Accelerating series convergence using Continued Exponentials

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The intent of this report is to study the properties of the one dimensional continued exponential map $x_{n+1} = a_n \exp(z * x_n)$, where $z, a_n \in \mathbb{C}$. A powerful method to rapidly accelerate the convergence of Taylor series using Continued Exponentials and Shanks Transform has been illustrated with a few examples.

I. INTRODUCTION

Continued functions like continued fractions and continued exponentials [1] [2] [3] [4] have become a popular domain of exploration in the study of Taylor Series. These representations of the function allow the Taylor Series to be analytically continued beyond the region of convergence. Bender and Vinson [5] proposed the continued exponential as a new method for summing a power series and illustrate some of the interesting convergence properties of the continued exponential. In this report, it is shown that the continued exponential when used in conjunction with Shanks transform can rapidly accelerate convergence and outperform the Continued Fractions and Pade Approximants.

A tetration of the general form is denoted as follows,

$$E(a_0, a_1, \ldots, a_n) = a_0^{a_1^{a_2^{\cdot^{\cdot}}}} \tag{1}$$

When the a_i's are a function of z, where $z \in \mathbb{C}$, one gets the continued exponential representation,

$$E(a_0e^z, a_1e^z, \ldots, a_ne^z) = a_0e^{a_1ze^{a_2ze^{\cdot^{\cdot^{\cdot}}}}} \tag{2}$$

This report is organized as follows: In Sec.II some basic theorems and properties involving continued exponentials are established. In Sec.III, it is shown that the coefficients of the Taylor series representation of the continued exponential can be modelled using a simple chaos game representation. Sec.IV discusses the limit cycle diagram that is produced by the continued exponential map when $a_i = c$, where $c \in \mathbb{C}$. Applying continued exponentials to accelerate the convergence of π, $\zeta(z)$, $\log(1 + z)$ and some Taylor series is demonstrated in Sec.V . Finally, I end this report with a small discussion on other continued functions and their limit cycle diagrams.

II. THEOREMS AND PROPERTIES ON CONTINUED EXPONENTIALS

Theorem 1 *The function $f = z^{z^{\cdot^{\cdot}}}$ converges when*
$e^{-e} \leqslant z \leqslant e^{1/e}$
and diverges for all other positive z outside this interval. There are in total three modes of behavior for the continued exponential

1. The sequence of hyperpowers $(z, z^z, z^{z^z}, \ldots)$ increase monotonically.
2. The sequence of hyperpowers decrease monotonically.
3. The sequence of hyperpowers oscillates. The hyperpowers converge but to a limit-cycle of period n.

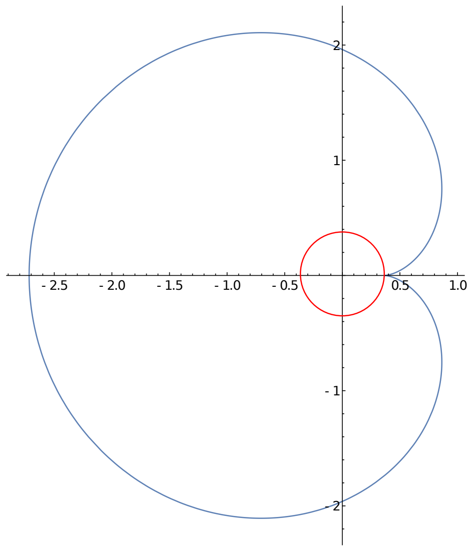

FIG. 1. Comparison of the region of convergence of Taylor series(Red) with that of the continued exponential(Blue)

Theorem 2 *The implicit expression for the boundary of the region of convergence of a continued exponential when $a_i = 1$ is given by:*

$$r = e^{-\cos t}$$
$$\theta = t - \sin t$$

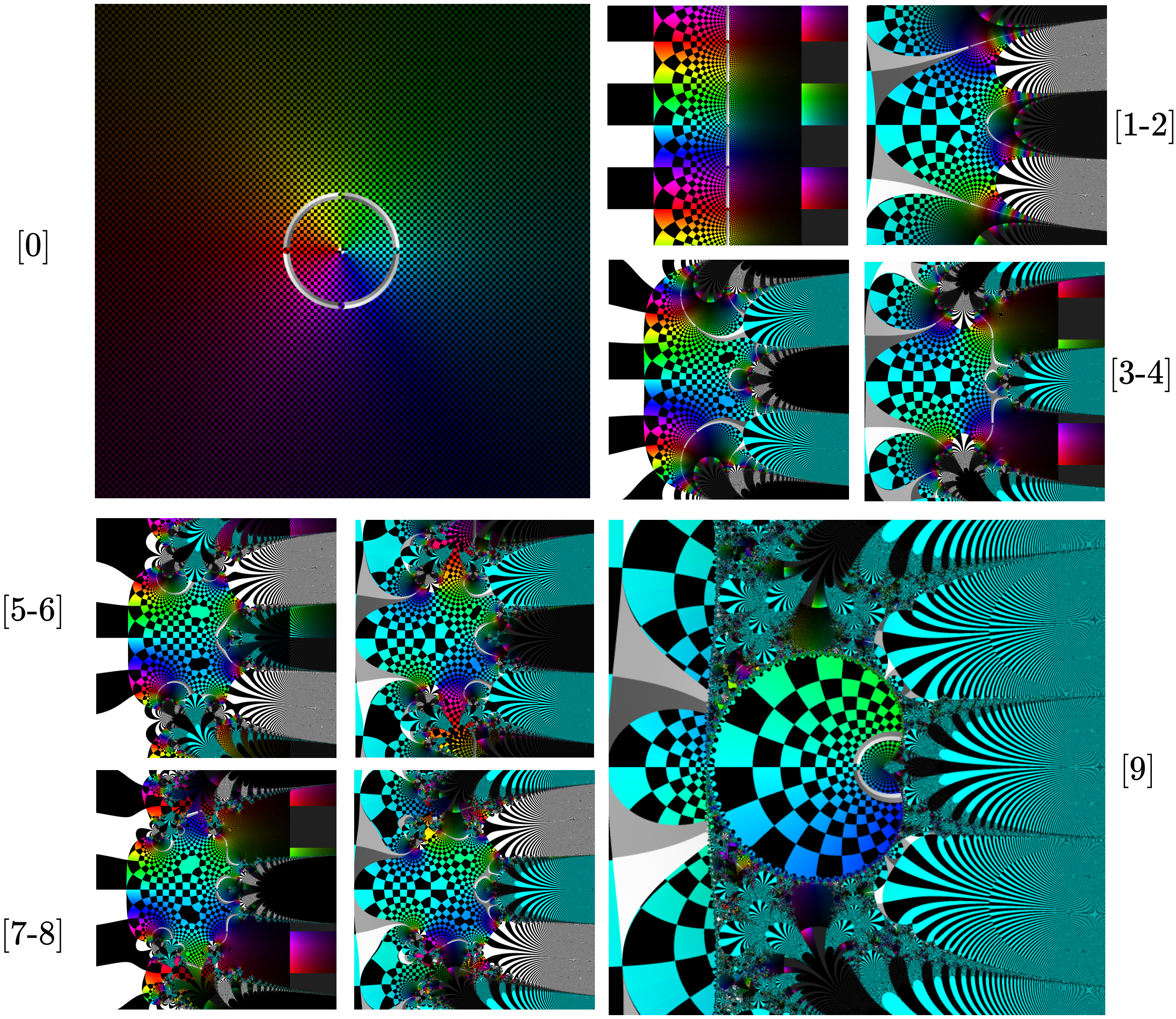

FIG. 2. Conformal mapping of the Image [0] under the exponential map. [1-8] demonstrates the emergence of limit cycle and the evolution of the cardioid region under repeated iterations. For example, the black and white region at the right side of image changes from black and white to black and blue every 3 iterations indication a 3-limit cycle behaviour, whereas the black and white region on the right exhibits a 2-cycle behaviour. [9] is the image after > 50 iterations of the map. A notable feature of [9] is the distinct cardioid shaped region at the center. After repeated mappings the color of the cardioid region becomes invariant under iteration (Limit cycle of period 1).

When we take all coefficients $a_i = 1$ in Eq.(2) to be equal to unity, the continued exponential has a simple Taylor series representation:

$$e^{ze^{ze^{\cdot^{\cdot^{\cdot}}}}} = \sum_{n=0}^{\infty} \frac{(n+1)^{n-1}}{n!} z^n \qquad (3)$$

For this simple case, the radius of convergence of the Taylor Series when compared with the continued exponential is plotted in Fig.1. The Taylor series converges in a disc of radius $\frac{1}{e}$ whereas the continued exponential converges in a larger cardioid region. In order to understand the structure of the continued exponential it is helpful to visualize the conformal map of the continued exponential (see Fig.2).

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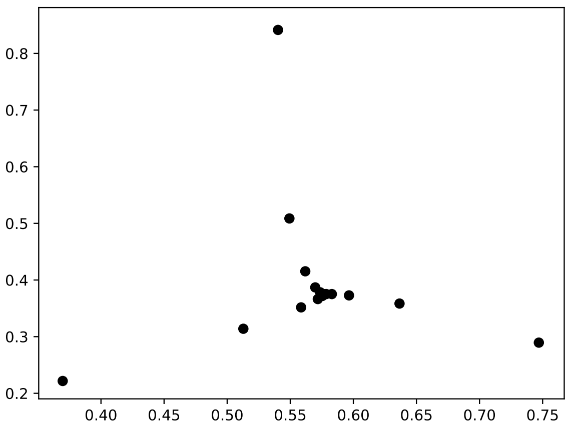

FIG. 3. Convergence of $E(e^i, e^i, \ldots, e^i) = 0.576412723031 + 0.374699020737j$ on the complex plane

A. Transcendental numbers

As a corollary of Theorem-2 it is interesting to discover the convergence of following continued transcendental quantities $z = i, log(i), \frac{i}{\pi}$. The convergence plot of $E(e^i, e^i, \ldots, e^i)$ is presented in Fig.3

$$i^{i^{i^{\cdot^{\cdot^{\cdot}}}}} = 0.885303089812... + 0.256298179656...j \tag{4}$$

$$e^{ie^{ie^{\cdot^{\cdot^{\cdot}}}}} = 0.576412723031... + 0.374699020737...j \tag{5}$$

$$e^{\frac{i}{\pi}e^{\cdot^{\cdot^{\cdot}}}} = 0.885302922632... + 0.256299537164...j \tag{6}$$

These are new transcendatal numbers which were not know before and they have now been added to the OEIS - A305200,A305202,A305208,A305210.

B. Electronic equivalence circuit

The electronic equivalent circuit for the continued exponential map can be constructed by using two operational amplifiers and is illustrated in Fig.4 .Treating the thermal voltage dissipation loss to be $V_T = 1V$ and the saturation current as a constant quantity the current output measured at B in circuit after every cycle will exhibit the same characteristics as an exponential map. If one measured the Voltage at B on the other hand it will exhibit the same characteristics as a Lambert-Z map.

$$V_B = I_s R e^{\frac{V_A}{V_T}}$$
$$I_B = I_s e^{\frac{V_A}{V_T}}$$

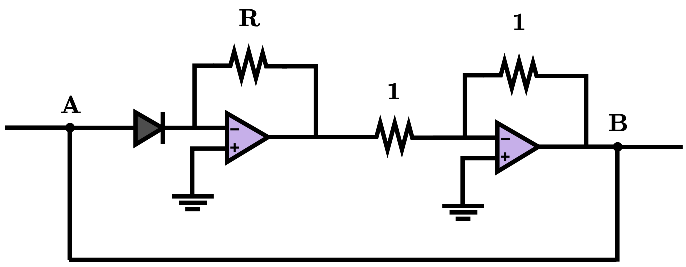

FIG. 4. Electronic equivalent circuit for the continued exponential map and the Lambert-Z map. Measuring the voltage at B gives the Lambert-Z map and the current measured at B gives the Exponential map

III. CHAOS GAME REPRESENTATION OF THE COEFFICIENTS

Consider a Taylor series of the form

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{7}$$

By comparing the coefficients of different terms of the Taylor series and the continued exponential we get that,

$$\begin{aligned}
c_0 &= a_0 \\
c_1 &= a_0 a_1 \\
c_2 &= a_0 a_1 a_2 + \frac{a_0 a_1^2}{2} \\
c_3 &= a_0 a_1 a_2 a_3 + \frac{a_0 a_1 a_2^2}{2} + a_0 a_1^2 a_2 + \frac{a_0 a_1^3}{6} \\
c_4 &= a_0 a_1 a_2 a_3 a_4 + \frac{a_0 a_1 a_2 a_3^2}{2} + a_0 a_1 a_2^2 a_3 + a_0 a_1^2 a_2 a_3 \\
&+ \frac{a_0 a_1 a_2^3}{6} + a_0 a_1^2 a_2^2 + \frac{a_0 a_1^3 a_2}{2} + \frac{a_0 a_1^4}{24} \\
c_5 &= \frac{a_0 a_1^5}{5!} + \frac{a_0 a_1^4 a_2}{6} + \frac{3 a_0 a_1^3 a_2^2}{4} + \frac{2 a_0 a_1^2 a_2^3}{3} \\
&+ \frac{a_0 a_1 a_2^4}{24} + \frac{a_0 a_1^3 a_2 a_3}{2} + 2 a_0 a_1^2 a_2^2 a_3 + \frac{a_0 a_1 a_2^3 a_3}{2} \\
&+ \frac{a_0 a_1^2 a_2 a_3^2}{2} + a_0 a_1 a_2^2 a_3^2 + \frac{a_0 a_1 a_2 a_3^3}{6} + a_0 a_1^2 a_2 a_3 a_4 \\
&+ a_0 a_1 a_2^2 a_3 a_4 + a_0 a_1 a_2 a_3^2 a_4 \\
&+ \frac{a_0 a_1 a_2^2 a_3 a_4^2}{2} + a_0 a_1 a_2 a_3 a_4 a_5 \\
&\vdots
\end{aligned}$$

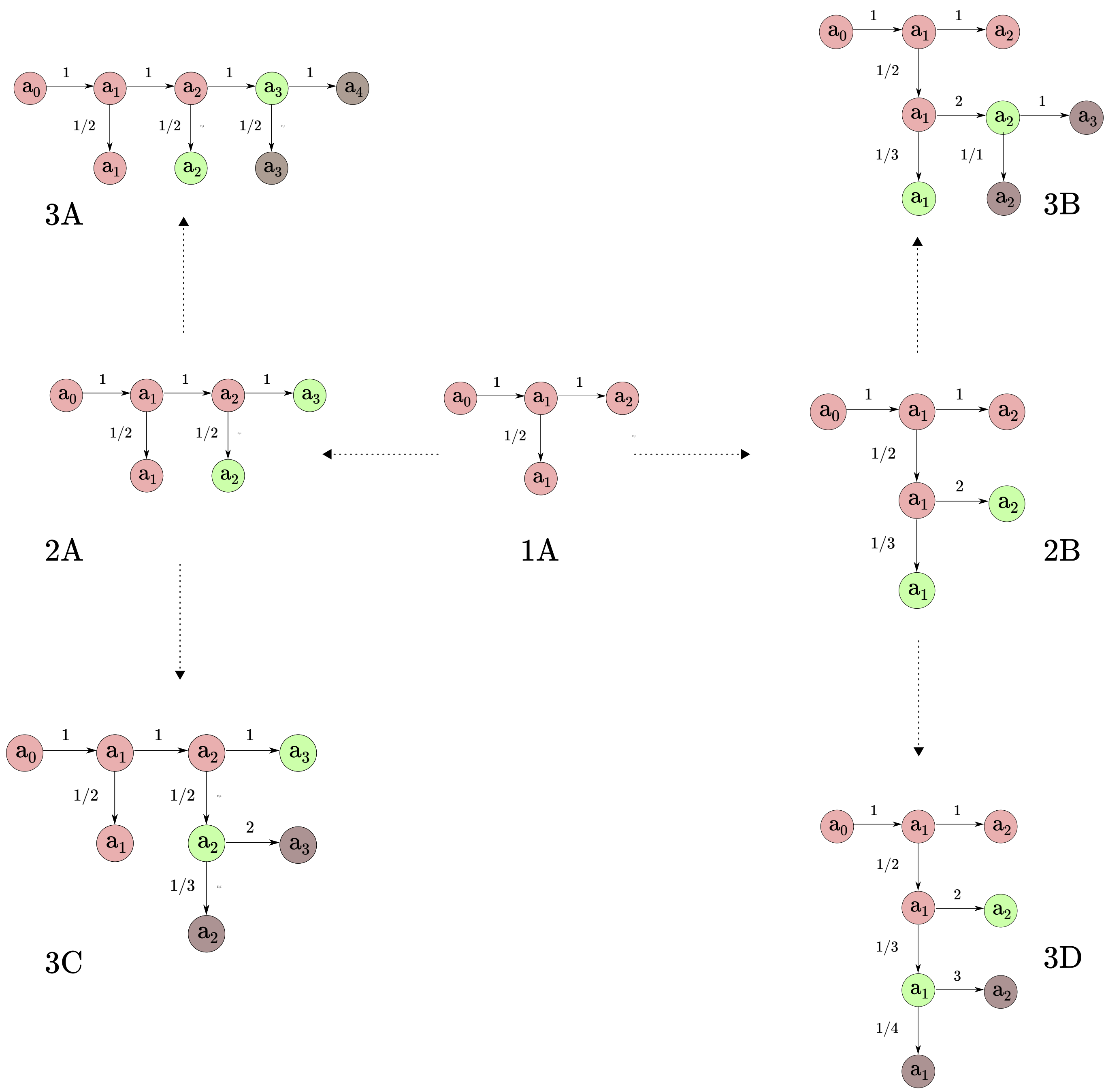

FIG. 5. Chaos game representation of the coefficients. In $1A$, taking the steps - (Right,Right) yields $a_0a_1a_2$ and steps - (Right,Down) yields the coefficient $\frac{a_0a_1^2}{2}$. Therefore, the coefficient $c_2 = 1A = a_0a_1a_2 + \frac{a_0a_1^2}{2}$. Similarly by using the rules of the game c_2 and c_3 are constructed: $c_2 = 2A + 2B$ and $c_3 = 3A + 3B + 3C + 3D$.

The combinatorial significance of these coefficients have been explored by Baker and Rippon and others. But these coefficients can also be constructed using a simple chaos game on a 2-D lattice tree. The rules of the game are as follows:

1. If you start at any vertex a_n, you always go to a_n and a_{n+1}.
2. If you enter a vertex horizontally with a weight n, you leave with weights 1 and $\frac{n}{2}$.
3. If you enter a vertex vertically with a weight $\frac{1}{n}$, then you leave with weights n and $\frac{1}{n+1}$.
4. Addition rule - I.
5. Addition rule - II.

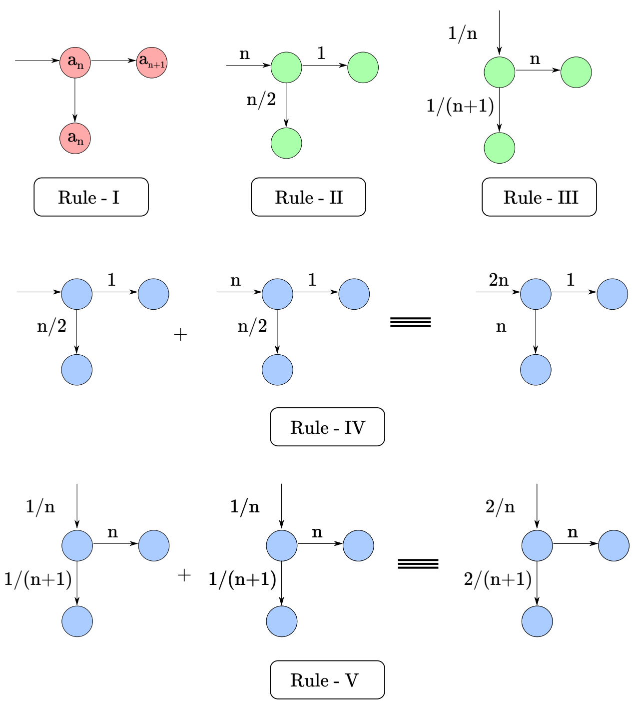

FIG. 6. Rules of the chaos game

Fig.5 demonstrates the construction of the coefficients c_2, c_3, c_4 (c_0 and c_1 are trivial) using the above five rules (Fig.6). The rest of the coefficients c_n's are constructed the same way.

IV. LIMIT CYCLE DIAGRAM

Outside the region of convergence, the continued exponentials either diverges monotonically or converges to a limit cycle of period n. The algorithm that was used to generate the number of limit cycles is listed as follows:

- Pick any arbitrary point $z \in \mathbb{C}$
- Iterate the map 200-500 times to eliminate transients.
- Store the next 500 values after performing a modulo m operation, i.e $z_1 = z_1 \pmod{m}$. This maps all the values $A \in [0, m)$.
- Using numpy's histogram function calculate the number of non-empty bins in the output and that would yield the number of cycles.

One chooses the value of m depending on the nature of the map one chooses to explore. For the continued exponential map it was found that $m = 2$ was the optimum choice. Fig.7 illustrates the continued exponential with $m = 1, 2$. Although both the images retain the same structure, the wrong colorings in $m = 1$ are because when a complex number z converges to $1, 0, 1, 0, \ldots$ which is a 2 limit-cycle choosing $m = 1$ wrongly classifies it as a 1-limit cycle.

The advantage of using the above algorithm besides the conventional ones is that it allows the detection of large limit cycle data without increasing the computational load by adjusting the parameters of the numpy's histogram function.

It also follows that if the power of the z-exponent is increased , the number of cusps in the limit cycle diagram also increases accordingly. Fig.8 plots the limit cycle diagram for the maps - $x_{n+1} = \exp(z * x_n)$, $x_{n+1} = \exp(z^2 * x_n)$ and $x_{n+1} = \exp(z^{10} * x_n)$. Red is region of absolute convergence, Orange color represents a region of limit cycle-2, Light shade of orange representing a region of limit-cycle-3 and so on. The dark blue outside the cardioid corresponds to the region where the continued exponential diverges.

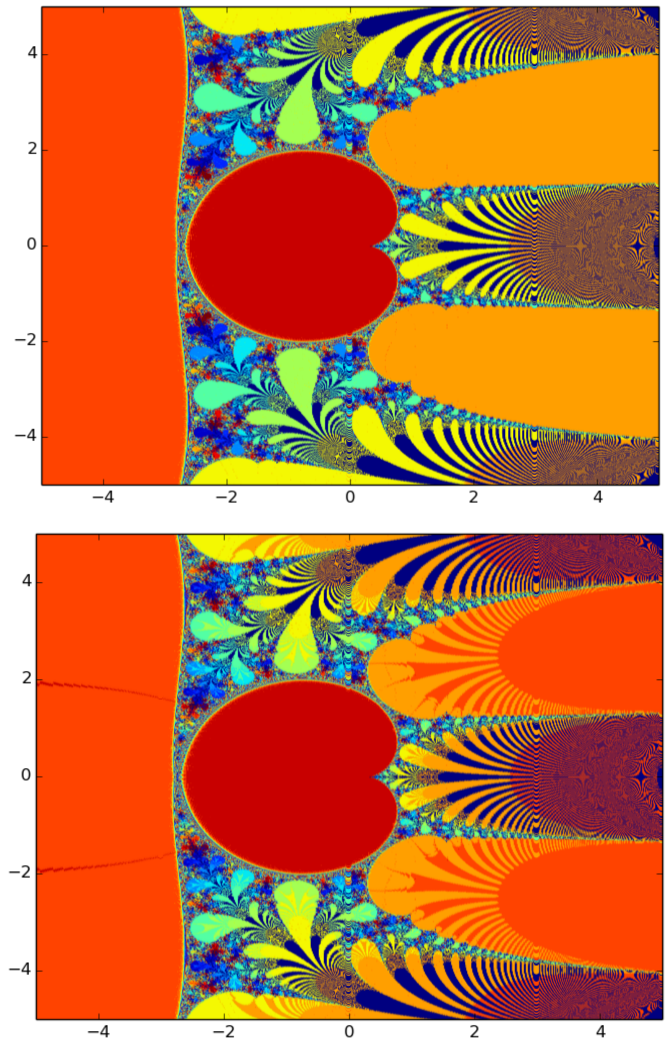

FIG. 7. Limit cycle diagram of the exponential map with $m = 2$(top) and $m = 1$(bottom)

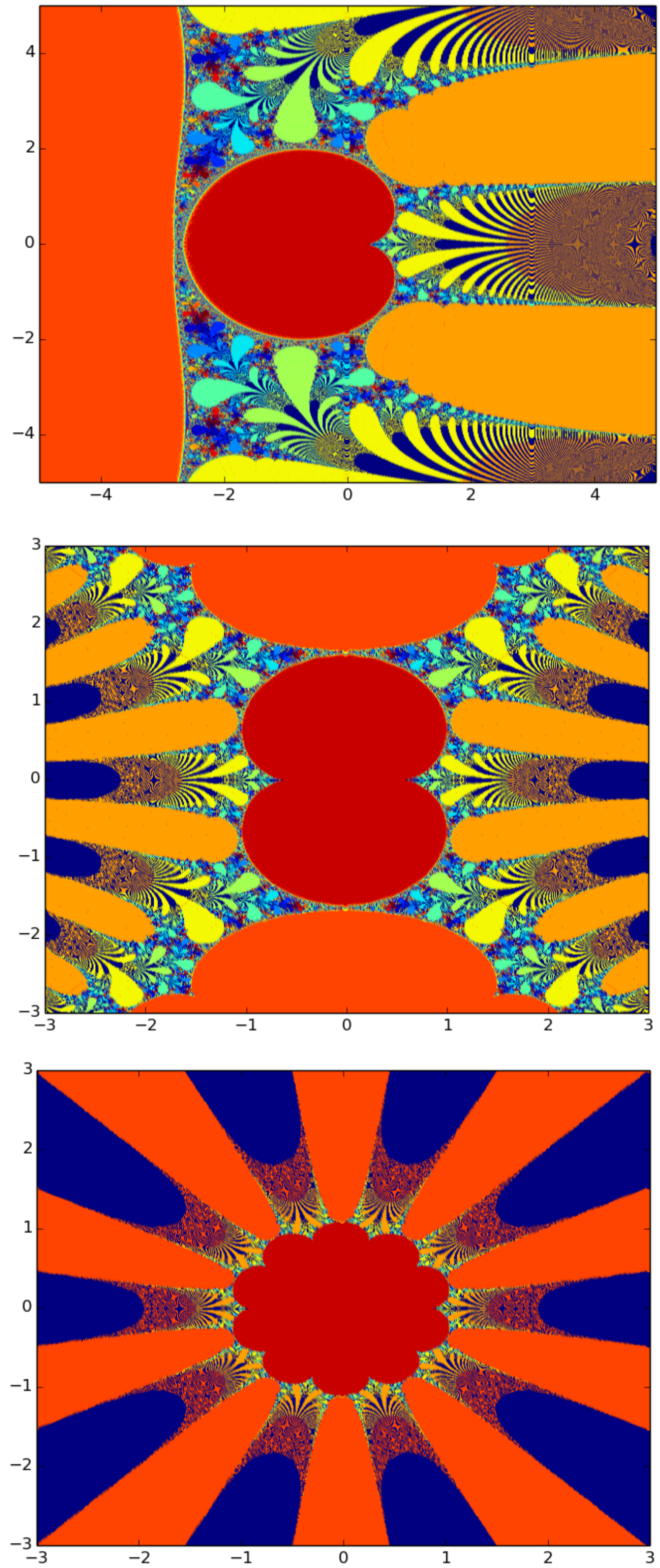

FIG. 8. The continued exponential maps $x_{n+1} = \exp(z * x_n)$, $x_{n+1} = \exp(z^2 * x_n)$,$x_{n+1} = \exp(z^{10} * x_n)$ with $Re(z) \in [-3, 3]$, $Im(z) \in [-3, 3]$.

V. APPLICATIONS TO PROBLEMS

Having had a solid understanding of the behaviour of continued exponentials and its remarkable properties, it is time to illustrate the usefulness of Continued exponentials and Shanks Transform in accelerating convergence of series with a few examples.

A. Accelerating the convergence for π

Let's consider the problem of finding the value of π using the following series

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 0.78539816339 \tag{8}$$

This series is known to be slowly convergent and therefore, acceleration techniques such as the Aitken's delta squared process are used to accelerate it's convergence. Converting this series into a continued exponential form yields the values for its coefficients. But remarkably when the continued exponentials representation in conjunction with the Shanks Transform is used, only from 8 terms of this expansion one can obtain the value of $\frac{\pi}{4}$ to upto 8 decimal points. A comparison of the partial sums of this series, Aitken's delta squared process and the continued exponential is illustrated in Table.1.

The continued exponential can also used to be further accelerate existing fast converging series. For example, estimating the value of π using the BaileyBorwein-Plouffe formula (BBP formula) allows for computing the nth digit of π directly without calculating the preceding digits. The following is a comparison of the results obtained using BBP, continued exponential and continued exponential with shanks transform.

$$\begin{aligned} \pi_{Actual} &= 3.14159265358979323846 \\ \pi_{BBP} &= \mathbf{3.14159265358979}11463 \\ \pi_{CE} &= \mathbf{3.1415926535897932}276 \\ \pi_{CE+Sh} &= \mathbf{3.14159265358979323846} \end{aligned}$$

B. log(2)

Yet another example of a extremely slow convergent series log(2). It is known that even if one tries to use Rectangular, Trapezoidal or Simpson's quadrature to compute the value of log(2) instead of naively adding the sum, only a few digits of the logarithmic function can be obtained. The rate of convergence of the function is very slow, but applying the continued exponential to 7 terms in the sequence yields us the right value upto 8 decimal precision (see Table.2)

TABLE I. Calculating the value of $\frac{\pi}{4} = 0.78539816339$ using Aitken's delta-squared process v/s Continued Exponentials

n	term	A	CE	S(CE)	S^2(CE)	S^3(CE)
1	-0.3333333333	0.78333333	0.7165313105737893	-	-	
2	0.2	0.78630952	0.8056428246139337	-	-	
3	-0.1428571429	0.78492063	0.7795538977546907	0.785383171125	-	
4	0.1111111111	0.78567821	0.7870604173842369	0.785399930384	0.785398055239	
5	-0.09090909091	0.78522034	0.7849282913678776	0.785397819009	0.785398157189	0.785398162673
6	0.07692307692	0.78551795	0.7855304163988939	0.785398221683	0.785398162292	
7	-0.06666666667	-	0.7853610343764056	0.78539815199		
8	0.05882352941	-	0.7854085683755658	-		

TABLE II. Accelerating the convergence of $\log(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = 0.69314718056$ using shanks and Continued Exponential.A_n is the nth partial sum of the series and it converges slowly.

n	A_n	CE	S(CE)	SS(CE)	SSS(CE)
1	1.0000000	0.7191967497444082	-		
2	0.5000000	0.6857283810599458	0.693146174201		
3	0.8333333	0.6952583599753418	0.693150321509	0.693148444292	
4	0.5833333	0.6925515796826819	0.69314689194	0.693147220051	0.693147183606
5	0.7833333	0.6933147356768786	0.693147254761	0.693147184673	
6	0.6166667	0.6931001655700353	0.693147167965		
7	0.7595238	0.6931603520385945	-		

C. Riemann Zeta Function and Test for convergence

Table.3 shows the performance of the continued exponential-Shanks transform when applied to $\zeta(4)$. In addition to accelerating convergence, one of the fundamental properties of the continued exponential (Theorem-2) is that it also monotonically diverges. Therefore, using this to evaluate $\zeta(1) = \infty$ yields a divergent function from ten terms of the series. Hence the continued exponential also acts as a efficient test for divergence of a series.

D. Continued Exponentials v/s Pade Approximants

Pade Approximants are extremely useful tools in analytically continuing a function to a larger region of convergence and also accelerating its convergence rate. In this subsection, let us consider the performance of the continued exponential with the Pade Approximants by considering the Taylor series represented by the continued exponential with all the coefficients $a_i = 1$.

$$e^{ze^{ze^{\cdot^{\cdot^{\cdot}}}}} = \sum_{i=0}^{\infty} \frac{(n+1)^{n-1}}{n!} z^n \tag{9}$$

The radius of convergence of the Taylor series of the function is $\frac{1}{e}$. From Sec.II we know that the radius of the convergence of the continued exponential is a cardioid. Using only 7 terms in the continued exponential expansion, it is possible to get an answer correct to 9 decimal places compared to the continued fractions which yields the right answer to 7 decimal places (Table.4).

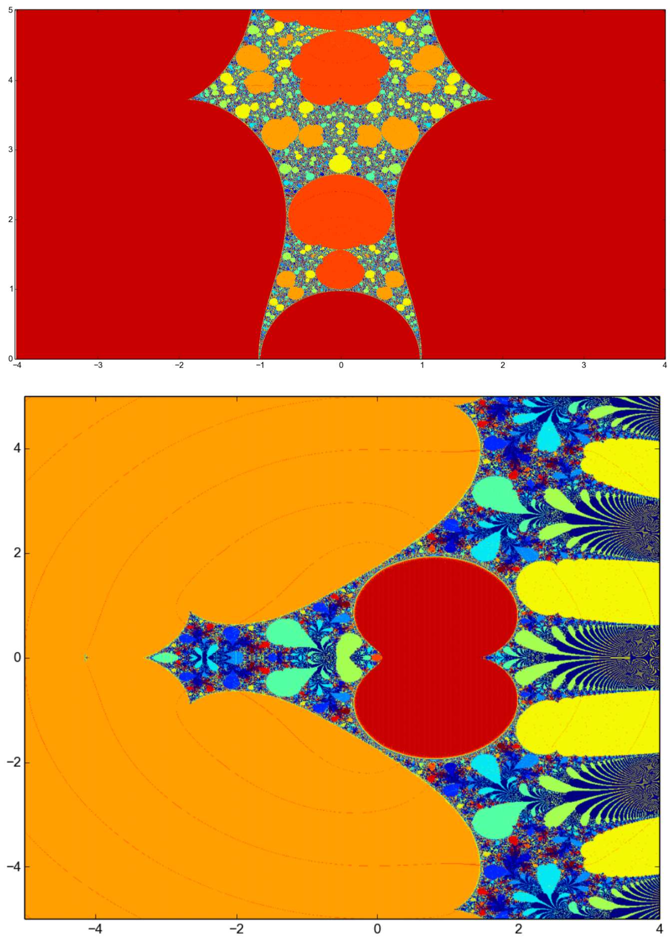

FIG. 9. (a) Ana's Map, (b) Comet Map

TABLE III. $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots = 1.0823232337$

n	Partial Sum	CE	S(CE)	S^2(CE)	S^3(CE)
1	1.0644944589	-	-	-	-
2	1.0765985126	-	-	-	-
3	1.080031458	1.0819533006	-	-	-
4	1.081263548	1.0821537913	1.08229663	-	-
5	1.0817803659	1.0822372034	1.0823097794	1.0823211373	-
6	1.082022856	1.0822760122	1.0823158735	1.0823221207	1.0823230265
7	1.0821467102	1.0823206268	1.0823189583	1.0823225922	-
8	1.0822143374	1.0823063365	1.0822956762	-	-
9	1.0822533137	1.0823124421	-	-	-
10	1.082276805	-	-	-	-

TABLE IV. Continued exponential-Shank v/s Pade Approximants. Actual value = 0.84457986749555

n	CE	S(CE)	S^2(CE)	S^3(CE)	Pade
1	0.8187307530780	-	-	-	0.83333333333333
2	0.8489575018686	0.844581504537	-	-	0.84615384615385
3	0.8438407387346	0.844579914244	0.844579867475	-	0.84444444444444
4	0.8447047273782	0.844579868829	0.844579867511	0.844579867262	0.84459732901659
5	0.84455877693024	0.844579867534	0.844579821066	-	0.84457826304384
6	0.8445834300364	0.844579867496	-	-	0.84458006597567
7	0.8445792657257	-	-	-	0.84457984859380

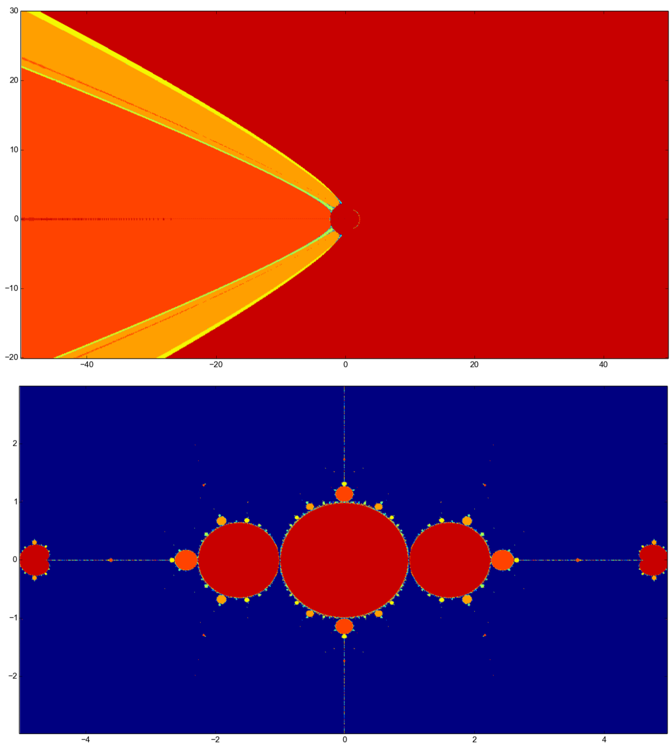

FIG. 10. (a) Iterated exponential map, (b) Sine map

VI. CONCLUSIONS

I would like to end this report with a couple of remarks on continued exponentials. As we can see continued exponentials offer an alternate means to analytically continue a Taylor series to a much larger region on the complex plane. From the observations of this report, it is also clear that converting a Taylor series into a continued exponential increases its rate of convergence. Inspired from this exploration of the continued exponential map, the following are some other interesting complex maps that need further investigation:

- Ana's map ($x_{n+1} = a_n \tanh(z * x_n)$)
- Comet map ($x_{n+1} = a_n \exp(1 + \log(z) * x_n)$)
- Iterated Exponential Map ($x_{n+1} = a_n^{x_n}$)
- Sine map ($x_{n+1} = a_n \sin(x_n)$)

The limit cycle diagram of these maps has been plotted in Fig.9 and Fig.10. and it is clear that they display similar fractal limit cycle behaviour outside the region of convergence that was observed with the exponential map. Exploring these is a topic of further research.

VII. ACKNOWLEDGEMENTS

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