Quantum Finite State Machines

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June 5, 2014

Stochastic Finite State Machine

Definition

A *stochastic finite-state machine* is a tuple $\{S, X, \{T^{(x)} : x \in X\}\}$ where

- S is a finite set of states, including a start state.
- X is a finite alphabet.
- $T^{(x)}$ are substochastic matrices where $T^{(x)}_{ss'}$ is the probability of transitioning from state s to state s' and emitting symbol x.

A *stochastic deterministic finite-state machine* is a stochastic-FSM such that for every $x \in X$, any row of $T^{(x)}$ has at most one nonzero entry (i.e. it is unifilar).

Quantum Finite State Machine

Definition

A *quantum finite-state machine* is a collection $\{Q, \langle\psi| \in \mathcal{H}, X, \{T^{(x)} : x \in X\}\}$ such that

- $Q = \{q_0, q_1, \ldots, q_{n-1}\}$ is a set of n states.
- The state vector $\psi \in \mathcal{H}$ belongs to an n-dimensional Hilbert space $\mathcal{H}$.
- X is a finite alphabet of output symbols.
- $T^{(x)} = U \cdot P(x)$ is an $n \times n$ transition matrix that is a product of a unitary matrix U and an orthogonal projection operator $P(x)$.
- The projection operators are mutually orthogonal and satisfy $\mathbb{I} = \sum_{x \in X} P(x)$.

A *quantum deterministic finite-state machine* is a quantum-FSM in which each matrix $T^{(x)}$ has at most one nonzero entry per row (unifilarity).

Properties of Quantum Finite-State Machines

We impose the following constraints on our quantum finite-state machines.

With each internal state q_i, we associate with it the canonical orthonormal basis vector $e_i = (0, \ldots, 1, \ldots, 0)$, which will also serve as the eigenbasis for the mutually orthogonal family of projections. That is,

$$P(x)e_i = \lambda_i e_i, \quad \lambda_i = 0, 1.$$

If $\lambda_i = 1$ then there is a transition into state q_i that emits symbol x. By mutual orthogonality, we know that all transitions into state q_i must emit symbol x.

Word Probabilites

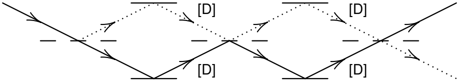
We use the density matrix representation to establish word probabilities. The stationary density matrix for any quantum-DFSM is given by

$$\rho = |Q|^{-1}\mathbb{I}.$$

If $w = w_1 w_2 \cdots w_k$, then $T(w) = T^{(w_1)} T^{(w_2)} \ldots T^{(w_n)}$ and

$$P(w) = \text{tr}(T^*(w)\rho T(w)).$$

Beam Splitter



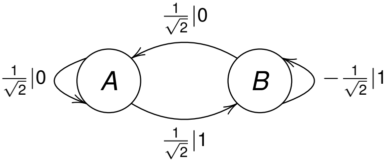
The above beam splitter is modeled by:

Quantum-DFSM

$$S = \{A, B\}, \quad X = \{0, 1\}$$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$P(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; P(1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$



Stochastic-DFSM

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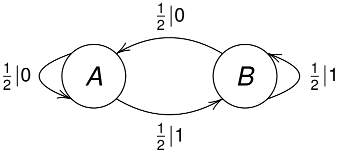
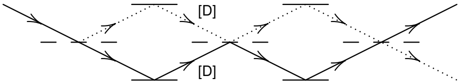


Remove Detectors. . .



If we remove every other set of detectors, interference occurs and the stochastic-DFSM is longer a valid description of the process. However, the quantum-DFSM still holds.

By noting that we are unsure of outcome of sites without detectors, we use

$$T(\lambda) = U(P(0) + P(1)) = U$$

to represent the unknown symbol at these sites. The probabilities

$$P(w_1\lambda w_2\lambda\ldots) = \mathrm{tr}(T^*(w_1\lambda w_2\lambda\ldots)\rho T^{(}w_1\lambda w_2\lambda\ldots))$$

correctly model the word distribution obtained after removing the detectors. This produces

Constructing Quantum from Classical?

Strategy:

- Construct stochastic-DFSM with bistochastic transition matrix T.
- Check if T is unistochastic, if not, modify machine.
- Construct a unitary matrix U such that $T_{ij} = |U_{ij}|^2$.
- Use classical machine associated with T to construct quantum-DFSM.

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Helpful observations for constructing quantum versions of stochastic-DFSM.

- $\Pr(x) = \mathrm{tr}(P(x)U^*\rho UP(x)) = \frac{\mathrm{rank}(P(x))}{|Q|}$.
- Classical states with multiple incoming symbols must be split.
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- A state with an incoming edge with probability one must have only one incoming edge.

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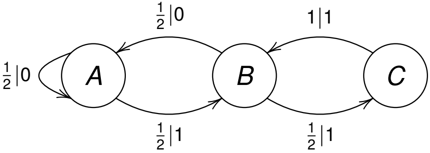
Example: The Odd Process

Recall the recurrent component of the Odd Process has ϵ-machine:



$$\Pr(0) = \frac{2}{5}, \quad \Pr(1) = \frac{3}{5}$$

- Quantum-DFSM must have a multiple of five states.
- The ratio of '0' states to '1' states is $2:3$.
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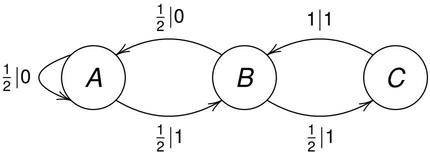

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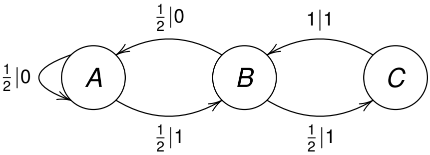

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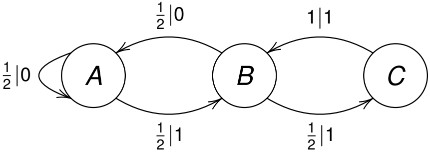

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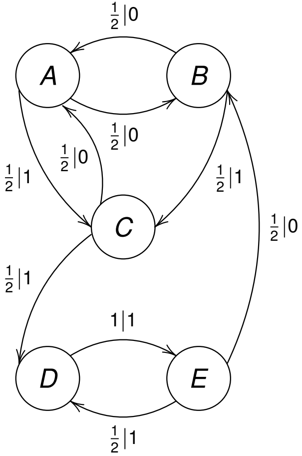

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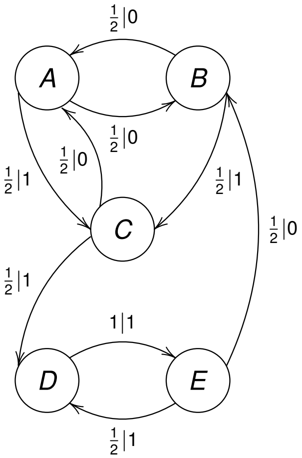


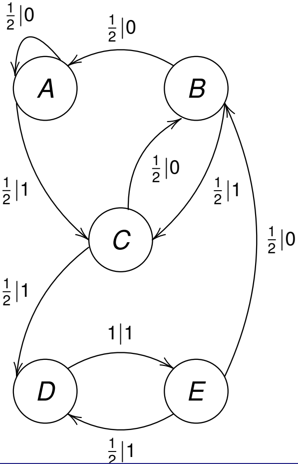
However, this has transition matrix

$$T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

which is bistochastic, but not unistochastic. :(

The Odd Process

So we try again...



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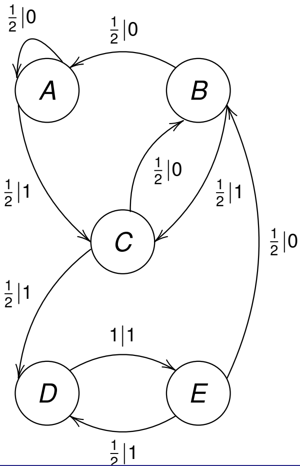

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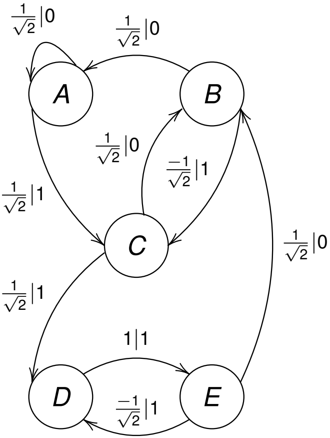
which is both bistochastic, and unistochastic. :D

The Odd Process

The quantum-DFSM for the Odd Process



This has unitary evolution and projection operators:

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$P(0) = |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|$$

$$P(1) = |e_3\rangle\langle e_3| + |e_4\rangle\langle e_4| + |e_5\rangle\langle e_5|$$

K. Wiesner, J. Crutchfield, *Computation in Finitary and Quantum Processes*, July 2007, `arXiv:quant-ph/0608206`