Robust Periodic Solution in One-dimensional Multi-state Edge Cellular Automata

PHY 256

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Background: One-Dimensional Cellular Automata

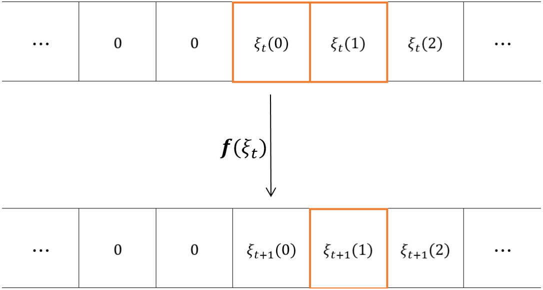
- **Lattice:** $\mathbb{Z}$
- **Site:** $x \in \mathbb{Z}$
- **Time:** $t \in \mathbb{N}$
- **Local state:** $\xi_t(x) \in \mathbb{Z}_n := \{0, 1, \cdots, n-1\}$
- **Global configuration:** $\xi_t = (\cdots, \xi_t(-1), \xi_t(0), \xi_t(1), \cdots) \in \mathbb{Z}_n^{\mathbb{Z}}$

$\cdots$	$\xi_t(-2)$	$\xi_t(-1)$	$\xi_t(0)$	$\xi_t(1)$	$\xi_t(2)$	$\cdots$

$$\Big\downarrow \, \boldsymbol{f}(\xi_t)$$

$\cdots$	$\xi_{t+1}(-2)$	$\xi_{t+1}(-1)$	$\xi_{t+1}(0)$	$\xi_{t+1}(1)$	$\xi_{t+1}(2)$	$\cdots$

The System: Edge Cellular Automata



The System: Edge Cellular Automata

- **Local dynamic:**

$$\xi_{t+1}(x) = f(\xi_t(x-1), \xi_t(x)) \tag{1}$$

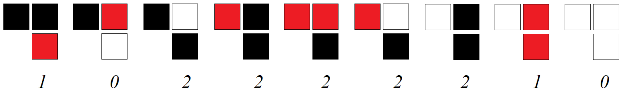
where

$$f : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n \text{ with } f(0,a) = a, \quad \forall a \in \mathbb{Z}_n \tag{2}$$

- **Initial configuration:** $\xi_0(x) = 0$ for all $x < 0$, i.e., semi-infinite.

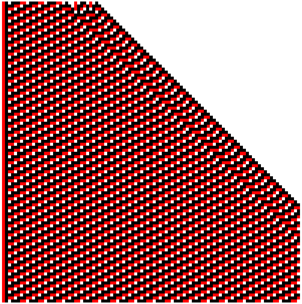
A 3-state Example: Rule *102222*

Local look up table:



A 3-state Example: Rule *102222*

Global evolution:



Main Question

- For a fixed n, there are n^{n^2-n} rules.
- $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$: Probability space.
- $\Omega_n =$ set of n^{n^2-n} rules. $\mathcal{F}_n = \sigma$-algebra. $\mathbb{P}(\{f\}) = 1/n^{n^2-n}$.
- $Y_n \subset \Omega_n$ contains rules with certain property.

- **Question:** What is $\lim_n \mathbb{P}(Y_n)$? Or what is the bound of $\lim_n \mathbb{P}(Y_n)$?

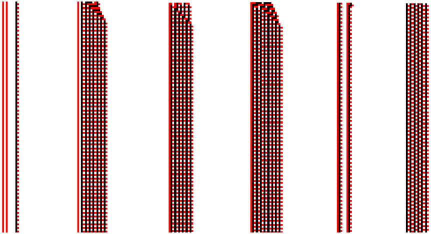
Main Properties

- Bounded growth
- Doubly periodic configuration
  - Existence of periodic solution
  - Existence of robust periodic solution

Def.: A rule f has bounded growth if, for all finite initial configuration A_0, there is a $K = K(A_0)$ such that $\xi_t^{A_0}(x) = 0$, for all t and $x > K$.

Example:

Figure: Rule *101220* from six finite initial configurations



Result of Bounded Growth

Question: What is $\lim_n \mathbb{P}(Y_n)$?

Proposition: With uniform distribution among n^{n^2-n} rules, the probability that a rule has growth velocity 1 is $1 - \frac{1}{n}$. So, the probability that a rule has bounded growth approaches to 0.

Proof

Fix an n. Let $g : \mathbb{Z}_n \to \mathbb{Z}_n$ be defined as $g(a) = b$ if $a\underline{0} \mapsto b$. Clearly, noting that $g(0) \equiv 0$, there are n^{n-1} such functions. Also note that a rule has growth velocity < 1 if and only if

$$\text{for any } a \in \mathbb{Z}_n, g^k(a) = 0 \text{ for some } k > 0, \tag{3}$$

where this k may depend on a. Note that there is a 1-1 correspondence between g's satisfying (3) and labelled trees with vertices $[n-1]$. See the following figure for a proof by example. There are n^{n-2} such trees by Cayley's formula and thus the result follows.

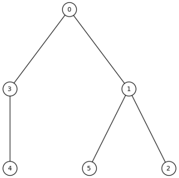


Figure: The labelled tree corresponding to the map $1 \mapsto 0$, $2 \mapsto 1$, $3 \mapsto 0$, $4 \mapsto 3$ and $5 \mapsto 1$.

Periodic Solution

- Handle: Finite configuration H with length h;
- Link: Finite configuration L with length l.
- Append m (can be ∞) Ls to H: $H + L + \cdots + L = HL^m$.

- **Def.:** Fix a rule f and run the CA starting with HL^m. If $\xi_\pi^{HL^m}|_{[0,h+ml]} = HL^m$, then $H + L$ is called a handle-link pair with period π.

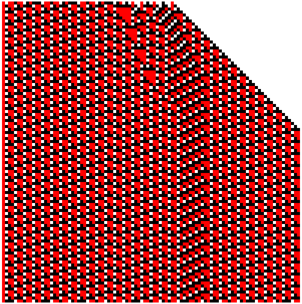
- **Intuition:** HL^m is doubly periodic with temporal period π and spatial period l.

A PS Example

Rule *122012*.

$\xi_0 = H + L + L + L + R$, where $H = 1$, $L = 011211021222$ and R is a random finite configuration. $\pi = 4$.

Figure: A PS of rule *122012*



Robust Periodic Solution

- **Def.:** (expansion velocity) Fix a handle-link pair $H + L$. Let A_0 be any configuration in the form of $H + L + R$, where R is any random semi-infinite configuration. Let

$$s_t = \max\{x | \xi_t^{A_0}(y) = \xi_t^{HL^\infty}(y), \forall y < x\}.$$

Then the expansion velocity in environment A_0 is

$$v(A_0) = \liminf_{t \to \infty} \frac{s_t}{t}$$

and the expansion velocity of $H + L$ is

$$v(H + L) = \inf v(A_0).$$

- **Def.:** A handle link pair $H + L$ is robust if $v(H + L) > 0$.

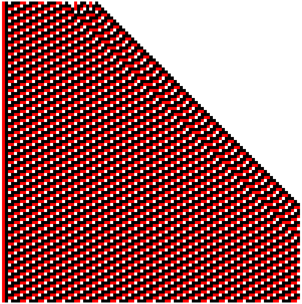
- **Intuition:** HL^m is doubly periodic and the spatial periodic part L grows into any environment with positive velocity.

A RPS Example

Rule *102222*.

$\xi_0 = H + L + L + L + R$, where $H = 1202$, $L = 221102$ and R is a random finite configuration. $\pi = 3$.

Figure: A RPS of rule *102222*



Main Tool: Induced Graph

- Fix a temporal period π.
- Directed graph on vertices $\{x_0 x_1 \cdots x_{\pi-1} | x_j \in \mathbb{Z}_n\}$.
- Construct an arc from $x_0 x_1 \cdots x_{\pi-1}$ to $y_0 y_1 \cdots y_{\pi-1}$ if and only if $f(x_{k \mod \pi}, y_{k \mod \pi}) \mapsto y_{k+1 \mod \pi}$ for all $k \in \mathbb{Z}_\pi$.
- Remove the labels that are not reachable from the label $00 \cdots 0$.

- **Intuition:** Provides with the info.: to extend the periodicity (with period π) from site k to $k+1$, what should we do.

Induced Graph: Example

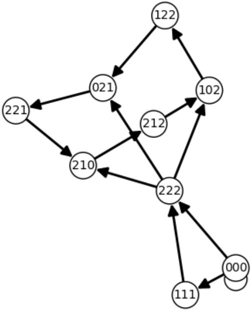


Figure: The induced graph ($\pi = 3$) of rule *102222* – the one a RPS was found. Labels having degree 0 are not shown.

Between RPS and Induced Graph

- **Theorem:** [Gravner] (rephrasing) An edge rule f has a PS of period π if and only if there is a cycle on the induced graph of f of period π; it has a RPS of period π if and only if there is a **faithful** cycle on the induced graph of f of period π.

- **Def.:** A cycle is *faithful* if each of its nodes has outer degree 1.

$\pi = 1$ Case

- Induced graphs are on vertices $\mathbb{Z}_n$.
- There is an arc from k to j if and only if $f(k, j) = j$, which holds with probability $1/n$ whenever $k \neq 0$.
- Reduces (almostly) to an Erdős-Rényi graph $G_{n,1/n}$: an random $n-$vertex directed graph such that each arc is formed with probability $1/n$.
- **Question:** What is the probability that a $G_{n,1/n}$ has a cycle? A faithful cycle?

Result for $\pi = 1$ Case

- **Proposition:** With uniform distribution among n^{n^2-n} rules, the probability that an n-state range-1 edge CA has at least a $\pi = 1$ PS approaches to 1 as n approaches to infinity. Consequently, the probability that an n-state range-1 edge cellular automata has at least a PS approaches to 1.

- **Proposition:** With uniform distribution among n^{n^2-n} rules, the probability that an n-state range-1 edge cellular automata has at least a $\pi = 1$ RPS approaches to $1/e$ as n approaches to infinity.

Not Solvable Yet

How about the general case?

Reference

Janko Gravner, David Griffeath. *Robust periodic solutions and evolution from seeds in one-dimensional edge cellular automata*. Theoretical Computer Science. Volume 466, 28 December 2012, Pages 64-86.