Entropy of Ising model on Bethe lattice:

how to define entropy in infinite system?

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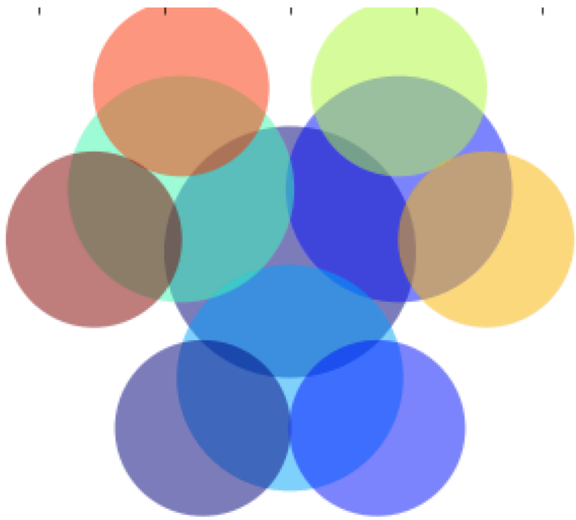
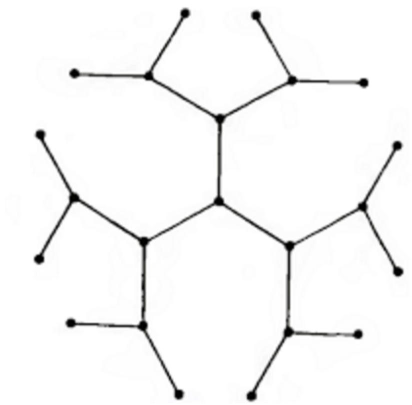
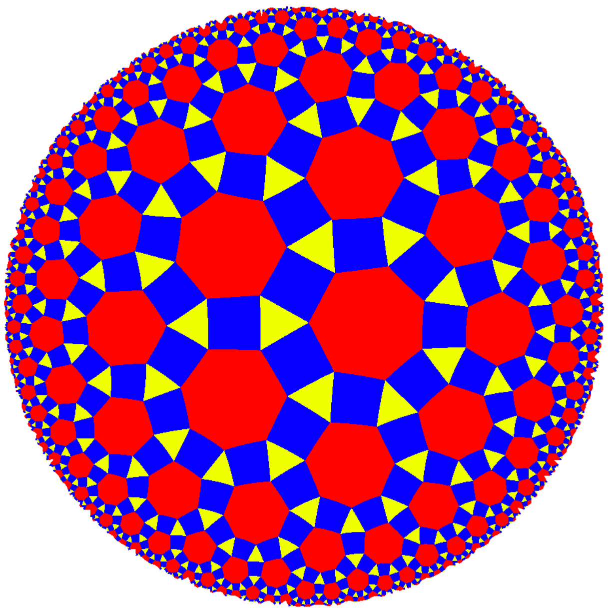
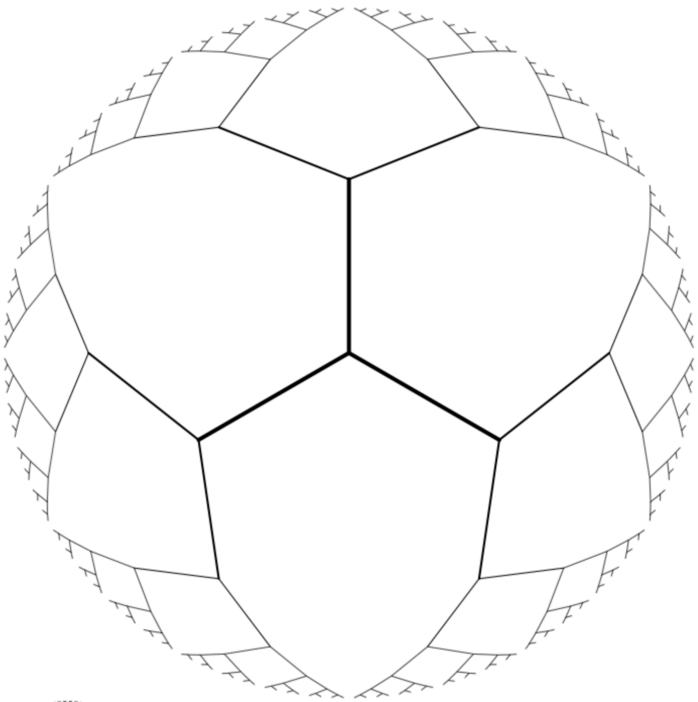


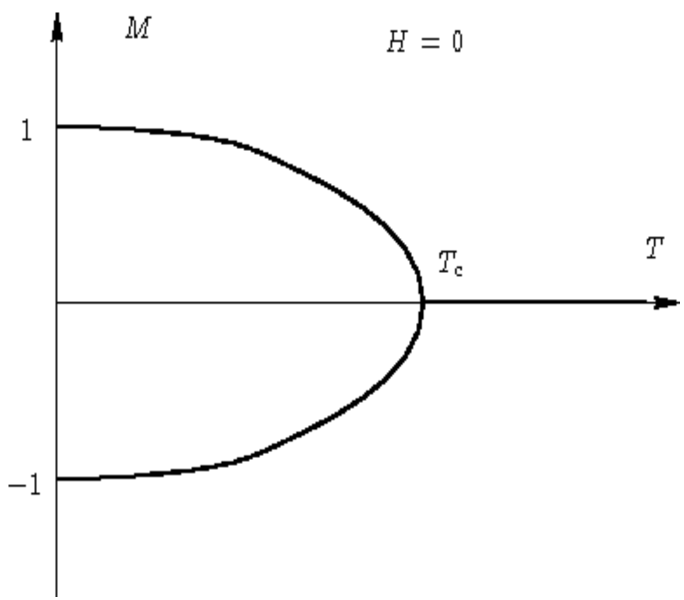


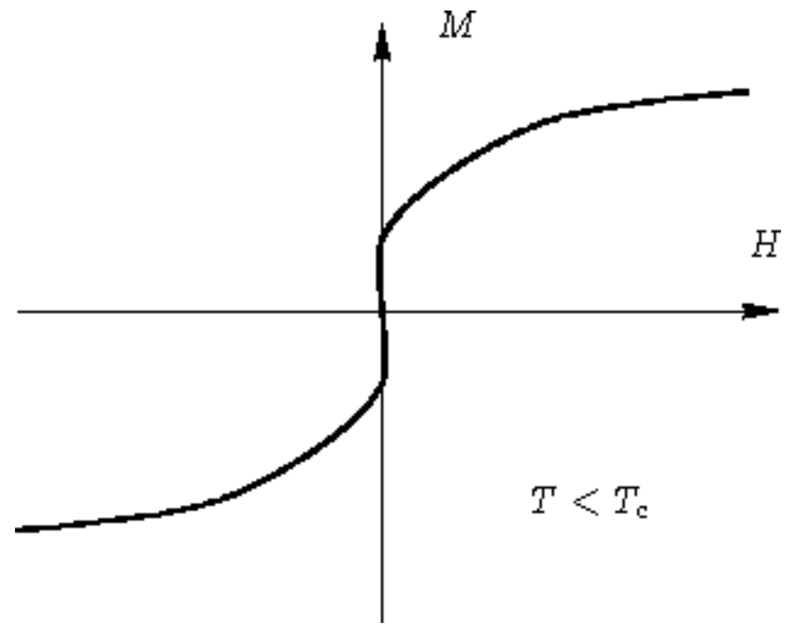




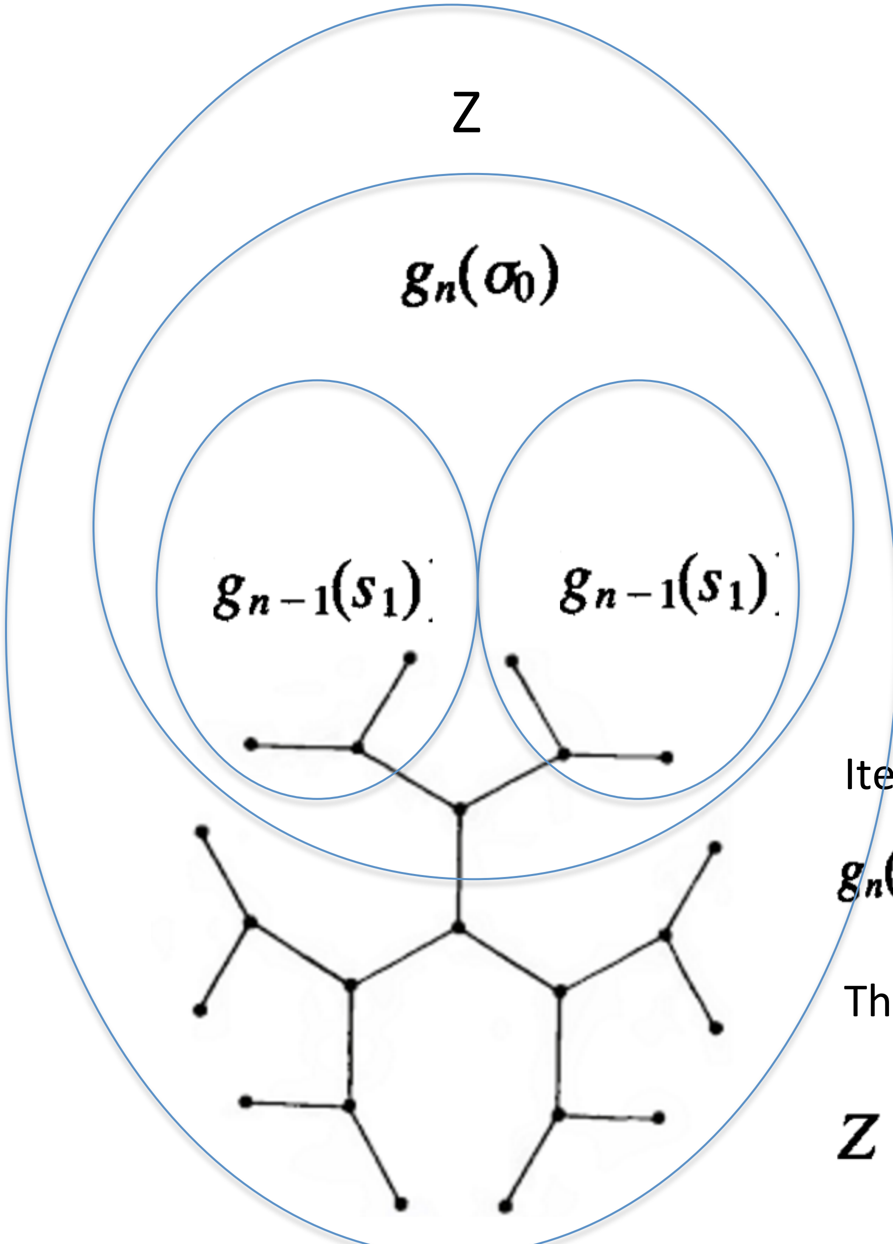
Critical phenomenon of Ising model on Bethe lattice

$$H(\sigma) = -\sum_{\langle i\ j \rangle} J_{ij}\sigma_i\sigma_j - \mu\sum_j h_j\sigma_j$$





Partition function for branches



Function $g_n(s)$ is the partition function of the branch with n shells and rooted at site s.

Iteratively:

$$g_n(\sigma_0) = \sum_{s_1} \exp(K\sigma_0 s_1 + h s_1)\,[g_{n-1}(s_1)]^{q-1}$$

The total partition function:

$$Z = \sum_{\sigma_0} \exp(h\sigma_0)\,[g_n(\sigma_0)]^q .$$

Why is there a critical temperature? Bifurcation!

Define $x_n = g_n(s = -1) / g_n(s = +1)$, from the iterator we have a map:

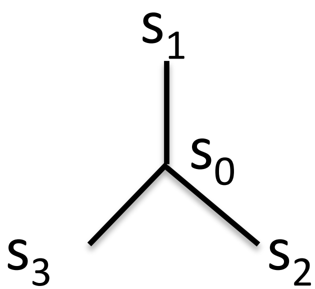
$$x_n = y(x_{n-1}), \text{ where:}$$

$$y(x) = [e^{-K+h} + e^{K-h} x^{q-1}]/[e^{K+h} + e^{-K-h} x^{q-1}].$$

With limit $n \rightarrow \infty$: $x_n = x_{n-1}$.
When $T > T_c$, there is one fixed point: $x_n = 1$
When $T < T_c$, there are three fixed point:
$x_n = 1$, $x_+ < 1$, $x_- > 1$

So when $H \rightarrow 0_+$, the spontaneous magnetism (mostly spin up) is associated with x_+
When $H \rightarrow 0_-$, the spontaneous magnetism (mostly spin down) is associated with x_-

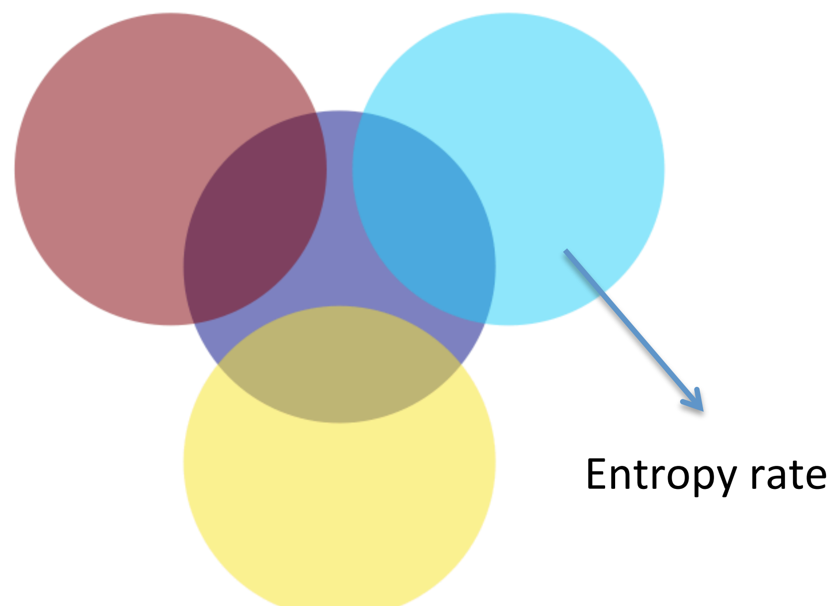
Information diagram for local sites neighborhood



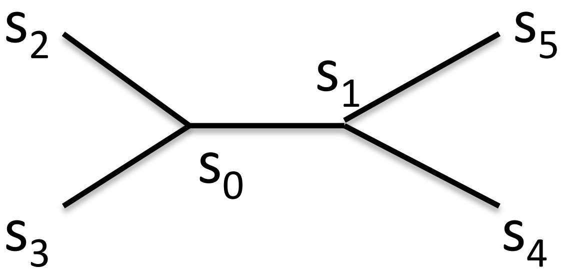
measure	bits
H[0|1,2,3]	0.233
H[1|0,2,3]	0.313
H[2|0,1,3]	0.313
H[3|0,1,2]	0.313
I[0:1|2,3]	0.037
I[0:2|1,3]	0.037
I[0:3|1,2]	0.037
I[1:2|0,3]	0.000
I[1:3|0,2]	0.000
I[2:3|0,1]	0.000
I[0:1:2|3]	0.006
I[0:1:3|2]	0.006
I[0:2:3|1]	0.006
I[1:2:3|0]	0.000
I[0:1:2:3]	0.002

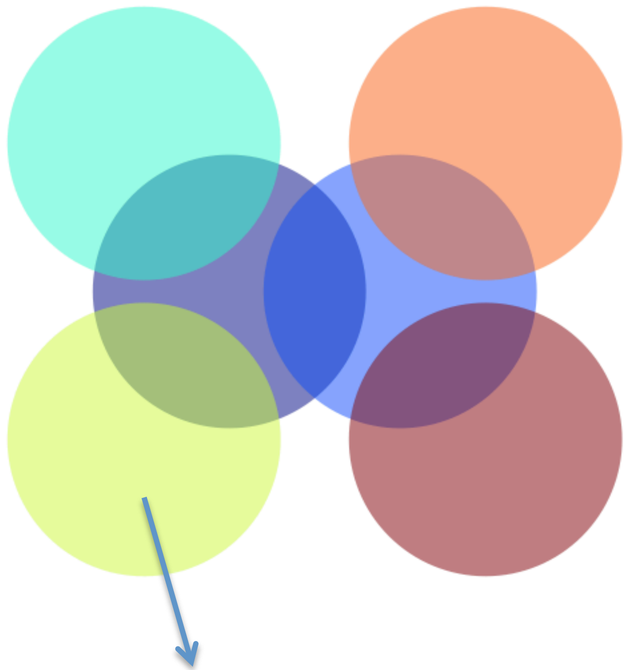
Sites i and j are shielded by the sites on all the possible paths between site i and j.

H [1 : 2 | 0] = 0
H [2 : 3 | 0] = 0
H [3 : 1 | 0] = 0



How about the second shell?



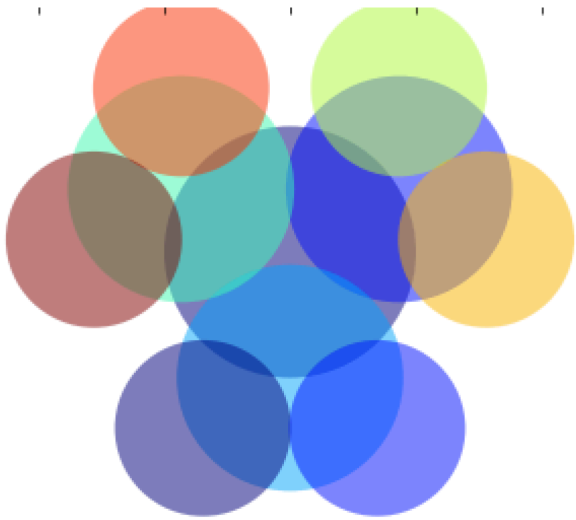


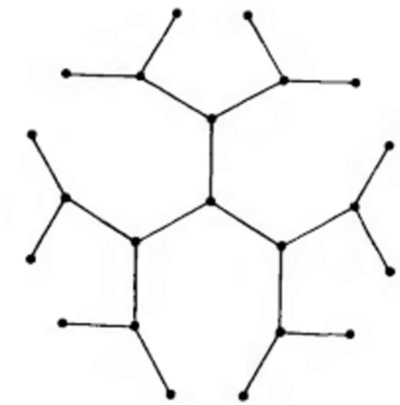
Entropy rate h'

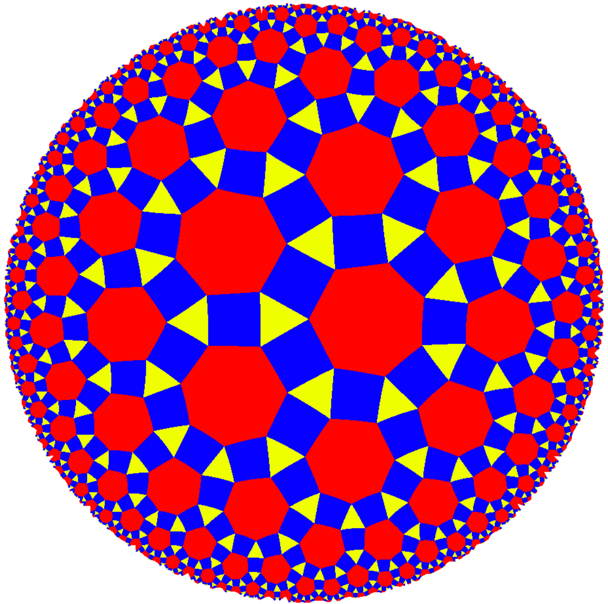
measure	bits
H[0|1,2,3,4,5]	0.233
H[1|0,2,3,4,5]	0.233
H[2|0,1,3,4,5]	0.313
H[3|0,1,2,4,5]	0.313
H[4|0,1,2,3,5]	0.313
H[5|0,1,2,3,4]	0.313
I[0:1|2,3,4,5]	0.026
I[0:2|1,3,4,5]	0.037
I[0:3|1,2,4,5]	0.037
I[0:4|1,2,3,5]	0.000
I[0:5|1,2,3,4]	0.000
I[1:2|0,3,4,5]	0.000
I[1:3|0,2,4,5]	0.000
I[1:4|0,2,3,5]	0.037
I[1:5|0,2,3,4]	0.037
I[2:3|0,1,4,5]	0.000
I[2:4|0,1,3,5]	0.000
I[2:5|0,1,3,4]	0.000
I[3:4|0,1,2,5]	0.000
I[3:5|0,1,2,4]	0.000
I[4:5|0,1,2,3]	0.000

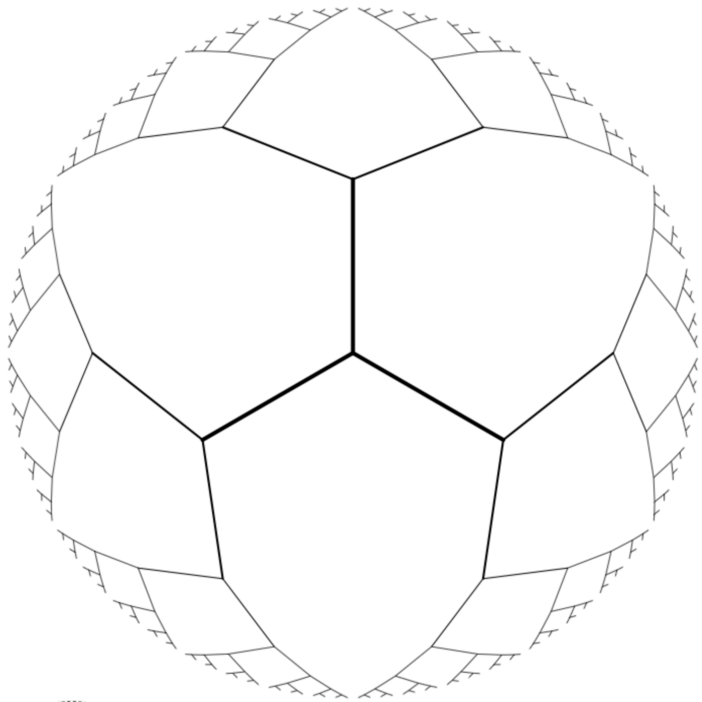
$$S = E(s_0) + (N-1) * h'$$

The same geometry structure





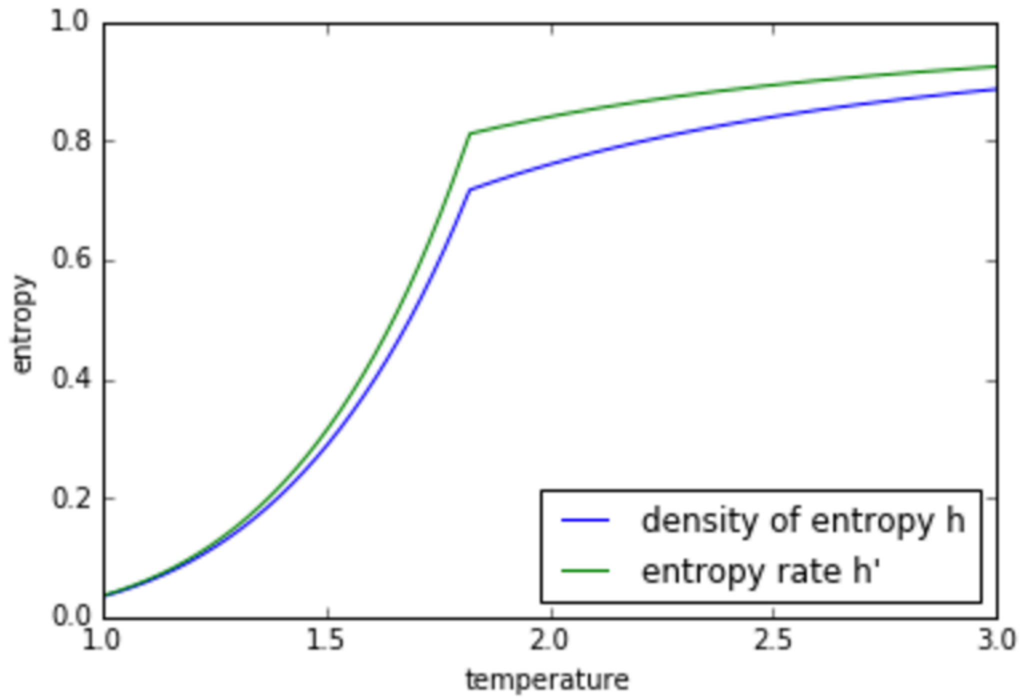




The entropy from thermodynamics

$$-\frac{\partial F}{\partial H} = \sum_i M_i$$

$$f_i/kT = -\tfrac{1}{2}Kq_i - h + \int_h^\infty [M_i(h') - 1]\, dh' .$$



Why are they different?
Center versus boundary?

Another new way to entropy near boundary

Trace the map

$$g_n(\sigma_0) = \sum_{s_1} \exp(K\sigma_0 s_1 + h s_1)\,[g_{n-1}(s_1)]^{q-1}$$

From $g_0(s) = 1$ numerically

$$Z = \sum_{\sigma_0} \exp(h\sigma_0)\,[g_n(\sigma_0)]^q\,.$$

Log(Z) for each site should equal to:
$[\, Z(n+1) - Z(n) \,] / (q*(q-1)^{n-1})$

The simulation in still under construction

Analytically, so far I only know

The expectation $< \log (g_n / g_n^{q-1}) >$ does not give a reasonable individual log(z) for each site

Also, if we set $g_n(1) = g_{n-1}(1)$, which does not make sense since partition function is an extensive quantity.
However, I can get a $g = g_n = g_{n-1}$ which give the same individual log(z) with thermodynamic method but with a factor of -2.

Conclusion

- Spins on Bethe lattice are shielded by the spin in between them.
- The information diagram for spins on sites away from boundary has the same geometry structure as Bethe lattice
- The entropy rate from the diagram does match the entropy from thermodynamic method
- The concept of individual entropy for a single site in an infinite fractal system is still mysterious.
- The study of map equation for partition function could be a solution

Acknowledge and thanks

- Code for Ising model on Bethe lattice is from: Ryan James
- Thanks Prof. Crutchfield, Ryan for discussion.

- Reference:
- Exact solved models in statistical mechanism. Rodney Baxter