When life gives you a Taylor Series, Make a continued exponential out of it.

Keerthi Vasan.G.C

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- Continued exponentials.

$$a_0 e^{a_1 z e^{a_2 z e^{\cdot^{\cdot}}}} \tag{3}$$

where $z \in \mathbb{C}$

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$$a_0 e^{a_1 z e^{a_2 z e^{\cdot^{\cdot^{\cdot}}}}} = a_0 + (a_0 a_1)z + (a_0 a_1 a_2 + \frac{a_0 a_1^2}{2})z^2 + \dots \tag{5}$$

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- Compare both series and solve for the coefficients a_i

$$c_0 = a_0 \tag{6}$$

$$c_1 = a_0 a_1 \tag{7}$$

$$c_2 = a_0 a_1 a_2 + \frac{a_0 a_1^2}{2} \tag{8}$$

$$\tag{9}$$

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- Is that progress ?

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$$\sum_{n=0}^{\infty} \frac{(n+1)^{n-1}}{n!} z^n = e^{ze^{ze^{\cdot^{\cdot^{\cdot}}}}} \tag{11}$$

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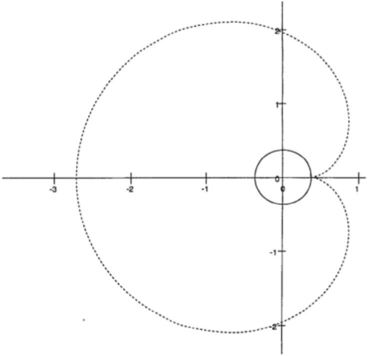
- And look the region of convergence of both in the Z plane:



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$$e^{\frac{i}{\pi}e^{\cdot^{\cdot^{\cdot}}}} = 0.885302922632 + 0.256299537164j \quad (14)$$

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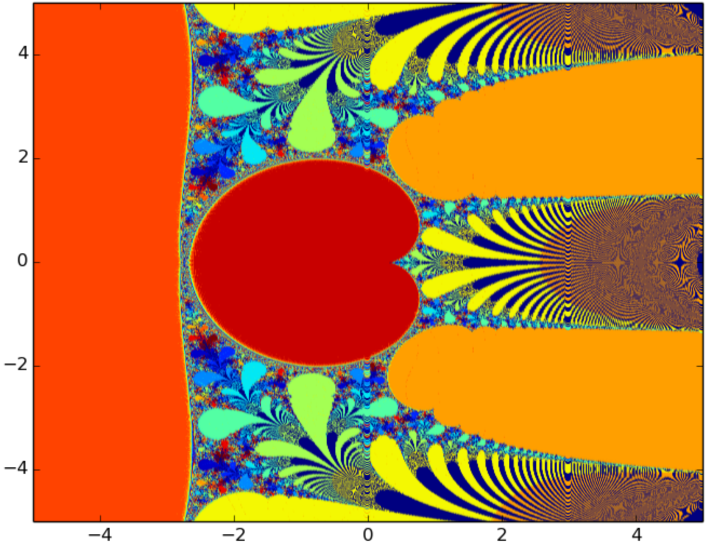
Limit Cycle Diagram



Taking advantage of rapid convergence and divergence property

Mantra:

- Taylor series $\rightarrow$ Continued Exponential

$$\sum_{n=0}^{\infty} c_n z^n \rightarrow a_0 e^{a_1 z e^{a_2 z e^{\cdot^{\cdot^{\cdot}}}}} \rightarrow a_0 = ..., a_1 = ..., a_2 = ... \quad (16)$$

- Partial sums of the continued exponential

$$a_0, a_0 e^{a_1 z}, a_0 e^{a_1 z e^{a_2 z}}, \ldots \quad (17)$$

- Take a weighted average of the continued exponential (Shanks Transform)

Finite Integration : $\int_0^1 \frac{dx}{1+x} = \log(2) = 0.69314718056$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \Big|_{x=1} = xe^{-0.5xe^{-0.41667xe^{\cdot^{\cdot^{\cdot}}}}}\Big|_{x=1} \tag{18}$$

Table : Accelerating the convergence of $\log(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ using shanks and Continued Exponential.

n	Partial Sum	CE	S^3(CE)
1	1.0000000	-	-
2	0.5000000	-	-
3	0.8333333	-	-
4	0.5833333	-	-
5	0.7833333	-	-
6	0.6166667	-	-
7	0.7595238	-	-

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1	1.0000000	0.7191967497444082	-
2	0.5000000	0.6857283810599458	-
3	0.8333333	0.6952583599753418	-
4	0.5833333	0.6925515796826819	-
5	0.7833333	0.6933147356768786	-
6	0.6166667	0.6931001655700353	-
7	0.7595238	0.6931603520385945	-

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Riemann Zeta Function : $\zeta(4)$

Table : $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots = 1.0823232337$

n	Partial Sum	CE	S^3(CE)
1	1.0644944589	-	-
2	1.0765985126	-	-
3	1.080031458	-	-
4	1.081263548	-	-
5	1.0817803659	-	-
6	1.082022856	-	-
7	1.0821467102	-	-
8	1.0822143374	-	-
9	1.0822533137	-	-
10	1.082276805	-	-

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3	1.080031458	1.0819533006	-
4	1.081263548	1.0821537913	-
5	1.0817803659	1.0822372034	-
6	1.082022856	1.0822760122	1.0823230265
7	1.0821467102	1.0823206268	-
8	1.0822143374	1.0823063365	-
9	1.0822533137	1.0823124421	-
10	1.082276805	-	-

Extracting more digits of π

Table : Calculating the value of $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 0.78539816339$ using Aitken's delta-squared process v/s Continued Exponentials

n	partial sum	Ai	CE	S^3(CE)
1	-0.3333333333	0.78333333	0.71653131057	-
2	0.2	0.78630952	0.80564282461	-
3	-0.1428571429	0.78492063	0.77955389775	-
4	0.1111111111	0.78567821	0.78706041738	-
5	-0.09090909091	0.78522034	0.78492829136	0.7853981632
6	0.07692307692	0.78551795	0.78553041639	-
7	-0.06666666667	-	0.78536103437	-
8	0.05882352941	-	0.78540856837	-

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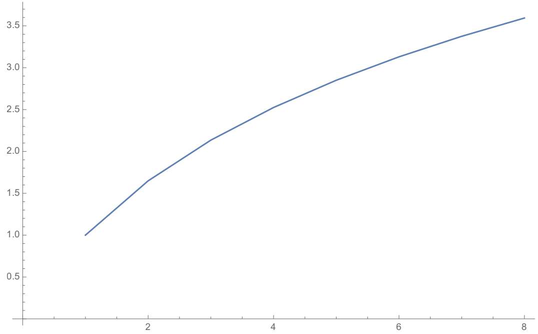
- Example: $\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots = \infty$
- Here's the continued exponential



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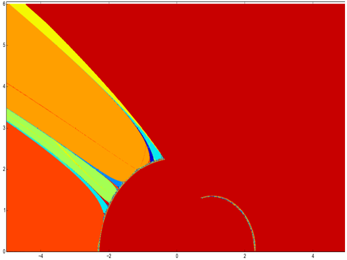
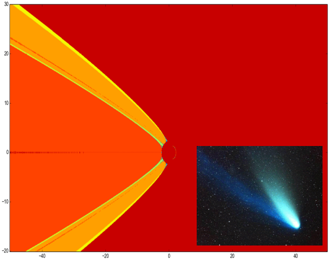
- QFT and perturbative methods you know only a few coefficients of the series

Some other interesting Limit cycle diagrams

Comet Map - $x_{n+1} = \log(1 + z * x_n)$





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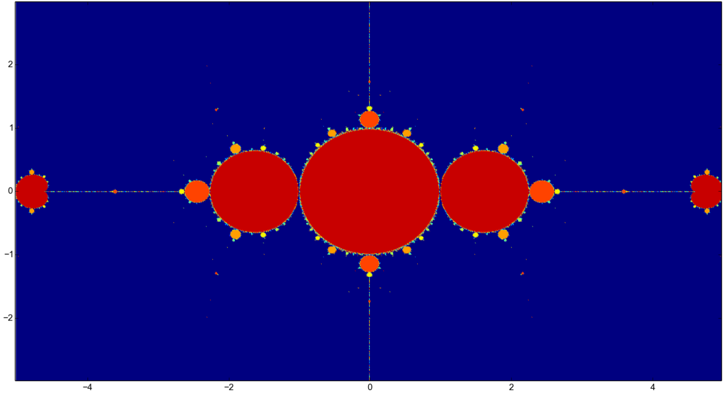
Sine Map - $x_{n+1} = \sin(z * x_n)$



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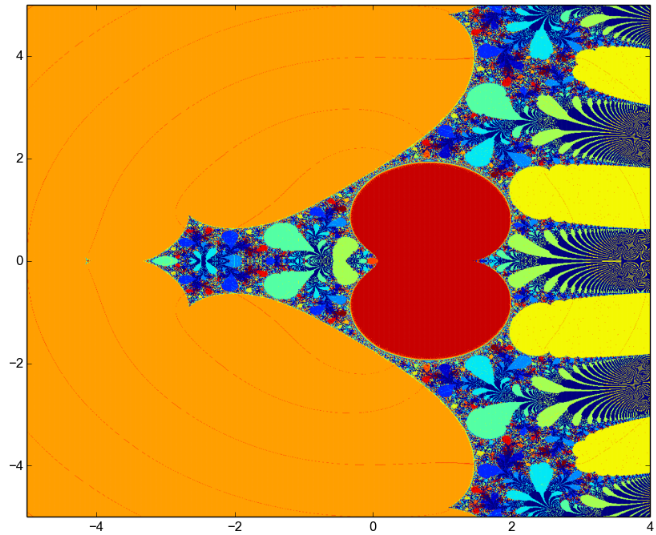
Lambert's Z Map - $x_{n+1} = z^{x_n}$



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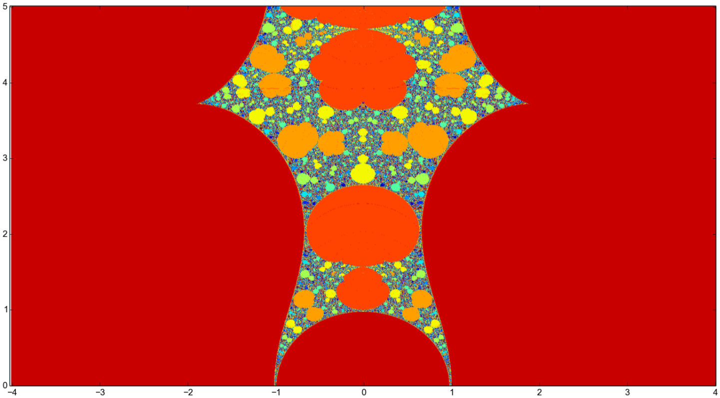
Ana's Map - $x_{n+1} = \sinh(z * x_n)$



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Thank you!