# Robust Periodic Solution in One-dimensional Multi-state Edge Cellular Automata 

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#### Abstract

We study one-dimensional range $1 / 2$ cellular automata with 3 and more states that advances toward a single direction. We investigate the properties including bounded growth and doubly periodic configurations that grows into any environment, namely robust periodic solutions. We also focus on the probability that such a configuration exists when the number of states is large.


## 1 Introduction

### 1.1 Motivation for the topic

In the recent monograph [3], one-dimensional range-2 edge cellular automata (CA) with two values starting with semi-infinite initial configuration are studied. Properties of bounded growth, existence of robust periodic solution and replication are investigated among 64 kinds of such CAs. Our goal is to generalize the results to the case of $n$-value CAs, and focus on the probabilistic behaviour of the existence of the doubly periodic configurations as the number of states $n$ approaches to infinity.

### 1.2 Why it is interesting

Doubly periodic behaviour has been widely investigated. Among regular doubly periodic configurations, a subset of them has the property that the spatial periodicity has positive growing velocity. Such configurations are called robust periodic solutions (RPS). See Section 3.4 for a formal definition and examples. In [3], among 64 range- 2 edge rules, 22 of them were found to have RPS. So, it is interesting to ask: how common are the robust periodic solutions when the number of states is large?

However, when the number of states is large, it is less possible to investigate the existence of RPS by enumerating all the rules: Even if we restrict the local rule to be range- $1 / 2$ (see (1)), there are still $n^{n^{2}-n}$ such rules, where $n$ represent the number of states. As a result, it is more realistic to investigate such behaviour in a probabilistic view.

### 1.3 Synopsis of project and results

In Section 2, we introduce our playground: the 1 -dimensional $n$-state edge cellular automata and the question to be investigated. In Section 3, we introduce the notations and define the main properties and tools. In Section 4, we present the main results. We will prove a lower bound of the probability of the existence of the RPS and present several experiment result.

## 2 Background

Let a configuration of an $n$-state CA at time $t \in \mathbb{N}$ be an element $\xi_{t}=\left(\xi_{t}(x)\right)_{x \in \mathbb{Z}} \in \mathbb{Z}_{n}^{\mathbb{Z}}$. We focus on the CAs with evolution

$$
\begin{equation*}
\xi_{t+1}(x)=f\left(\xi_{t}(x-1), \xi_{t}(x)\right) \tag{1}
\end{equation*}
$$

with restriction

$$
\begin{equation*}
f(0, a)=a, \quad \forall a \in \mathbb{Z}_{n} \tag{2}
\end{equation*}
$$

and initial configuration $\xi_{0}$ satisfying $\xi_{0}(0) \neq 0$ and $\xi_{0}(x)=0$ for all $x<0$.

Equation (1) characterizes the range- $1 / 2$ nature: the value of a cell at time $t$ depends only on the value of this cell and its left cell at time $t-1$; the edge property is defined by equation (2) together with the restriction of the initial configuration: since $0 \underline{a} \mapsto \underline{a}$ for all $a$, an edge with constant $a$ is formed at the leftmost non-zero cell of $\xi_{0}$.

Clearly, fix the number of states $n$, there are $n^{n^{2}-n}$ such one-dimensional range- $1 / 2$ edge CAs rules. For $n=3$, it is possible to enumerate the existence of a property rule by rule, while for $n \geq 4$, it is more practical to analyse the existence of a property in the probabilistic sense. More precisely, consider the probability space $\left(\Omega_{n}, \mathscr{F}_{n}, \mathbb{P}_{n}\right)$, where $\Omega_{n}$ contains all of the $n^{n^{2}-n}$ rules for a fixed $n ; \mathscr{F}_{n}$ is the $\sigma$-algebra of $\Omega_{n}$ containing all of the subsets of $\Omega_{n} ; \mathbb{P}_{n}$ assigns uniform probability $\mathbb{P}_{n}(\{f\})=1 /\left|\Omega_{n}\right|=1 / n^{n^{2}-n}$ for all $f \in \Omega_{n}$. Let $Y_{n} \subset \Omega_{n}$ be the set of the $n$-state rules that have a specific property, e.g., having at least a robust periodic solution. Our goal is to investigate $\lim _{n \rightarrow \infty} \mathbb{P}_{n}\left(Y_{n}\right)$.

## 3 Preliminaries

In this section, we briefly introduce our notation, and then present some definitions regarding the properties to be discussed later, some of which are borrowed from [3].

### 3.1 Notation

Let $\mathbb{Z}_{n}=\{0,1, \cdots, n-1\}$ and $\mathbb{N}=\{0,1, \cdots\}$. A configuration of an $n$-state CA at time $t \in \mathbb{N}$ with initial configuration $A_{0}$ is an element $\xi_{t}^{A_{0}}=\left(\xi_{t}^{A_{0}}(x)\right)_{x \in \mathbb{Z}} \in \mathbb{Z}_{n}^{\mathbb{Z}}$. The initial configuration $A_{0}$ can also be omitted if it does not cause any confusion. We call a configuration finite if it has finitely many non-zero values or it has finite length. $\left[\begin{array}{ll}m & C\end{array}\right]$ represent the configuration formed by appending $m$ $C$ 's consecutively, say, $\left[\begin{array}{ll}3 & 120\end{array}\right]=120120120$.

A rule of an $n$-state range- $1 / 2$ edge CA , or simply edge CA , is a function $f: \mathbb{Z}_{n} \times \mathbb{Z}_{n} \rightarrow \mathbb{Z}_{n}$ with $f(0, a)=a$ for all $a \in \mathbb{Z}_{n}$. Denote $a \underline{b} \mapsto c$ if $f(a, b)=c$. For a 3 -state edge CA, the nomenclature is straight forward: rule $a b c d e f$ represents the rule such that $2 \underline{2} \mapsto a, 2 \underline{1} \mapsto b, 2 \underline{0} \mapsto c, 1 \underline{2} \mapsto d$, $1 \underline{1} \mapsto e$ and $1 \underline{0} \mapsto f$.

### 3.2 Induced Graph

A cycle with (temporal) period $\pi$ is a configuration restricted to the spatial interval $[0, N],\left.\xi_{t}\right|_{[0, N]}$,
 period $\pi$ is a directed graph on vertices $\left\{x_{0} x_{1} \cdots x_{\pi-1} \mid x_{j} \in \mathbb{Z}_{n}\right\}$ that is constructed in the following
 all $k \in[\pi-1]$; delete the labels and the adjacent edges that cannot be reached from the label


Figure 1: The induced graph $(\pi=3)$ of rule 102222.
$00 \cdots 0$ ( $\pi 0$ 's). With this definition, self-loops are allowed in a induced graph. The induced graph is a coarser version of the definition of stage I label tree in [3], in the sense that we do not rule out equivalent labels (labels that are shift of each other, e.g., 01 and 10,0123 and 2301 etc.).

Intuitively, an induced graph provides with the information about how to extend the temporal period $\pi$ on the spatial position $k$ to $k+1$, i.e., to the right. For a simple example, the induced graph of period 3 of the 3 -state rule 102222 is shown in Fig. 1. For instance, at the spatial position -1 , the values of the cells are $000000 \cdots$ (viewed vertically and as having period 3 ), then to extend this period, the values at the spatial position 0 has to be one of $000000 \cdots, 111111 \cdots$ or $222222 \cdots$. If the states are $102102 \cdots$ at a spatial position $k$, to extend this period 102 behaviour to $k+1$, the values at the spatial position $k+1$ has to be $122122 \cdots$.

### 3.3 Growth Velocity

If the initial configuration $A_{0} \in \mathcal{A}_{0}:=\{$ configurations with only finitely many non-zero states $\}$, we may investigate the growth velocity of the (right) boundary of $\xi_{t}$. We define $s_{g}(\xi)$ as the site of the rightmost non-zero state in the configuration $\xi$. Hence, for a initial configuration $A_{0}$, we define its growth velocity starting with $A_{0}$ to be

$$
v_{g}\left(A_{0}\right)=\limsup _{t \rightarrow \infty} \frac{s_{g}\left(\xi_{t}^{A_{0}}\right)}{t},
$$

and the growth velocity of an edge CA is

$$
v_{g}=\sup _{A_{0} \in \mathcal{A}_{0}} v_{g}\left(A_{0}\right)
$$

A CA is said to have bounded growth if its growth velocity is 0 , i.e., there exists an integer $K=$ $K\left(A_{0}\right)$ such that $s_{g}\left(\xi_{t}^{A_{0}}\right)<K$, for all $t \geq 0$ and all $A_{0} \in \mathcal{A}_{0}$.

### 3.4 Robust Periodic Solution

Let $H$ and $L$ be configurations with finite length $h$ and $l$, respectively. Also, let the first (leftmost) position of $H$ be non-zero. Form the initial configuration $\xi_{0}$ by appending $H$ with infinitely many $L$, denoted by $H L^{\infty}$. Run a fixed CA rule $f$ starting with $\xi_{0}$ until time $\pi$. If $\xi_{\pi}=\xi_{0}$, we call $H$ a handle and $L$ a link and we say we found a periodic solution $H L^{\infty}$ or periodic handle-link pair $H+L$ of the CA rule $f$. Intuitively, a handle-link pair is a doubly periodic configuration with temporal period $\pi$ and spatial period $l$. Note that the handle $H$ does not participate the spatial periodic behaviour.

Some of the periodic solutions are of particular interests. For a periodic handle-link pair $H+L$ of a rule $f$, if one runs the CA starting with $\xi_{0}=H+L+R$, then for any $m \in \mathbb{N}$ there exists a time $t_{m}$ such that $\xi_{t_{m}}=H+L^{m}+R^{\prime}$, where $X$ and $X^{\prime}$ are arbitrary $n$-state configurations. That is, The periodic handle-link pair $H+L$ grows into any environments. Formally, fix a handle-link pair $H+L$. Let $A_{0}$ be any configuration in the form of $H+L+R$, where $R$ is any random semi-infinite configuration. Let

$$
s_{t}=\max \left\{x \mid \xi_{t}^{A_{0}}(y)=\xi_{t}^{H L^{\infty}}(y), \forall y<x\right\} .
$$

be the furtherest spatial position such that the configuration (at time $t$ ) starting with $A_{0}$ and starting with $H L^{\infty}$ agree. Then the expansion velocity in environment $A_{0}$ is

$$
v\left(A_{0}\right)=\liminf _{t \rightarrow \infty} \frac{s_{t}}{t}
$$

and the expansion velocity of $H+L$ is

$$
v(H+L)=\inf v\left(A_{0}\right) .
$$

If $v(H+L)>0$, then the handle-link pair $H+L$ is called a robust periodic solution (RPS) or robust periodic handle-link pair. See Fig. 2 for a 3 -state case illustration, where a white, red and a black cell represent 0,1 and 2 , respectively. The periodic solution is generated by the 3 -state rule 122012 with initial configuration $\xi_{0}=1 \quad\left[\begin{array}{ccc}3 & 011211021222^{\prime} s\end{array}\right] \quad R$, where $R$ is a random finite configuration. Restricting on the first $1+3 \times 12=37$ sites, it is a doubly periodic configuration with temporal period 4 and spatial period 12 , where the spatial periodicity starts after the handle 1. The robust periodic solution is generated by the 3 -state rule 102222 with initial configuration $\xi_{0}=1202\left[\begin{array}{ll}3 & 221102^{\prime} s\end{array}\right] \quad R$, where $R$ is a random finite configuration. It is a doubly periodic configuration with temporal period 3 and spatial period 12 and the spatial periodicity "grows" into the right environment.

(a) A periodic solution

(b) A robust periodic solution

Figure 2: Periodic solution vs robust periodic solution.

## 4 Main Result

### 4.1 Bounded Growth

It is clear that a bounded growth rule cannot have a RPS. However, the following result describes that a rule has 1 as the growth velocity with high probability.
Proposition 1. With uniform distribution among $n^{n^{2}-n}$ rules, the probability that a rule has growth velocity 1 is $1-\frac{1}{n}$.

Proof. Fix an $n$. Let $g: \mathbb{Z}_{n} \rightarrow \mathbb{Z}_{n}$ be defined as $g(a)=b$ if $a \underline{0} \mapsto b$. Clearly, noting that $g(0) \equiv 0$, there are $n^{n-1}$ such functions. Also note that a rule has growth velocity $<1$ if and only if

$$
\begin{equation*}
\text { for any } a \in \mathbb{Z}_{n}, g^{k}(a)=0 \text { for some } k>0, \tag{3}
\end{equation*}
$$

where this $k$ may depend on $a$. Note that there is a 1-1 correspondence between $g$ 's satisfying (3) and labelled trees with vertices $\mathbb{Z}_{n}$. See Fig. 3 for a proof by example. There are $n^{n-2}$ such trees by Cayley's formula and thus the result follows.


Figure 3: The labelled tree corresponding to the map $1 \mapsto 0,2 \mapsto 1,3 \mapsto 0,4 \mapsto 3$ and $5 \mapsto 1$.

### 4.2 Robust Periodic Solution

The existence of a periodic solution is strongly connected to the structure of the induced graph of a rule. Formally, it is easy to see that

Proposition 2. An edge rule $f$ has a period handle-link pair with temporal period $\pi$ if and only if there is a cycle (including self-loop) in the induced graph of $f$.

The existence of RPS relies on the existence of a special kind of cycles in the induced digraph. Specifically, in a directed graph, we call a cycle faithful if the nodes on the cycle have outer degree 1. Now, we rephrase the proposition 2.2 in [3] as

Proposition 3. [3] An edge rule $f$ has a robust periodic solution with temporal period $\pi$ if and only if ther is a faithful cycle (including self-loop) in the induced graph of $f$.

Proposition 4. With uniform distribution among rules, the probability that an n-state range-1 edge cellular automata has at least a constant robust periodic solution approaches to 1 /e as $n$ approaches to infinity.

Proof. Fix a positive integer $K \ll n$. We compute $\mathbb{P}\left(\exists\right.$ a faithful cycle of length $\leq K$ on $\left.G_{n, 1 / n}\right)$ as $n$ goes to $\infty$. Let ( $k, i$ ) represent a $k$-cycle (a cycle with $k$ nodes) on a random digraph $G_{n, 1 / n}$ with index $i$, where $i=1,2, \ldots,\binom{n-1}{k}:=N_{k}$. (Here note that the vertex 0 always has outer_degree $(v)=$ n.)

Let $\Gamma$ be the set of all cycles of length $\leq K$. Let $\Gamma^{(k)}$ be the set of of $k$-cycles for a fixed $k$. Let $\Gamma_{(k, i)} \subset \Gamma$ be the set of cycles that have at least a joint vertex with a fixed cycle ( $k, i$ ), excluding $(k, i)$ itself and $\Gamma_{(k, i)}^{(l)}$ be such cycles of length $l$.

Let

$$
I_{(k, i)}= \begin{cases}1 & \text { if }(k, i) \text { is faithful } \\ 0 & \text { otherwise }\end{cases}
$$

be the indicator of the goodness of cycle $(k, i)$. Let $p_{(k, i)}=\mathbb{E} I_{(k, i)}=\left[\frac{1}{n}\left(\frac{n-1}{n}\right)^{n-1}\right]^{k}=\left(\frac{p}{n}\right)^{k}$, where $p=\left(\frac{n-1}{n}\right)^{n-1}$, be the probability that $(k, i)$ is faithful. Let $W_{K}=\sum_{k=1}^{K} \sum_{i=1}^{N_{k}} I_{(k, i)}$ be the random variable of the number of good cycles of length $\leq K$ in $G_{n, p}$. Let $\lambda_{K}=\mathbb{E} W_{K} \rightarrow \sum_{j=1}^{K}\left(j e^{j}\right)^{-1}$ as $n \rightarrow \infty$.

To apply the Stein-Chen method

$$
d_{T V}\left(W_{K}, \lambda_{K}\right) \leq \min \left(1, \lambda_{K}^{-1}\right)\left[\sum_{(k, i) \in \Gamma} p_{(k, i)}^{2}+\sum_{(k, i) \in \Gamma} \sum_{(l, j) \in \Gamma_{(k, i)}}\left(p_{(k, i)} p_{(l, j)}+\mathbb{E} I_{(k, i)} I_{(l, j)}\right)\right],
$$

first note that $\min \left(1, \lambda_{K}^{-1}\right)=O(1)$ and $\mathbb{E} I_{(k, i)} I_{(l, j)}=0$ for $(k, i) \in \Gamma$ and $(l, j) \in \Gamma_{(k, i)}$ since two cycles are both non-faithful whenever they have joint vertices.

$$
\text { Also } \sum_{(k, i) \in \Gamma} p_{(k, i)}^{2}=\sum_{k=1}^{K} \sum_{i=1}^{N_{k}}\left(\frac{p}{n}\right)^{2 k}=\sum_{k=1}^{K} \frac{(n-1) \ldots(n-k)}{k!}\left(\frac{p}{n}\right)^{2 k}=o(1) .
$$

Last we need to bound

$$
\begin{align*}
& \sum_{(k, i) \in \Gamma} \sum_{(l, j) \in \Gamma_{(k, i)}} p_{(k, i)} p_{(l, j)} \\
= & \binom{n-1}{1}\left[\sum_{(1, j) \in \Gamma_{(1, i)}^{(1)}}\left(\frac{p}{n}\right)^{2}+\sum_{(2, j) \in \Gamma_{(1, i)}^{(2)}}\left(\frac{p}{n}\right)^{3}+\cdots+\sum_{(K, j) \in \Gamma_{(1, i)}^{(K)}}\left(\frac{p}{n}\right)^{K}\right]+ \\
& \binom{n-1}{2}\left[\sum_{(1, j) \in \Gamma_{(2, i)}^{(1)}}\left(\frac{p}{n}\right)^{3}+\sum_{(2, j) \in \Gamma_{(2, i)}^{(2)}}\left(\frac{p}{n}\right)^{4}+\cdots+\sum_{(K, j) \in \Gamma_{(2, i)}^{(K)}}\left(\frac{p}{n}\right)^{K+1}\right]  \tag{4}\\
& +\cdots+ \\
& \binom{n-1}{K}\left[\sum_{(1, j) \in \Gamma_{(K, i)}^{(1)}}\left(\frac{p}{n}\right)^{K+1}+\sum_{(2, j) \in \Gamma_{(K, i)}^{(2)}}\left(\frac{p}{n}\right)^{K+2}+\cdots+\sum_{(K, j) \in \Gamma_{(K, i)}^{(K)}}\left(\frac{p}{n}\right)^{2 K}\right]
\end{align*}
$$

For a fixed pair of $(k, l)$,

$$
\binom{n-1}{k} \sum_{(l, j) \in \Gamma_{(k, i)}^{(l)}}\left(\frac{p}{n}\right)^{l+k}=\binom{n-1}{k}\left[\sum_{j=1}^{l}\binom{k}{j}\binom{n-1-k}{l-j}\right] l!\left(\frac{p}{n}\right)^{l+k}=O\left(\frac{1}{n}\right) .
$$

So, $d_{T V}\left(W_{K}, \lambda_{K}\right)=o(1)$. So, $\mathbb{P}(\exists \operatorname{good}$ cycle of length $\leq K)=\mathbb{P}\left(W_{K}>0\right)=1-e^{-\lambda_{K}}$. Let $W=\lim _{K} W_{K}$ be the total number of good cycles. Note that $\left\{W_{K}>0\right\} \uparrow\{W>0\}$. So, $\mathbb{P}(W>0)=\lim _{K} 1-e^{-\lambda_{K}}=e^{-1}$.

Fig. 4 displays the simulation result for number of states from 2 to 50 . Within each number of state, 10,000 rules were uniformly distributed sampled. The horizontal line is the asymptotic probability $1 / e$.


Figure 4: The probabilities of existing a period-1 RPS.


Figure 5: The probabilities of existing a RPS of period $>1$.

For larger period $k \geq 2$, we present a simulation result in Fig. 5 for the number of states ranging from 2 to 10. Each curve (except the highest one) represents a specific period $k$ for $k=2,3, \cdots, 6$.

Fix a period $k$ (thus a curve), the $y$-axis represents the probability of existing a period- $k$ RPS but no period- $j$ RPS for all $j<k$ given the number of states $n$ (the $x$-axis). The highest one represents the probability of existing a period- $k(k \leq 6)$ RPS given the number of states $n$, namely the sum of the other curves. For, period 2, we run 10,000 simulations within each number of state, while for larger period, we run 1,000 .

It is shown from the graph that the asymptotic probability of existing a period-2 RPS but no period 1 RPS is approximately 0.1 . And the asymptotic probability of existing a period $\leq 6$ RPS is approximately $0.2+1 / e$. Hence, the future work will be to find out what exactly this asymptotic probability is.

## 5 Conclusion

In this project, we investigate the existence of robust period solutions for a certain type of multistate CA. We proved a lower bound $(1 / e)$ of the existence of the probability and presented a simulation result for larger number of states. The future work will be to investigate the exact probability of existing such configuration.

## References

[1] A. D. Barbour, L. Holst, S. Janson. Poisson Approximation. Oxford Science Publications. 1992.
[2] David P. Feldman, James P. Crutchfield. Synchronizing to Periodicity: The Transient Information and Synchronization Time of Periodic Sequences. arXiv:nlin/0208040.
[3] J. Gravner, D. Griffeath. Robust periodic solutions and evolution from seeds in one-dimensional edge cellular automata. Theoretical Computer Science. Volume 466, 28 December 2012, 64-86.
[4] J. Gravner, D. Griffeath, The one-dimensional Exactly 1 cellular automaton: replication, periodicity, and chaos from finite seeds, Journal of Statistical Physics 142 (2011), 168-200.
[5] W. Hordijk, C. Rohilla Shalizi, J. P. Crutchfield. Upper Bound on the Products of Particle Interactions in Cellular Automata. Physica D 154 (2001): 240-258.
[6] K. Sutner. De Bruijn Graphs and Linear Cellular Automata. Complex Systems 5 (1991) 19 30.

