Control of Synchronized States in Networks of NEMS

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Recent advances in design and implementation of Nanoelectromechanical Systems (NEMS) enable us to explore non-linear phenomena in unprecedented ways. In particular we can study the behaviors of non-linear elements that are coupled together in simple network configurations. Here we consider NEMS that, with appropriate feedback, act as nonlinear Duffing-like oscillators, allowing the dynamics of each oscillator to be modeled by its slow-time envelope. In a network of coupled oscillators, these complex envelopes are coupled via a diffusive term, with the real and imaginary parts of the coefficient corresponding to coupling of the oscillators' velocities and displacements, respectively. We apply methods for the control of nonlinear systems to these coupled NEMS oscillators. We focus on the existence of stable synchronized limit cycles and our ability to guide the system between these attracting states via control of a small number of oscillators' natural linear frequencies.

I. INTRODUCTION

Control of complex networks is a subject with many exciting application, ranging from power grid management to drug target selection. Such networks tend to have large numbers of elements connected by discrete links and can therefore be represented by a network (graph).

Nonlinear network control is neither linear controllability nor statmech.

II. BACKGROUND

Roukes, et al. have developed NanoElectroMechanical Systems (NEMS) with such varied applications as ¡current applications¿. Intended applications. They have developed both beams and membranes with precise and accurate physical parameters.

At the mechanical level, NEMS can be represented as euler beams.

$$Euler - Bernoulli \tag{1}$$

The deflection in the piezoelectric beam is recorded as an electric potential. By filtering the signal, a single mode of vibration can be isolated. So long as the nonlinearity is sufficiently small, the notion of a linear vibration mode is well founded and is represented by the standard oscillator equation with a single nonlinear term. This system is typically studied as driven by a periodic forcing function

$$\ddot{x} + \frac{\omega_0}{Q}\dot{x} + \omega_0^2 x + \tilde{\alpha}x^3 = \tilde{g}\cos(\omega_D t) \tag{2}$$

When the system is weakly damped and the driving is appropriately weak and near to the system's linear resonance frequency, this system is characterized by fast oscillations within a slowly varying envelope. The separation of time scales is characterized by the frequency dissipation Q. Additionally nondimensionalizing time by a reference frequency ω near to the natural linear frequency, the slow time variable is defined as $T = \frac{\omega t}{Q}$. The fast time scale oscillations are then factored out of the dynamics, defining a slow time envelope $A(T)$.

$$x(t) = x_0 \Re\left[A(T)e^{i\omega t}\right], \quad T = \frac{\omega}{Q}t \tag{3}$$

This envelope A is complex and may be written as a real magnitude and phase $ae^{i\phi}$. Upon substituting this solution and expanding perturbatively in Q^{-1}, the slow-time envelope equation is derived by requiring that terms leading to unbounded solutions (resonant driving of an undamped system) must precisely cancel. In this derivation, the driving function takes the form $ge^{i(\omega_D-\omega)t}$. However, the NEMS of interest are not externally driven. Instead, the resonators are driven by their own signal that isolated and manipulated. By saturating this signal at a particular magnitude, the driving function takes the form $\frac{1}{2}e^{i\phi} = \frac{A}{2|A|}$.

$$\frac{\partial A}{\partial T} = -\frac{1}{2}A + \frac{i\delta}{2}A + i\alpha|A|^2 A + \frac{A}{2|A|} \tag{4}$$

With a self-driving feedback loop, the systems sustain oscillations independent of any external power frequency. This frequency independence distinguishes limit-cycle *oscillators* from periodic $resonators$.

Coupling: Combination of signals and rotation/diminishment of signal. some words about dissipative coupling. Reactive coupling dynamics,

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words, and envelope.

$$\frac{dA_i}{dT} = f(A_i) + \sum_{j \in \mathcal{N}_i} \left(K_{ij} + i\beta_{ij}\right)\left(A_j - A_i\right) \quad (5)$$

Two oscillators have been previously coupled electronically by the Roukes group. Figure 1 shows the schematic for this set-up. In particular, the three colored boxes represent the three free system parameters: $\Delta\omega$, the natural frequency difference; α, the nonlinearity; and β, the reactive coupling constant.

As of June 2015, eight NEMS have been fabricated with the intent to build several coupling networks. (the first two of which being complete and ring.)

III. DYNAMICAL SYSTEM

The particular system under study is that of eight identical oscillators reactively coupled in a ring topology as represented in Figure 2. That is, each oscillator is reactively coupled to two "adjacent" neighbors. All natural frequencies are identical, so we choose all slow-time natural frequencies to be zero. These conditions give the following equation of motion for oscillator i, where the indices are taken $\mod 8$.

$$\frac{dA_i}{dT} = -\frac{A_i}{2} + i\alpha|A_i|^2 A_i + \frac{A_i}{2|A_i|} + i\beta\Big(A_{i-1} - 2A_i + A_{i+1}\Big) \quad (6)$$

The amplitudes and phases of each envelope can be isolated, giving the following form.

$$\frac{da_i}{dT} = -\frac{a_i - 1}{2} - \frac{\beta}{2}\left[a_{i+1}\sin\phi_{i+1,i} - a_{i-1}\sin\Delta\phi_{i-1,i}\right] \quad (7)$$

$$\frac{d\phi_i}{dT} = \alpha a_i^2 + \frac{\beta}{2}\left[\frac{a_{i+1}}{a_i}\cos\phi_{i+1,i} + \frac{a_{i-1}}{a_i}\cos\Delta\phi_{i-1,i} - 2\right] \quad (8)$$

Noting that the governing equation depends on phase differences, and not the phases themselves, as well as the uniformity of oscillators, we expect the steady states to have identical descriptors at each node. That is, we try states where all nodes oscillate with the same amplitude $a_i = a$, and all coupling edges support the same phase difference: $\Delta\phi_{i,i+1} = \Delta\phi$. under these conditions, the fixed amplitudes must be unity, and the amplitudes must be integer multiples of $\pi/4$, such that $8\Delta\phi = 0 \mod 2\pi$.

$$a_i^* = 1 \quad (9)$$

$$\frac{d\phi_i^*}{dT} = \alpha + \beta\cos\frac{n\pi}{4}, \quad n \in \mathbb{Z} \quad (10)$$

There are eight unique states of this prescription, enumerated in Figure 3.

Writing the amplitudes and phases as a combined 16 element vector, we linearize about an arbitrary point on these limit cycles.

$$\frac{d}{dT}\begin{pmatrix} d\vec{a} \\ d\vec{\phi} \end{pmatrix} = \frac{1}{2}\left(\begin{array}{c|c} -\mathcal{I} - \mathcal{M}\beta\sin\Delta\phi & \mathcal{L}\beta\cos\Delta\phi \\ \hline 4\alpha\mathcal{I} - \mathcal{L}\beta\cos\Delta\phi & -\mathcal{M}\beta\sin\Delta\phi \end{array}\right)\begin{pmatrix} d\vec{a} \\ d\vec{\phi} \end{pmatrix} \quad (11)$$

There is at least one zero eigenvalue of the linearization matrix, corresponding to the eigenvector $(\vec{0}, \vec{1}\,)$ in the direction of the limit cycle itself, as expected, which confirms that linearization of a non-fixed point on the limit cycle is reasonably meaningful. For each of the eight allowed $\Delta\phi$'s, the eigenvalues of this linearization allow for classification of its linear stability: if any have a positive real part, the state is unstable; if the rest have negative real parts, the state is stable. If there are additional eigenvalues with zero real part, then linear stability may not correspond to asymptotic stability.

Combining the envelopes to a vector $d\vec{A}/dT = g(\vec{A}\,)$, we see that the governing dynamic is equivariant with respect to rotations of the complex plane.

$$\frac{de^{i\theta}\vec{A}}{dT} = g\left(e^{i\theta}\vec{A}\right) = e^{i\theta}g\left(\vec{A}\right), \quad \forall\theta \in \mathbb{R} \quad (12)$$

Equivariant bifurcation theory should be able to be applied to help classify limit cycle stability, but I have not yet worked out the details.

IV. METHODS

In these systems, control input is reasonably implemented constrained to envelope phases. Oscillator phases may be shifted experimentally by detuning the natural frequencies very briefly. If the detuning occurs over a time scale much faster than the slow time scale, this corresponds to an instantaneous shift of the envelope's phase.

In these systems of NEMS, the physical energy of any oscillator is incredibly low simply due to their size, so minimizing an energy cost of the control signal is not an immediately useful quantity. On the other hand, the ability to implement such a phase kick of a single oscillator has notable experimental overhead: calibrating yet another "knob" of the system. As such, rather than consider control signals that may affect all phases, we study the ability to move between the known limit cycles by controlling a small number of phases.

The ability to control from one state to another is quantified by the fraction of available phase space that is in

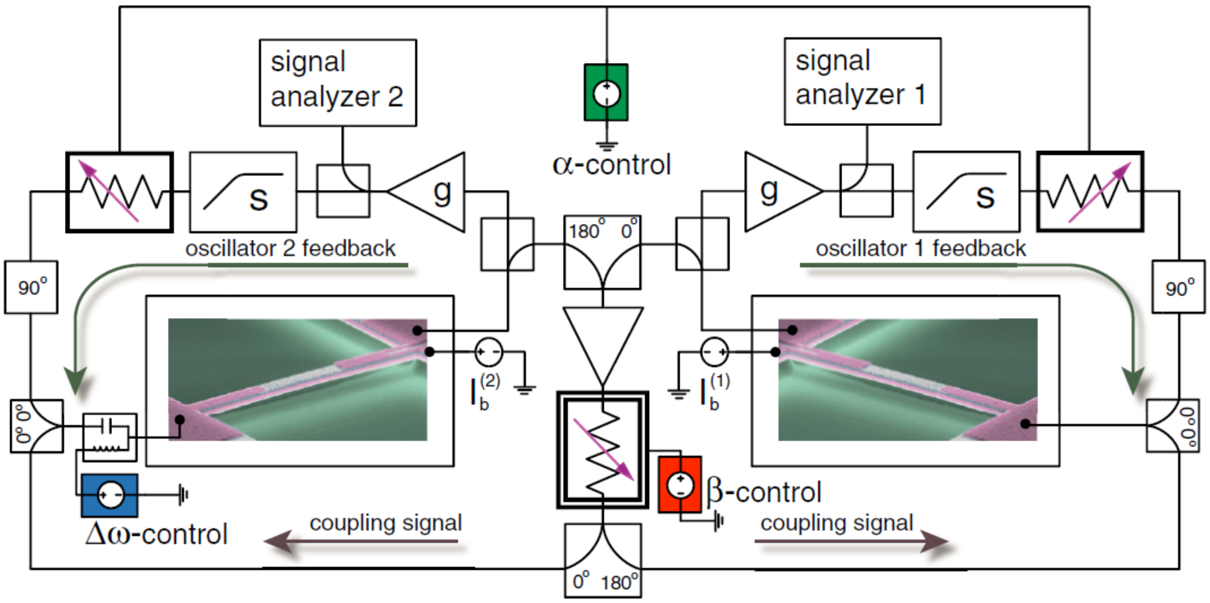


FIG. 1: Schematic of two reactively coupling oscillators, from Matheny, et al. PRL 2014

FIG. 2: Topology of Reactive Rings

the target limit cycle's basin of attraction. If the relevant basin slices are reasonably convex, then this fraction corresponds to a sort of precision required of the phase kicks in order to move to the target basin. This qualitative assumption will need some quantified justification. For now, the ability to move the system from a particular limit cycle into the basin of another limit cycle is represented by an edge in a directed graph, weighted by the discussed fraction of available phase space.

Basins of attraction were categorized by computer simulation of the system dynamics up to a time cut off at 1000 slow-time units.

V. RESULTS

At strong coupling, the most limit cycles are linearly stable, so we start in that regime. The discovered network is shown in Figure 4a. In phase synchronization: $\Delta\phi = 0$ is stable here, but the only other limit cycle that can be reached by moving a single oscillator's phase is out-of-phase synchronization $\Delta\phi = \pi$. Once in this limit cycle, no others can be reached. Starting in either $\Delta\phi = \pm\pi/2$, the basin of $\Delta\phi = \pi$ is not accessible.

With weak coupling, we can get from $\pm\pi/2 \to \pi$, but only through $\pm\pi/4$.

With weak coupling and control of the node opposite on the ring from the first, we can get directly from $\pm\pi/2 \to \pi$.

Only with control of adjacent nodes' phases can we get to anything other than π from 0 or leave π.

VI. CONCLUSIONS

TBD.

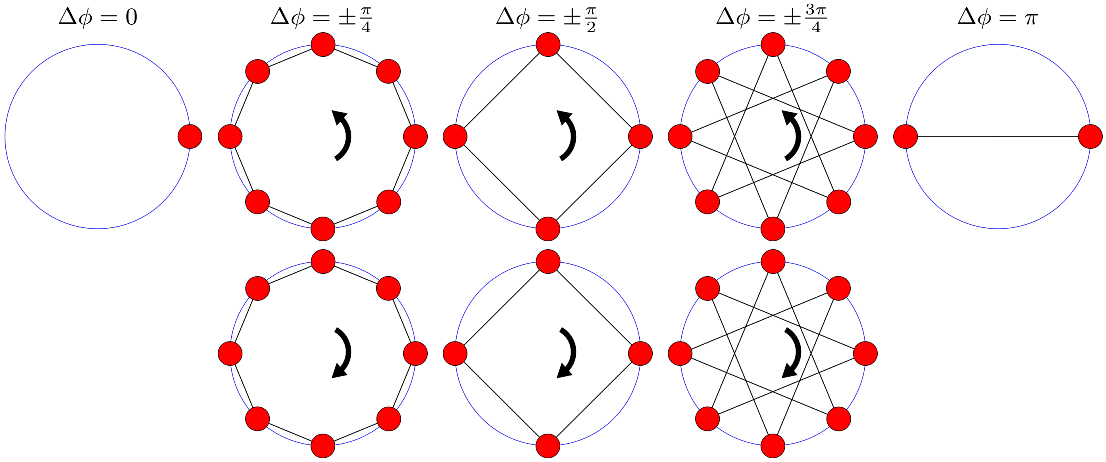
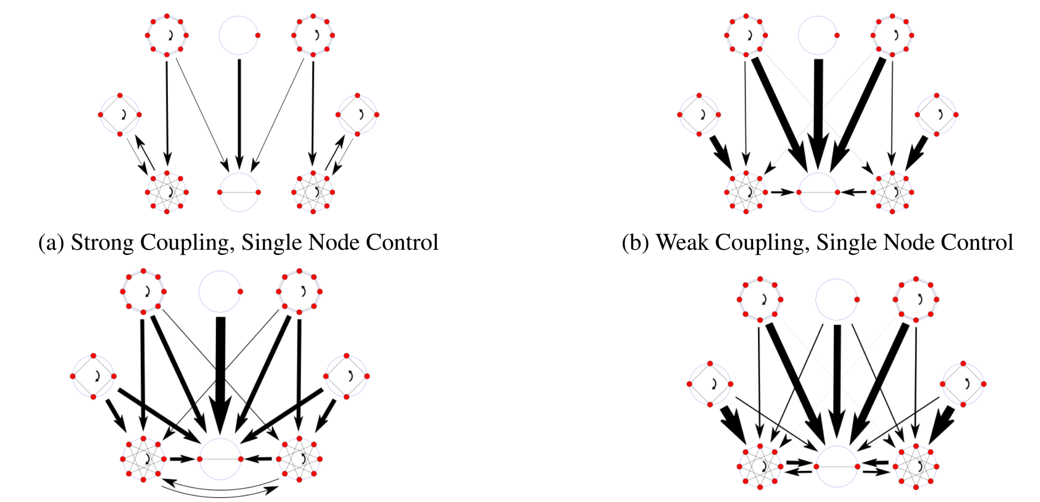


FIG. 3: Caption



(a) Strong Coupling, Single Node Control

(b) Weak Coupling, Single Node Control

(c) Weak Coupling, Opposite Node Control

(d) Weak Coupling, Adjacent Node Control

FIG. 4: Caption

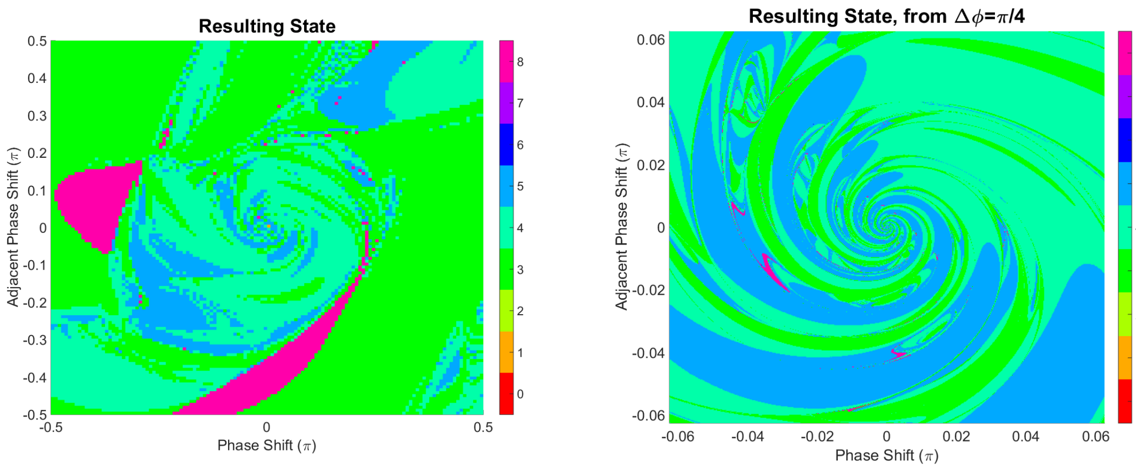


FIG. 5: A map of basins of attraction of states $\Delta\phi = n\pi/4$, centered at a point on the unstable $\Delta\phi = \pi/4$ limit cycle ($\alpha = 0.3$, $\beta = 0.5$) and slicing through the two phases of adjacent oscillators. The second plot is rather zoomed in and gives strong visual evidence for fractal basin boundaries near this unstable limit cycle.