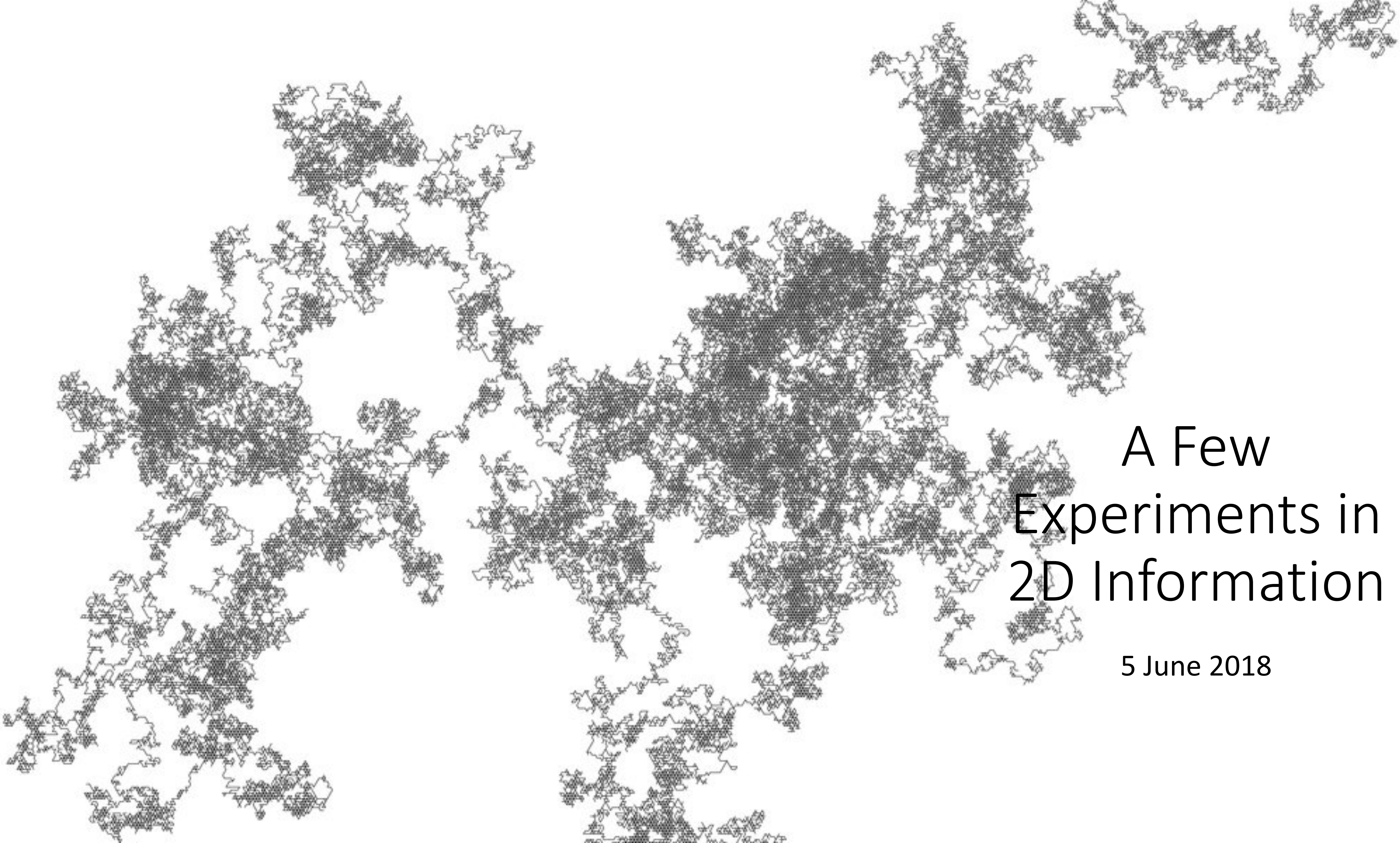


A Few Experiments in 2D Information

5 June 2018

Background

- Interests: Probability Theory/Mathematical Physics
- Modern Probability Theory: Random Matrices, Percolation, Universality results…
- An Interesting Example: Self-Avoiding Walks (SAW) and the Connective Constant

$$c_n \approx \mu^n n^{\gamma-1}$$

- Honeycomb lattice: $\mu = \sqrt{2+\sqrt{2}}$ (Duminil-Copin/Smirnov, 2011)
- $\gamma = 43/32$ (conjectured) is universal: only depends on dimension

Background

- Past project idea: SAWs as an approximation to information theory in 2 dimensions
- Self avoiding assumption is necessary to avoid unnatural correlations
- Gather information measures well understood in the 1D setting along paths in a grid
- Possible suggestion: Delve deeper into study of SAWs using tools of IT

Motivation and Challenges

- Generalize measures of information such as Excess Entropy or Entropy Rate
- Classical IT : Discrete Time Stochastic Processes

$$\{X_t\} \longrightarrow \{X_{m,n}\}$$

- For topological reasons 2D case is more interesting
- Applications to image processing
- No clear generalization: Use well known 1D techniques
- Theoretical Difficulties: No canonical way to scan a lattice
- Computational Difficulties:

$$c_n \approx \mu^n n^{\gamma-1}$$

- We are making sampling assumptions about the weight of each path

Literature

Lempel, Ziv (1986)

- Assymptotic compressibility: Analogue of Entropy Rate
- Does not capture geometrical/causal structure
- Peano-Hilbert curve construction

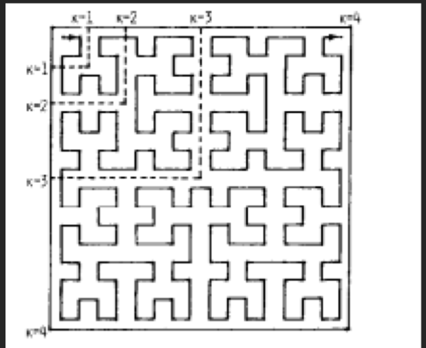
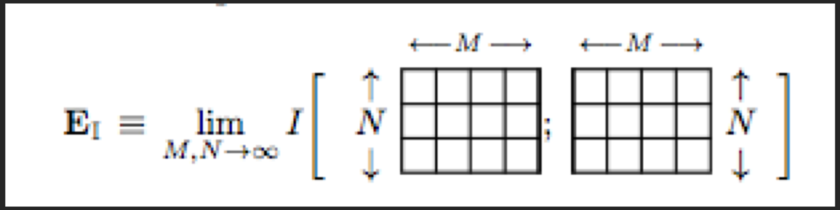


Fig. 4: Standard scan for I_{2^k}; $k = 1, 2, 3, 4$.

Crutchfield, Feldman (2002)

- Generalizations of Excess Entropy
- Works well in the setting of the Ising Model
- E_I can distinguish periodic states that are structurally distinct; e.g. checkerboard and left diagonal pattern



Methods: Measures

- **Block entropy:**

$$H(L) = \sum_{s^L \in \mathcal{A}^L} \mathbb{P}(s^L) \log \mathbb{P}(s^L)$$

- **Entropy rate:**

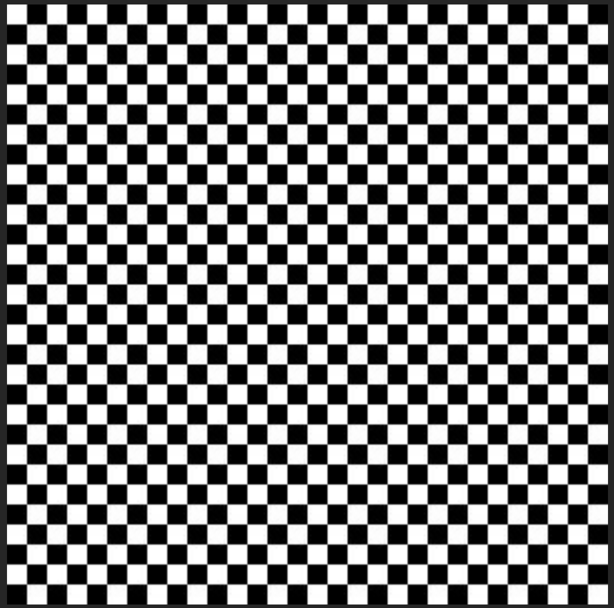
$$\lim_{L \to \infty} \frac{\boldsymbol{H(L)}}{\boldsymbol{L}}$$

- **Excess entropy :**

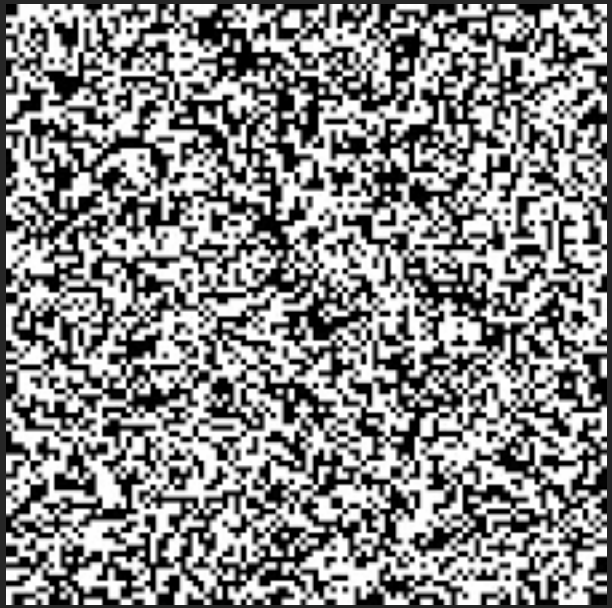
$$\boldsymbol{E} = \boldsymbol{I}(\boldsymbol{X}_{\leftarrow}; \boldsymbol{X}_{\rightarrow}) = \lim_{L \to \infty} \boldsymbol{I}(\boldsymbol{X}_{\boldsymbol{0}}, \dots \boldsymbol{X}_{\boldsymbol{L-1}}; \boldsymbol{X}_{\boldsymbol{L}}, \dots \boldsymbol{X}_{\boldsymbol{2L-1}})$$

Methods: Grid

Checkerboard: No randomness in next step



White Noise: Next step is 1 or -1 with probability 1/2



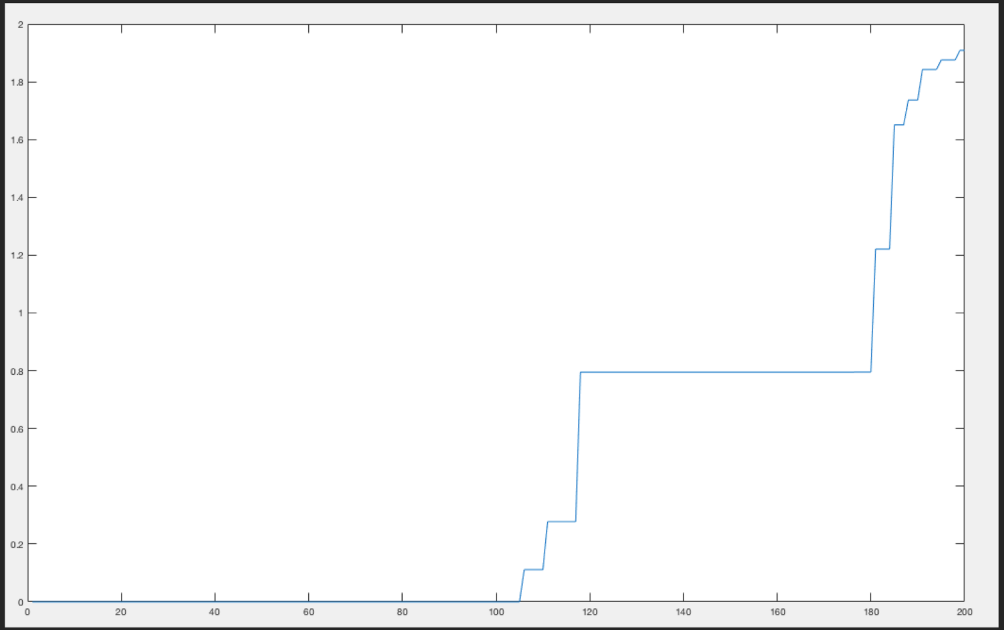
Methods: Generating Paths

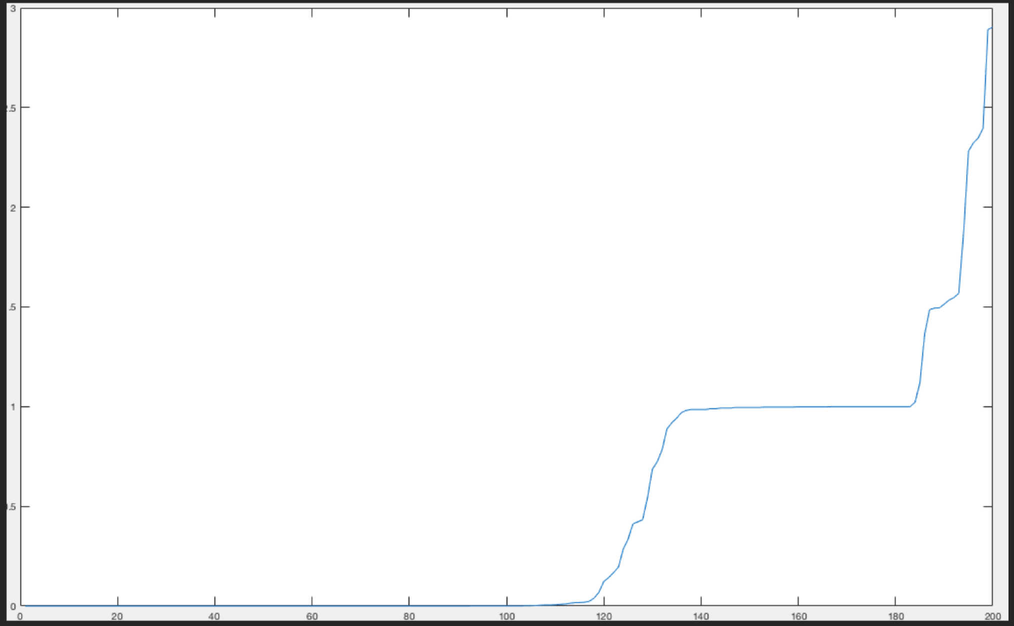
- State emitting HMM: We identify the symbols 1/2/3/4 with directions (resp.) U/R/D/L
- No theoretical reason to use this; just a convenient way to generate SAWs.
- We can control potentially interesting aspects like length of walks or drift.
- MATLAB built-in tool: hmmgenerate

```
TRANS = [.5 .5; .5 .5];

EMIS = [1/4, 1/4, 1/4, 1/4;
1/4, 1/4, 1/4, 1/4];

[seq,~] = hmmgenerate(25,TRANS,EMIS);
for n=1:25
```

Experiements: Spectrum of Information

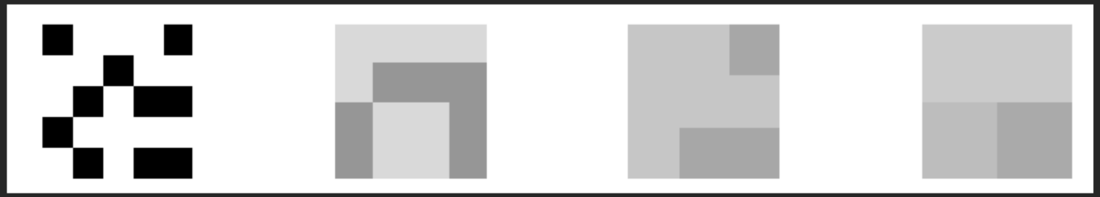
200 Walks generated by the same HMM on a 3x3 checkerboard vs random configuration: With a periodic structure fewer values are attained and the random configuration appears as a “smoothed” version of the checkerboard spectrum.

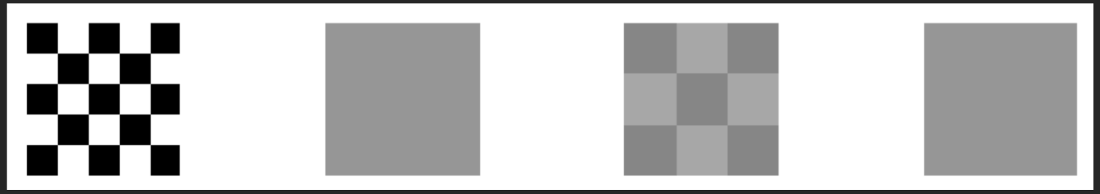




Experiments: Sliding Window

- Idea: 2D Analogue of entropies of length n substrings in written text.
- Method: For each grid site consider reassign its value by the average of an $m \times m$ square (Mollification/Heat Eq.)
- Intuition: More disordered configurations should have persistently higher entropies at different resolutions.





Remarks

- Lack of generality in the definitions might mean that, in practice, structural information has to be measured on a case by case basis
- Information measures of structural complexity might not be enough; incorporate notions of difficulties of learning and synchronizing to patterns

Acknowledgments and References

Code and suggestions: Jordan Snyder

[1] Lempel, Abraham, and Jacob Ziv. "Compression of two dimensional data." Information Theory, IEEE Trans actions on 32.1 (1986): 2.8

[2] Feldman, David P. and James P. Crutchfield. "Structural information in two dimensional patterns: Entropy convergence and excess entropy." Physical Review E 67.5 (2003): 051104

[3] J. P. Crutchfield and D. P. Feldman, "Regularities Unseen, Randomness Observed: Levels of Entropy Convergence", CHAOS 13:1 (2003) 25-54.