Information Along Time-like and Space-like Paths in One-dimensional Cellular Automata

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Abstract

Cellular Automata offer a spacially and temporally discrete example of spacially-extended dynamical systems. These are, at minimum, two-dimensional arrays of data. Information measures are best-developed for one-dimensional sequences of data, and therefor 1-D sequences must be extracted from cellular automata if such measures are to be applied. Due to the inherent limit on the spacial extent of local state information for Elementary cellular automata for a single time step, some 1-D sequences of data can contain elements that were never in causal contact. Cellular automata were generated for the Rule 18, Rule 30, and Rule 110 cases, and 1-D sequences were extracted along paths of different velocities. Bayesian inference was utilized to estimate the entropy rate and statistical complexity of candidate ε-machines for each sequence, and these information measures were plotted against velocity.

Introduction

Motivation

Cellular Automata (CA) comprise a subset of dynamical systems which are the focus of much study. Information metrics used to study such dynamical systems are most understood in one dimensional cases. As CA are comprised of at least one spacial dimension as well as a temporal dimension, 1-D sequences of data must of extracted from them for analysis with conventional techniques.

Due to CA having limits on the spacial extent of influence in a single time step, some binary sequences extracted from the 2-D lattice consist of elements that were never in causal contact, whereas some sequences are. This could potentially alter conclusions drawn from information analyses applied to such sequences. I therefor study the changes in two information measures for selected CA.

Why its interesting

Our own universe has a limit to the spacial extent of influence on an event given a set time: the speed of light. This dynamic is shared with cellular automata. I have therefor incorporated some terminology typically applied to physical spacetime discussions. “Time-like” describes a path taken through spacetime which connects causally-connected events, i.e. a path with velocity less than the speed of light. “Space-like” describes a path with velocity greater than the speed of light; these paths connect events that are not causally connected.

Though the “Elementary” CA studied here only one spacial dimension (1+1 dimensional) and our own universe has three (3+1 dimensional), studies of the differences between time-like and space-like sequences of events could shed light on properties of our own spacetime.

Synopsis

In order to study these effects, 1000 x 1000 cell CA were generated from a uniformly random initial global state according to three different dynamical rules: 18, 30, and 110. From this resulting 2-D array, a single 1-D array of 0s and 1s was selected by picking a random initial cell at time = 1 and drawing a line through the CA. The inverse slope of this path defined the “velocity” of the path. Bayesian inference methods were then applied to the resulting sequence to select the most likely generating ε-machines, which were then used to estimate the entropy rate and the statistical complexity for the sequence. After averaging over several start-cells, the information metrics were plotted against the velocities of their sequences.

Rule 18 showed interesting structure for all velocities, with slightly different structure for timelike paths. Rule 30 resulted in sequences best interpretted as unbiased coin flips for all finite velocities. Rule 110 appeared as an unbiased coin flip for finite spacelike velocities, but showed large variety in structure for timelike paths.

Background

Cellular Automata

CA can be considered the simplest of spacially-extended deterministic systems. Spacetime is divided, in both the spacial and temporal dimensions, into discrete cells. These cells can themselves be in discrete states. The number of states each cell can be in defines the alphabet size. This paper explores the properties of binary (alphabet size of 2) one dimensional CA (one dimension of space and one of time). The entire CA can be thought of living in a state space defined by all possible values for each cell at a given time.

The local dynamic which evolves the global state can be defined in many ways. In the case of Elementary CA, explored here, each cell's evolution is depended on it's own past as well as that of it's neighboring cells. Since information can only travel one cell away during one time step, we say the speed of light is 1. These three cells, expressing a binary alphabet, yield eight unique configurations. How each of these configurations evolves is defined by the "rule" of the CA.

Wolfram [1] enumerated these rules in the following manner. The 8 possible configurations are arranged in reverse numerical order, and the central cell's time-evolved state is assigned to each. The evolved states are then read like a binary number. For example, the following dynamical rule is known as "Rule 18":

111	110	101	100	011	010	001	000
0	0	0	1	0	0	1	0

Wolfram also categorized elementary CA into 4 classes. Class 1 included CA that quickly homogenize; these were not studied in this project due to lack of features. Class 2 CA evolve towards a periodic pattern, and example of which is Rule 18:

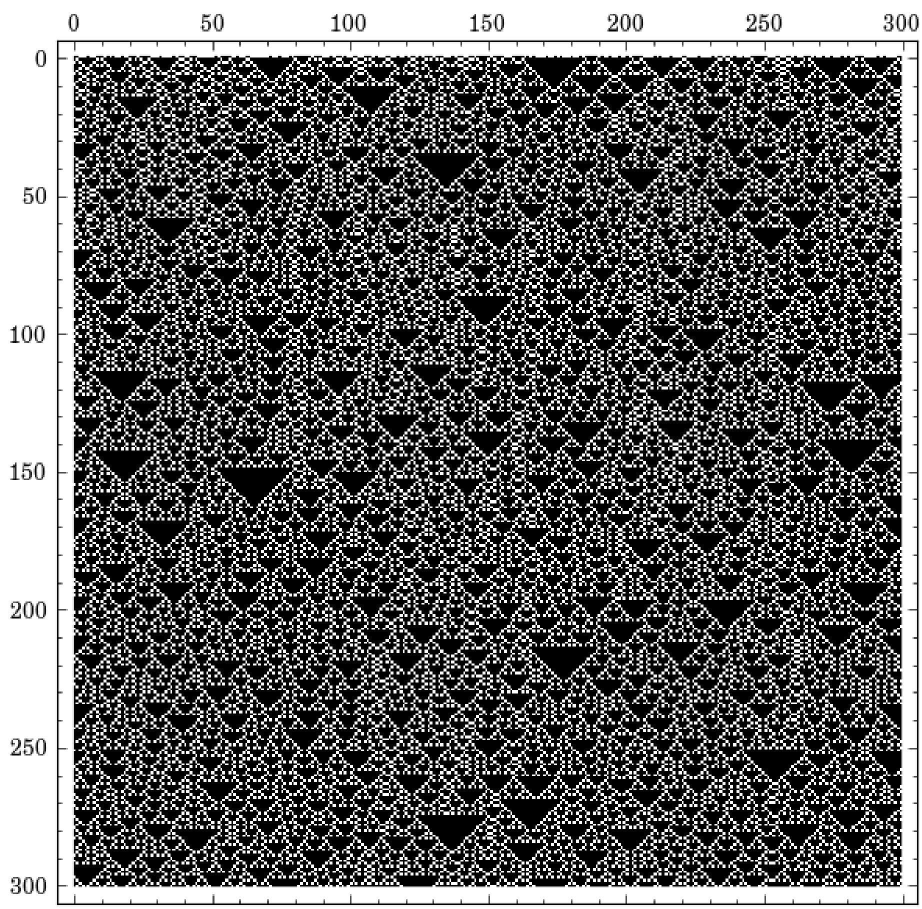

Figure 1: Rule 18: Space on the horizontal axis, time descending on the verticle axis. Cell of state "0" are shown in black, state "1" in white

Class 3 CA evolve chaotically, and though they typically have transient small-scale structure, many aspects of their behavior is random. Rule 30 is such a CA

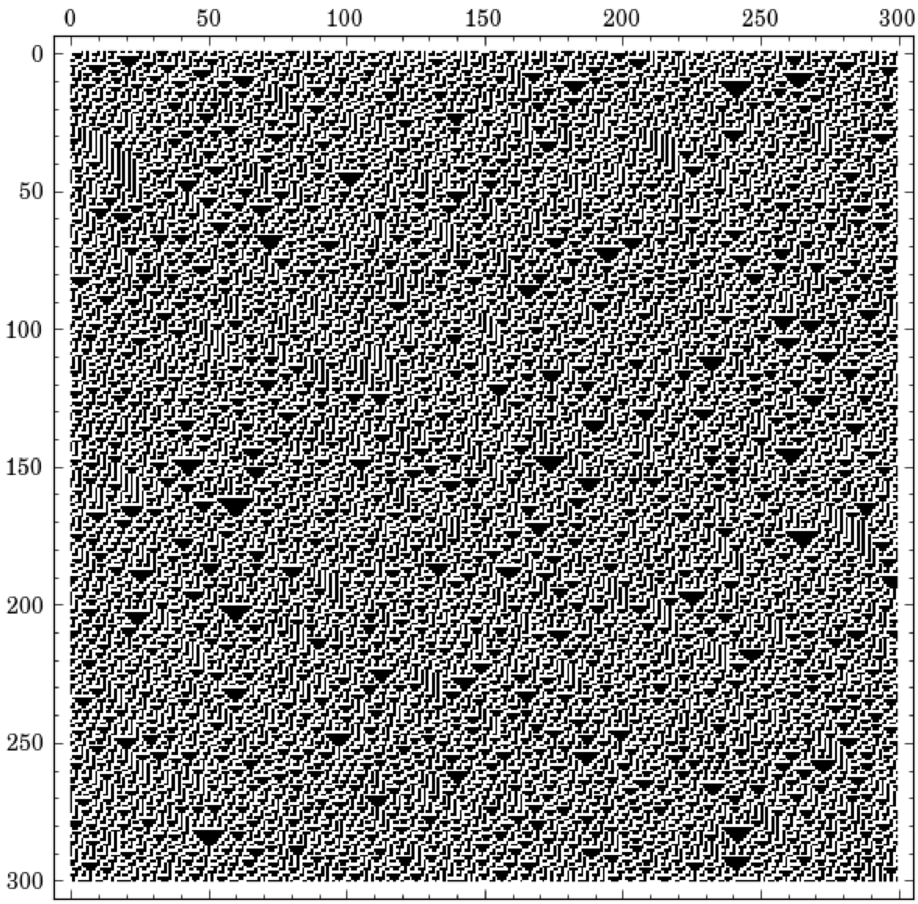

Figure 2: Rule 30

Finally, class 4 CA do not typically evolve to periodicity, but their small-scale structures do persist and move across the lattice, interacting with each other in complex ways. An example of this is Rule 110:

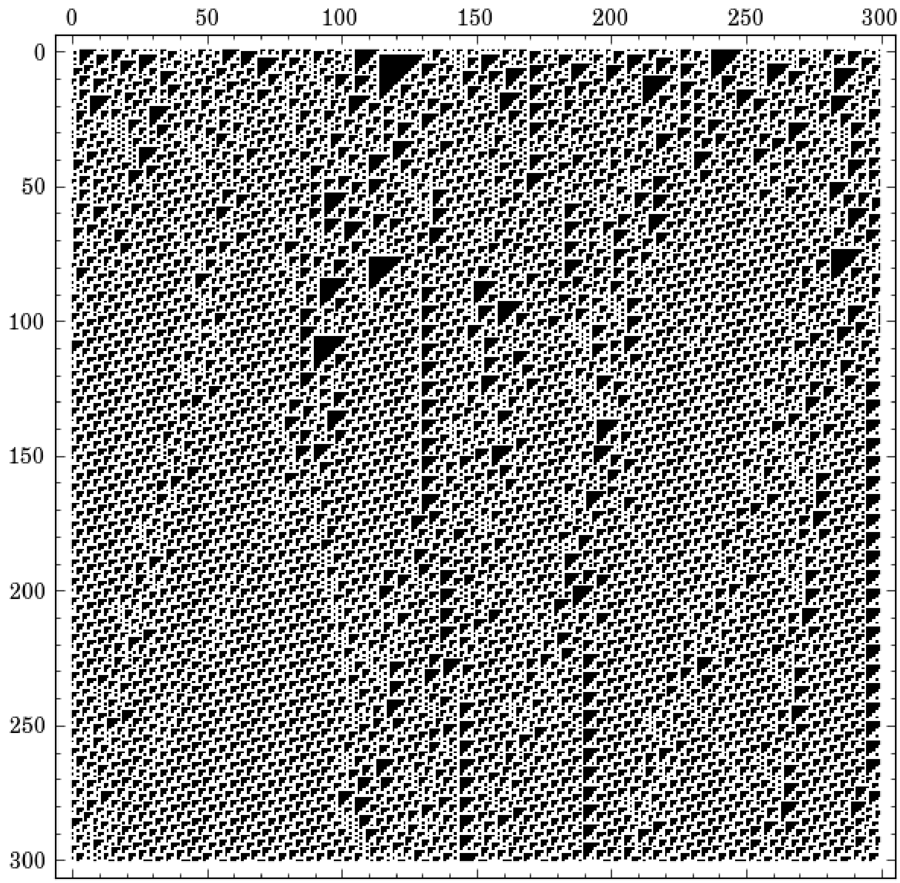

Figure 3: Rule 110

For this project, one CA of types 2, 3, and 4 were analyzed.

ε-Machines

An ε-machine is a representation of a process that generates a sequence of symbols via a Hidden Markov model. Hidden causal states transition to each other according to a transition matrix *T*, and each transition emits a 1 or 0 in the case of our binary alphabet. Specifically, of all Hidden Markov models that could generate an observed sequence, the ε-machine is the minimal machine [2]. For a given sequence of data, the generating ε-machines can be recovered in several ways. For our analysis, Bayesian inference was utilized, and is further explained below.

Methods

All work was performed in the Sage environment utilizing some prebuilt packages of the CMPy server.

The generation of a CA is straightforward. The initial global state is created by randomly assigning each of 1000 cells a 0 or 1 with equal probability. Each subsequent time step is generated by applying the dynamical rule of interest, until the desired length as been reached. In our case 1000 time steps were calculated. Periodic spacial boundary conditions are applied in order to determine the evolution of cells on the spacial ends of the lattice. This produced a 1000 x 1000 lattice of cells. CA were generated for rules 18, 30, and 110.

1-D binary sequences are then gathered from the CA by randomly selecting an initial cell and proceeding in a straight line through the spacetime according to a given velocity. For example, a velocity of 0 selects a sequence from the same spacial position in each time step, while a velocity of 1 moves to the right by one cell every time step. Space-like velocities included the integers 2-10, as well as the "infinite" velocity of a horizontal sampling. Time-like velocities were taken between 0 and 1, at steps of 0.1.

Once a sequence of data was isolated, the Bayesian inference methods from [4] were applied to find the most likely generating ε-machines. First, all 1, 2, 3, and 4 state ε machines were considered with the prior distribution suggested in [4], with prior distribution parameter $\beta = 4$ to prefer less complex machines. The data sequence was then used to discard those machines that were topologically incompatible; if the sequence contained forbidden words for a given machine, that machine was thrown out.

To observe any information differences in the ε-machines reconstructed from different velocity sequences, the entropy rate and statistical complexity were calculated. For an ε-machine with transition matrix T_{ij}, there is a distribution among causal states such that

$$\mu_i = \sum_j T_{ij} \mu_j$$

This is known as the stationary distribution.

The entropy rate h_μ measures the difference in uncertainty in the appearance of length L sequences and length L+1 sequences. For a hidden Markov model, this can be calculated using the transition matrix and stationary state [3]:

$$h_\mu = -\sum_{ij} \mu_i T_{ij} \log_2(T_{ij})$$

The statistical complexity C_μ of a hidden Markov model is a measure of how "complicated" the connections between causal states are. For example, a model with a single causal state has a statistical complexity of 0. It can also be calculated from the stationary distribution [2]:

$$C_\mu = -\sum_i \mu_i \log_2(\mu_i)$$

These information measures, chosen due to their ease of calculation, were found for 500 different candidate ε-machines selected from the Bayesian posterior distribution. These values were then averaged together to produce the estimate of h_μ and C_μ of the sequence.

The Bayesian inference and information measure calculations were repeated for sequences formed from 25 different starting cells in the CA, all of the same velocity, and the results from each sequence were averaged. This average, as well as the standard deviation of these 25 samples, was plotted against the velocity of the sequences and is presented below. Not considered was the Bayesian confidence interval for each h_μ and C_μ measured.

Results

In the Rule 18 case, the left-right symmetry of it's evolution rules can be seen in the apparent symmetry under sign change of velocity in the information metrics. The entropy rate has minima at velocities of 0 and ±∞, though small local minima exist at v= ±0.5. All other spacelike paths result in fairly uniform entropy rate. The statistical complexity demonstrates large uncertainty for spacelike paths, while timelike paths appear much less uncertain. v = ±1 appears as minima, while ±0.5 once again seems to play an interesting role as local maxima.

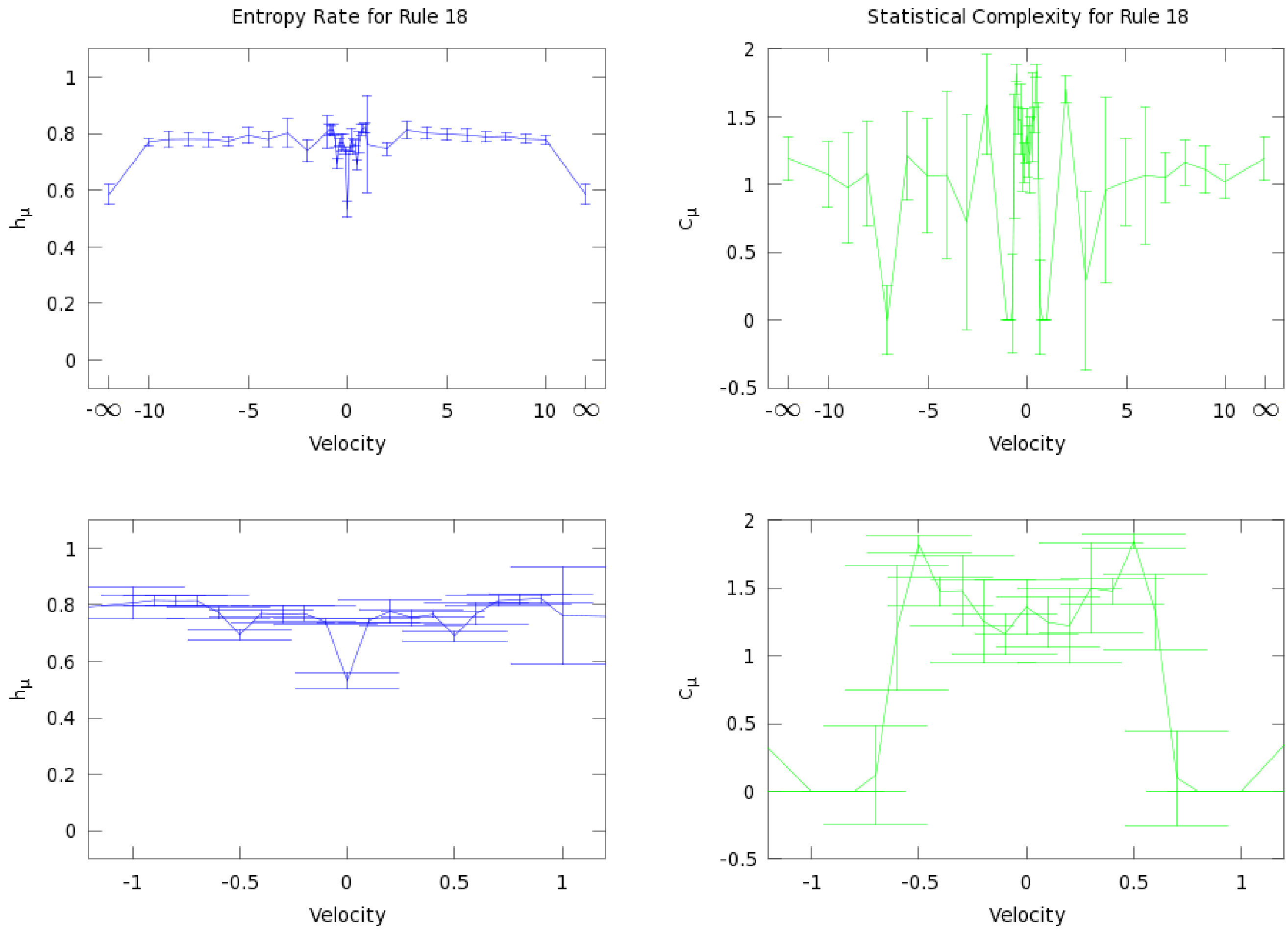

Figure 4: Results of analysis of Rule 18

For Rule 30, every path appears as $h_\mu = 1$ and $C_\mu = 0$ (a fair coin flip), except the infinite velocity sequence. It is interesting that the "class 3" property of randomness seems to extend to all paths that travel through time.

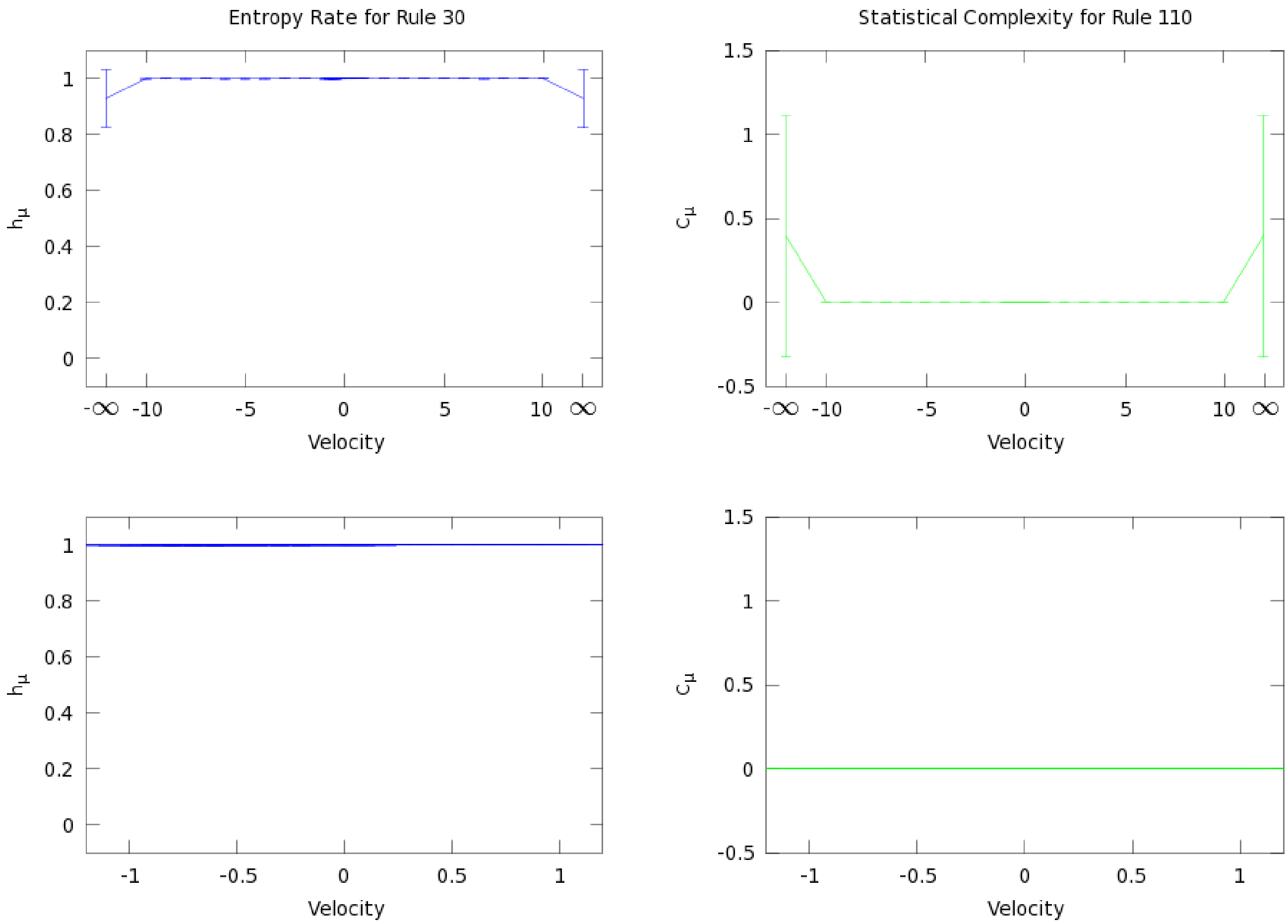

Figure 5: Results of analysis of Rule 30

The final case, Rule 110, appears as almost a combination of the previous two results. For spacelike paths of finite velocity, the information metrics show an unbiased coin flip, much like Rule 30. However, in both entropy rate and statistical complexity, the error becomes much larger for timelike sequences.

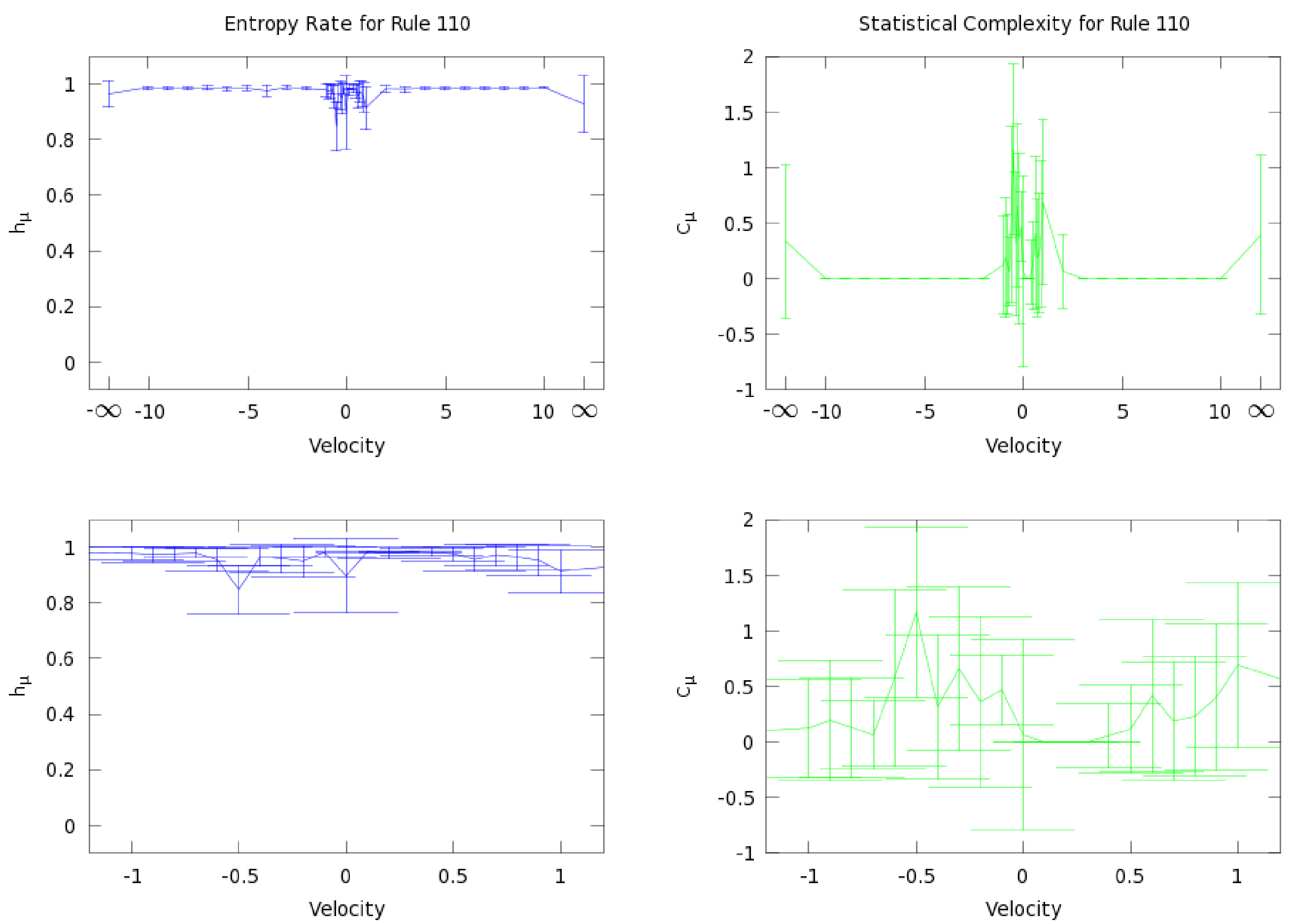

Figure 6: Results of analysis of Rule 110

Conclusion

Velocity dependence on information metrics was observed, and this dependence was different for each of the three examined CA. Common to all three CA, infinite velocities displayed different information metrics than other spacelike paths. Our class 3 example, Rule 30, demonstrated randomness in all finite velocity paths, with no preference for space- or time-like velocities. However, both rules 18 (a class 2 CA) and 110 (a class 4 CA) demonstrated a sensitivity to causality. Rule 18 shows less error in metrics derived from timelike paths, while the opposite it true for rule 110, which seems to contain many different ε-machines in it's timelike examples, but only one, the unbiased coin, for spacelike.

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