{t_i}^{t_f} dt\{\frac{1}{2m}(\frac{dx}{dt})^2 - V(x)\}$. We may reconigse that this quantity is, in fact, the so-called action S from classical mechanics. We then retrieve the path integral formula:

$$K(x_f, t_f; x_i, t_i) = \lim_{N\to\infty} \int dx_N \int dx_{N-1} \ldots \int dx_1 (\frac{m}{2\pi i\hbar\delta t})^{\frac{N+1}{2}} \exp[\frac{i}{\hbar}S]. \tag{13}$$

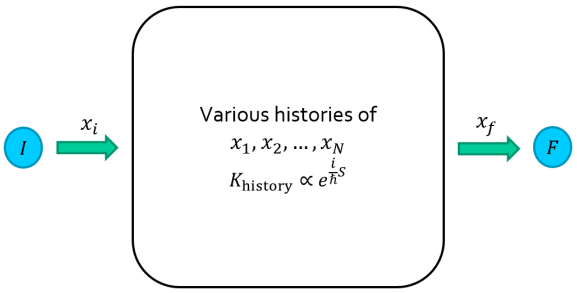

FIG. 3: Schematic diagramme of the path integral

B. State Machine Construction

So how would one construct a state machine through this formalism? It is not difficult to see how a Markov chain for each time step can be implemented, as that is already in the structure of the path integral through the time slicing and kernel chaining. The remaining question is how the observable x can be treated. x is a continuous observable whose value for our particle evolves analytically, i.e. treated through differential calculus.

Each path is essentially a time series of observable output x. We may then, for starters, consider each path as a time series of length $N+2$, the number of time slices plus the predetermined initial and final values. From this time series, we could perhaps use the methods we have for a word of length $N+2$ (e.g. future morph parsing, block entropies etc.) to construct and analyse a presentation for a given quantum 1D system.

If we resort to our usual methods of fixed-length word analysis, however, we run into the problem of an infinite sized alphabet of observables. The only way we can construct every possible history for a continuous, real-valued differentially evolving observable x is if we have access to the entire real line as our alphabet. The real numbers are, somewhat unfortunately, known to be uncountably infinite, so any attempt at constructing transition matrices or computing entropy rates of whatever state machine we construct will lead to difficulties not tractable by our discrete state machines. Even if we attempt to turn to measures such as differential entropy, subtleties in information theory of continuous variables such as the changes in differential entropy under a change of variables [5], suggest there may be a more effective method to construct a minimal state machine.

Here is where I suggest to replace fixed word length with fixed alphabet size by the same procedure used in time slicing – we slice up spacetime into an $M \times N$ lattice of rectangular cells, with $M, N \to \infty$ to achieve the continuum limit. For computational purposes, this can be approximated through very large M, N values. For each time step, the particle's position cell is updated according to the infinitesimal kernel Equation 10 of the system, akin to cellular automaton analyses to spacetime processes [6]. Note here, however, that this lattice is set up for the observable, not the internal state of the particle in the system.

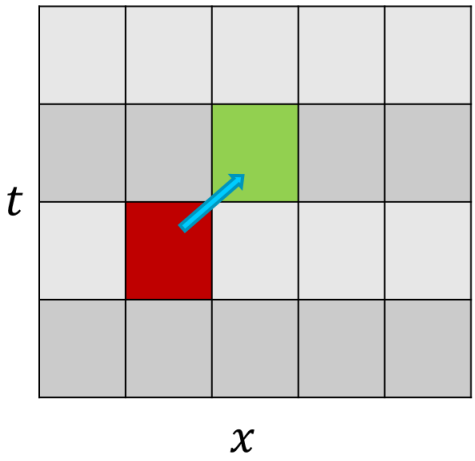

FIG. 4: Slicing of $1+1$ spacetime into $M \times N$ rectangular cells. To achieve continuity, send $M, N \to \infty$.

Instead of looking for words of the same length to count as paths of the particle, we examine words of varying length but of a size M alphabet for x that may start with x_i and end with x_f. This would most easily be found through an algorithm starting off with the classical path of the given Hamiltonian. The classical path corresponds to one unique path of x values through the lattice, the path length depending on the alphabet size M. Then, all other histories can be retrieved as continued appending of available symbols, under continuity so the next position symbol must be that of the position cell adjacent to it. The appending would continue the word ends in x_f again, and that would be a non-classical path to add to our inventory of paths. Repeat the process for each distinct valid word and, in principle, we should a good approximation to a collection of all possible paths.

The actual number of paths is, of course, infinite. It is here we take the key insight of how the path integral formalism agrees with classical physics. Note that each path carries in its kernel a phase factor of $\exp[\frac{i}{\hbar}S]$. We can split up the action S into a classical and non-classical portion such that path carries $\exp[\frac{i}{\hbar}(S_{\text{classical}} + S_{\text{non-classical}})]$. The classical action is the local extremum with respect to deviations, whereas the non-classical action is a deviation away from it. It turns out that, the further a path deviates away from its classical trajectory, the more rapid the oscillations become in the phase $\exp[\frac{i}{\hbar}]$. Eventually, for certain deviations these oscillations are so rapid they essentially look like

extremely rapid sinusoidal oscillations that go in opposite directions, cancelling each other out [3]. This is how the classical limit is achieved by instead of considering only extreme deviations from the classical path, we send $\hbar \to 0$ such that all deviations from the classical path oscillate so quickly they cancel out.

Thus, with enough iterations of the word-finding algorithm, a good approximation for the path integral should be achievable, as the other extreme, longer words will have contributions to the kernel that just cancel each other out. A sort of perturbation theory, if you will.

V. RESULT

A. The Kernel of the Free Particle

We now apply the laid out methods for the 1D free particle.

The free particle is special in its absence of the potential, so it has continuous symmetry with respect to the position x. This fact allows the calculation of its kernel $K(x_f, t_f; x_i, t_i)$ to be not too difficult. Consider the infinitesimal kernel of the free particle:

$$K(x_{j+1}, t_{j+1}; x_j, t_j) = \sqrt{\frac{m}{2\pi i\hbar\delta t}} \exp[\frac{i}{\hbar}(\frac{m}{2}\frac{(x_{j+1} - x_j)^2}{\delta t})]. \tag{14}$$

There are two ways to quickly reach the result here. The first is to observe that Equation 14 is a Gaussian. Repeated integrations over x_j across the real line just gives another Gaussian in the form of the remaining variables, so we can procedurally see the form of the finite time kernel to be same as the kernel, giving us the same Gaussian with the endpoints just taken to the interval endpoints.

The second, cheekier way of reaching the result is by the positional symmetry of the system. The system, defined by the Hamiltonian, essentially looks the same everywhere in space, so no matter where you look at the particle or how close you zoom into it, its behaviour should be the same. In other words, this positional invariance of the system immediately allows us to carry out the infinitesimal kernel to the finite interval, as an infinitesimal zoom onto the particle should not change its behaviour with respect to more macroscopic scales, and vice versa.

Hence, the free particle kernel is:

$$K(x_f, t_f; x_i, t_i) = \sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}} \exp[\frac{i}{\hbar}\left(\frac{m}{2}\frac{(x_f - x_i)^2}{t_f - t_i}\right)]. \tag{15}$$

B. A Presentation of the Free Particle State Machine

By the uncertainty principle (Equation 2), our measurement of the position x should be represented as a probability distribution centred around x with some error σ. For our purposes in position space, we are sacrificing our knowledge of the momentum in exchange of accuracy in position, so here let us take the "ballistic" approximation, where σ is very small and we may essentially take the distribution as a sharp and narrow peak.

The classical path of the free particle between two points x_i and x_f on the real line is simply the straight line with uniform velocity from x_i to x_f. This classical path, on our 1+1 lattice of size $M \times N$ for very large M, N, is then just a straight positive diagonal line connecting the cell (x_i, t_i) to (x_f, t_f). Based on this linear classical path, we may as well emphasise symmetry and construct the lattice into size $N \times N$ by equating the position slice number M to the time slice number N.

On this lattice, all paths are words of length $L \geq N + 2$ that begins with x_i and ends with x_f. Each output is characterised not by a real-valued probability, but by a complex infinitesimal kernel. Continuity in x and t is also imposed, meaning the next cell occupied by the observable time series has to be diagonally adjacent to the current one.

In the case of the free particle, not only is the exact result for the finite kernel known and relatively straightforward, but the infinitesimal kernel has no dependence on which cell the observable currently corresponds to, and depends only on the size of the cell, which is a constant for our lattice. This prompts our free particle state machine to be a Markov chain machine, whose presentation is depicted in the following Figure 5.

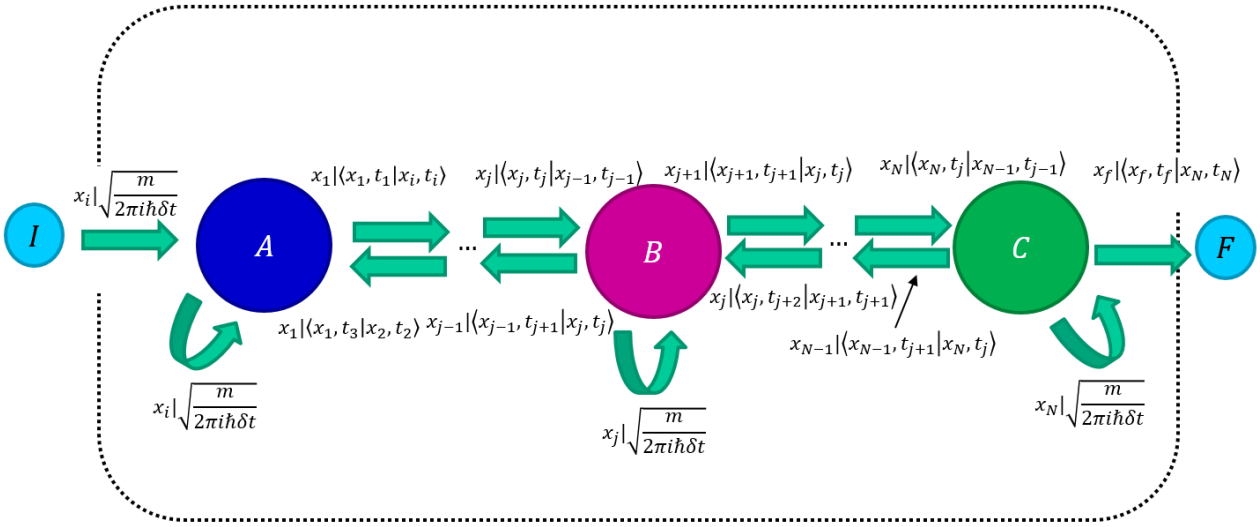

FIG. 5: A presentation of the free particle state machine.

C. Discussion

Immediately an important question arises and remains unanswered: is this presentation the minimal one for the free particle? Each state does correspond to a set of observable histories (ending in their particular coinciding observable) printed through the same infinitesimal kernel. This does seem to comply well with the definition of a causal state, only with kernels instead of real probabilities. Whether this is a successful equivalent of an ϵ-machine is, unfortunately, not quite demonstrated as of the moment.

Several tools from information theory should be employable in answering this. Of course, a major issue is that this presentation uses complex values kernels and not real-valued probabilities as information theory as we know should. Exactly how the tools we have at our disposal may be translated into complex kernels is another entire, fascinating topic one may pursue in its own right, regardless of whether it turns out to be somewhat straightforward or not. In any case, the question of minimality for this presentation is one that should immediately follow in this study.

A much more interesting example to find presentations from path integral constructions would be the harmonic oscillator, arguably the most important 1D single particle quantum system. It is the fundamental system with bound states, which one may suspect would manifest as recurrent states of the state machine of the quantum process. If we consider our current free particle presentation and extend t_f to infinity, it seems that no state is truly recurrent – we can consider this as an extremely large collection of transient states, and the final "recurrent" state is never quite reached, only asymptotically approached. Of course, whether or not this is actually true is another matter in need of addressing.

Furthermore, considerations of relativistic quantum mechanics and quantum field theory would be especially interesting, in particular through the picture of the observables mapped onto a spacetime lattice resembling cellular automaton theories. Special relativity adds certain constraints on spacetime evolution and the theory of observables for a quantum particle. Implications of these constraints would be an excellent topic of further pursuit.

It also seems that, in general, state machines of quantum systems beyond single particles are not truly Markov chains [7]. Many of the processes studied in our course, of course, have not been Markov, so there is not too much trepidation in pursuing that topic at some point in the future.

And of course, paraphrasing the words of a certain postdoc, it's not believable unless some code has been run to provide empirical calculations that indeed this presentation properly executes the free particle process. The primary method would be to set up the code to execute the word-finding algorithm laid out in Section IV B to some number of words, and sum up the calculated kernels resultant from each word. This result should be, within some margin of error, the expected result Equation 12.

VI. CONCLUSION

The path integral formalism indeed gave many insights into how a presentation of the state machine may be constructed without even defining quantum causal states. In exchange, we have resorted to kernels, or probability amplitudes, which provide distinct phase factors for each possible path between spacetime points of interest. Is this presentation the minimal one? Again, not answered. Is this a definitive end for the topic of path integrals as state machine constructions? Certainly not. Does this problem open up the way into a deeper investigation of quantum state machines, as set out in the beginning of this project? It indeed does.

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