Pathwise Information Theory in Two Dimensions

Jordan Snyder

Department of Mathematics
University of California, Davis

Natural Computation and Self-Organization, Spring 2014

Outline

Using Information Theory to study Spatial Organization

- Basic formulation of IT supposes a time series of data
    - In one space dimension, we don't have to change anything
- Fundamental difference in higher dimensions: no shielding as in "past/future"
    - Information can flow however it wants!

Some Methods

- Cover lattice with a space-filling curve to find 2D entropy density (Lempel/Ziv 1986)
  - Computes a quantity well, but discards spatial data
- Use a full dimensional "notch template" (Feldman/Crutchfield 2002)
  - Presupposes a particular structure (optimality only shown for a specific case)

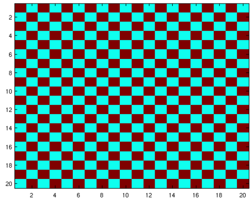
Some Methods

- Cover lattice with a space-filling curve to find 2D entropy density (Lempel/Ziv 1986)
  - Computes a quantity well, but discards spatial data
- Use a full dimensional "notch template" (Feldman/Crutchfield 2002)
  - Presupposes a particular structure (optimality only shown for a specific case)

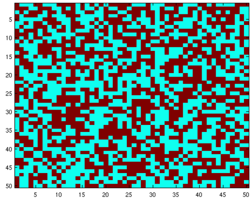
Towards a Solution

- **Assumption**: Spatially extended interacting systems communicate along 1D paths
- Consider all possible ways to parse a 2D grid into 1D paths, do ordinary IT, and collect results
    - Block Entropies $H(p_L)$, Myopic entropy rates $h_\mu(p_L)$
    - "Past/Future" Mutual information $I(p_{0:\frac{L}{2}}; p_{\frac{L}{2}:L})$. Notation: $I(p_L)$

Boring Cases

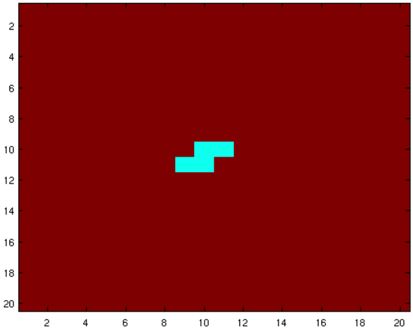


- 1x1 Checkerboard
  - Mutual Information = 1 for all paths

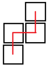


- White noise - each site ± 1 with equal probability
  - Mutual information $\sim 10^{-3}$ for all paths

Lone Notch

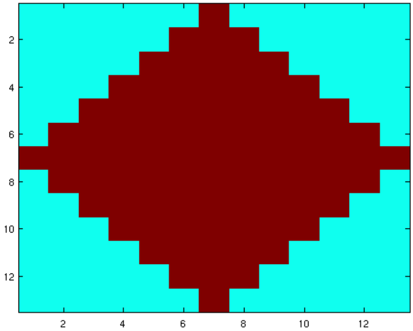


- Consider all paths visiting 4 sites
- Max MI on $dRL \sim rRL$
  - Pathwise MI is maximized *across* boundaries:



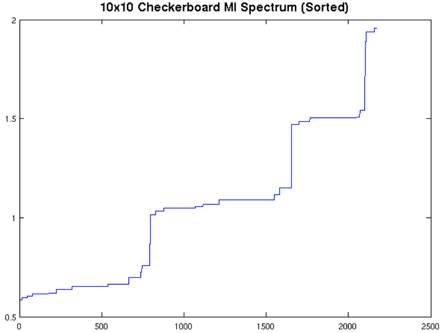
  - Second highest on paths that follow the notch: $rLR \sim lLR$

Diamond



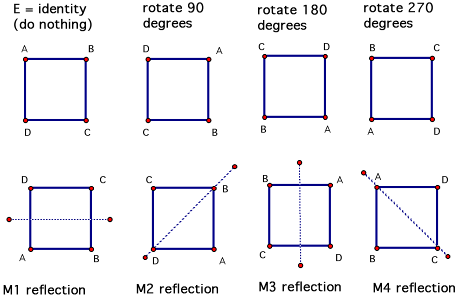
- Eight of the 36 four-site paths maximize mutual information:
  - $rLR \sim uRL$, $dLR \sim rRL$, $dRL \sim lLR$, $lRL \sim uLR$, where $\sim$ denotes equality after diagonal reflection
  - MI spectrum could be used to probe spatial symmetries

10x10 Checkerboard



- Mutual Information for each eight-symbol path
- Intriguing “staircase” structure - is there some simple reason for the strong clumping?

Symmetry



- The group D4 has an action on paths
- How does this action affect information measures?

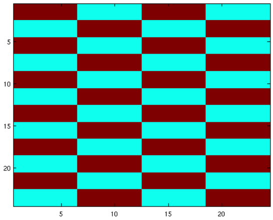
Quantifying Symmetry

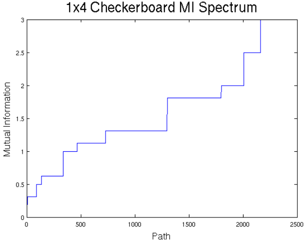
What are meaningful computations to do? For example,

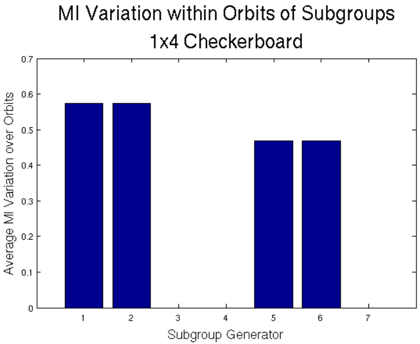
- Consider a subset P of paths with similar MI, and find their stabilizer; $\{f \in D_4 | f(p) \in P \, \forall p \in P\}$
- For each $f \in D_4$ compute $\langle |I(p) - I(f(p))| \rangle$
- For each subgroup of D_4, partition the paths into *orbits*, and investigate how MI varies within orbits

Shapes with Partial D_4 Symmetry

Checkerboard with 1x4 Blocks

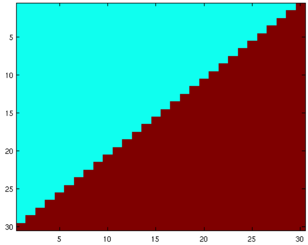


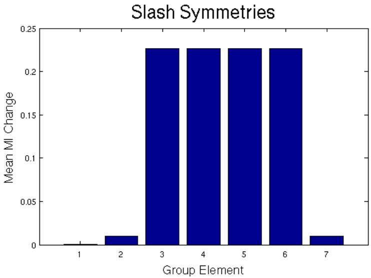


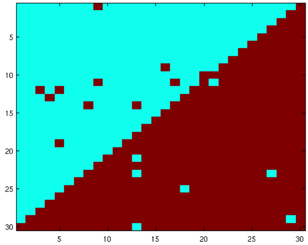


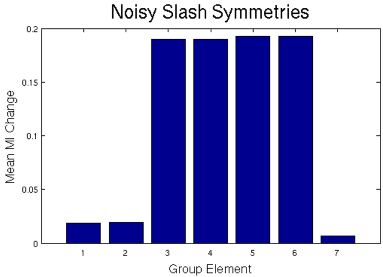
Labelling: $1 = \text{Ref}_{x=-y}$, $2 = \text{Ref}_{x=y}$, $3 = \text{Ref}_x$, $4 = \text{Ref}_y$, $5 = \text{Rot}_{90}$, $6 = \text{Rot}_{270}$, $7 = \text{Rot}_{180}$

(More) Shapes with Partial D_4 Symmetry









Breaking assumptions

- Full-dimensional templates as opposed to paths - measure how clumps communicate with clumps
  - Main difficulty: enumerating shapes. Could do with portions of space-filling curves, in the spirit of Lempel and Ziv
- More dimensions / arbitrary networks
  - Main difficulties: path enumeration, larger/stranger isometry groups
- Inhomogeneous systems
  - Main difficulty: need many equivalent samples, and obtain MI spectrum *for every site*

References

Lempel, A. and Ziv, J.
Compression of Two-Dimensional Data.
IEEE Transactions on Information Theory, Vol. IT-32, No. 1, January 1986.

Feldman, D. and Crutchfield, J.
Structural Information in Two-Dimensional Patterns: Entropy Convergence and Excess Entropy
Santa Fe Institute Working Paper 02-11-065, December 5, 2002.