Computational Mechanics for Mathematical Approximation Processes

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PHY 256B

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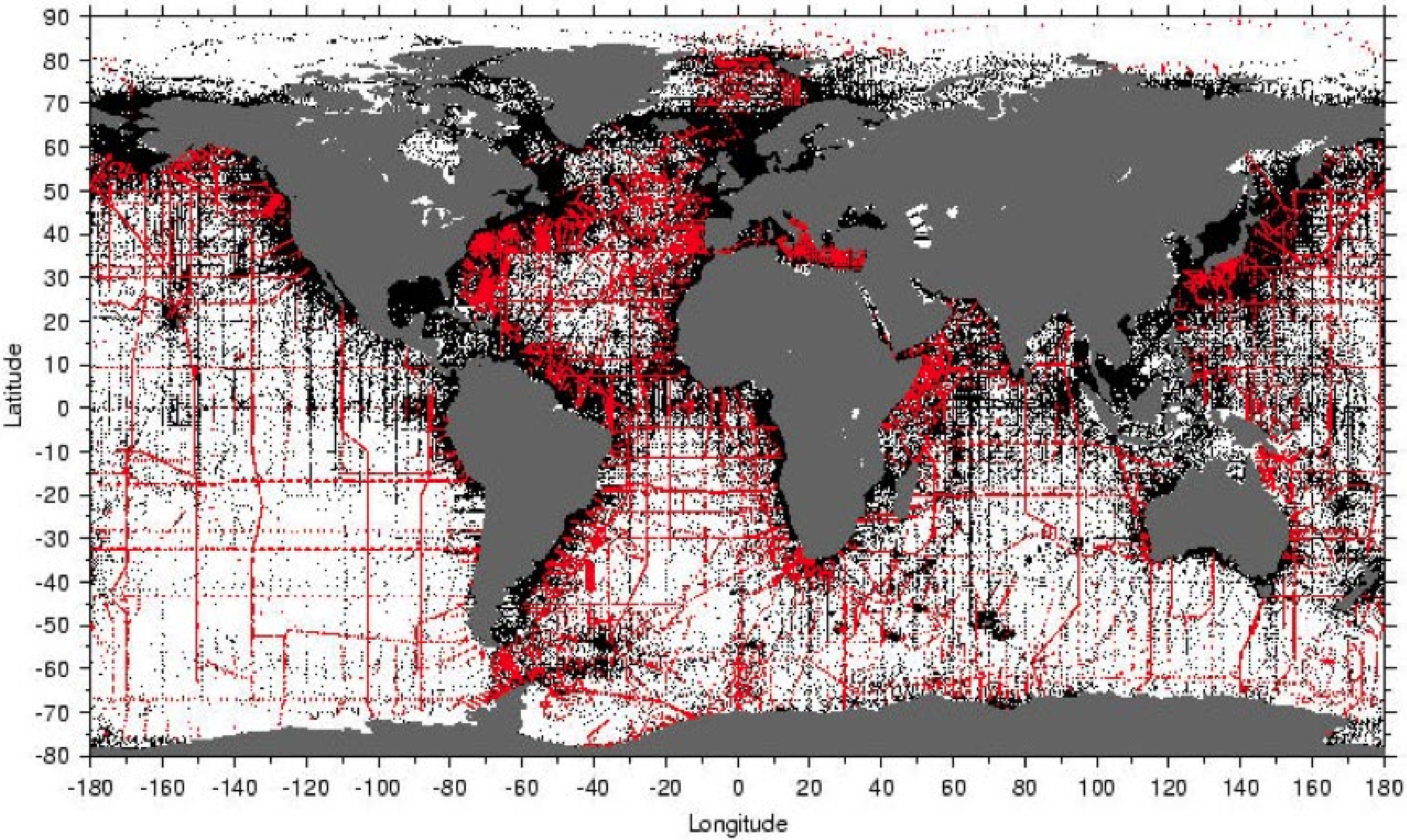
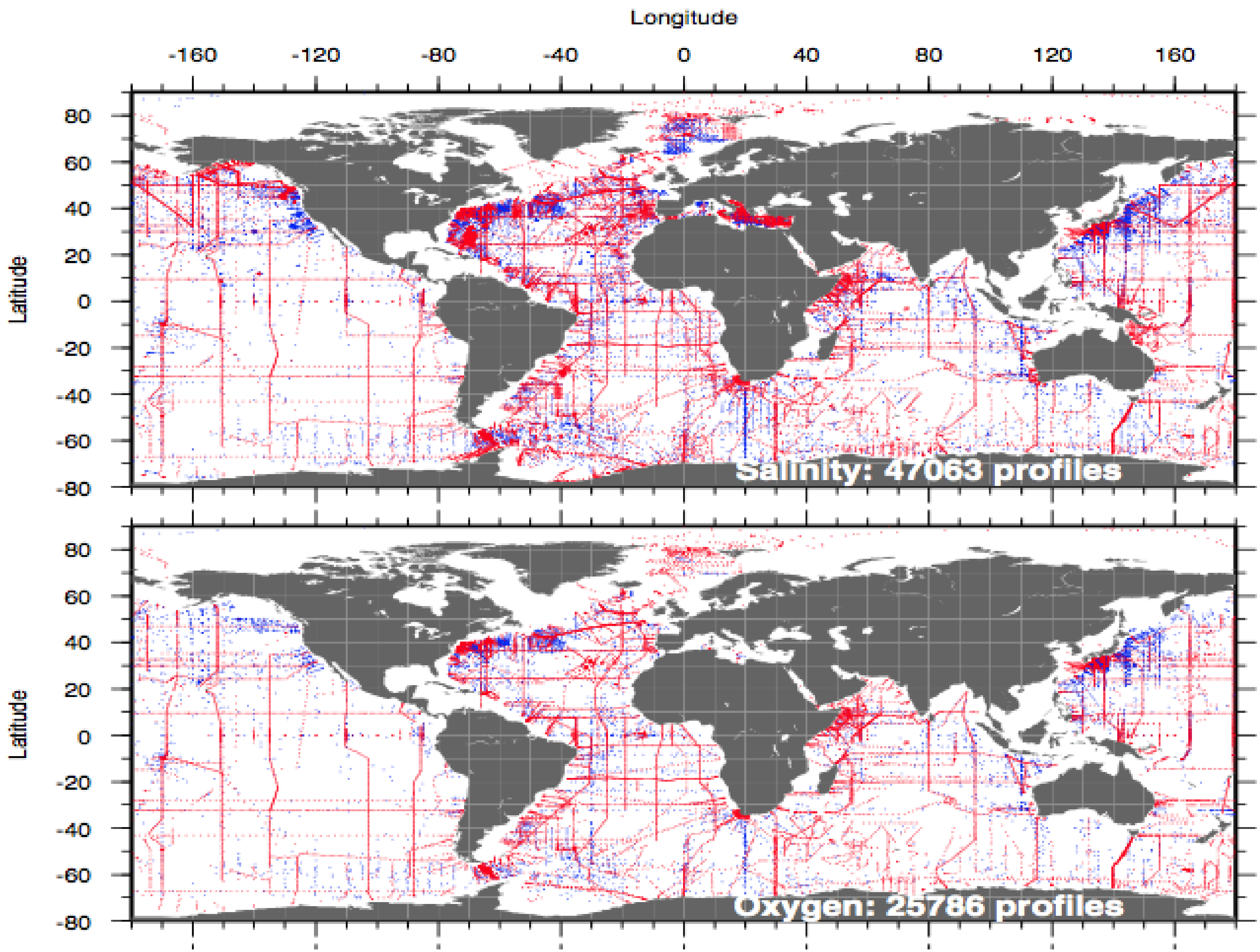
Motivation

- Can mathematical approximation processes be viewed as dynamic systems?
  - These systems seem to have interesting nonlinear structure (Gibbs & Runge phenomena)
- If so, can computational mechanics be used to develop an increased understanding of the information processing properties of these methods?

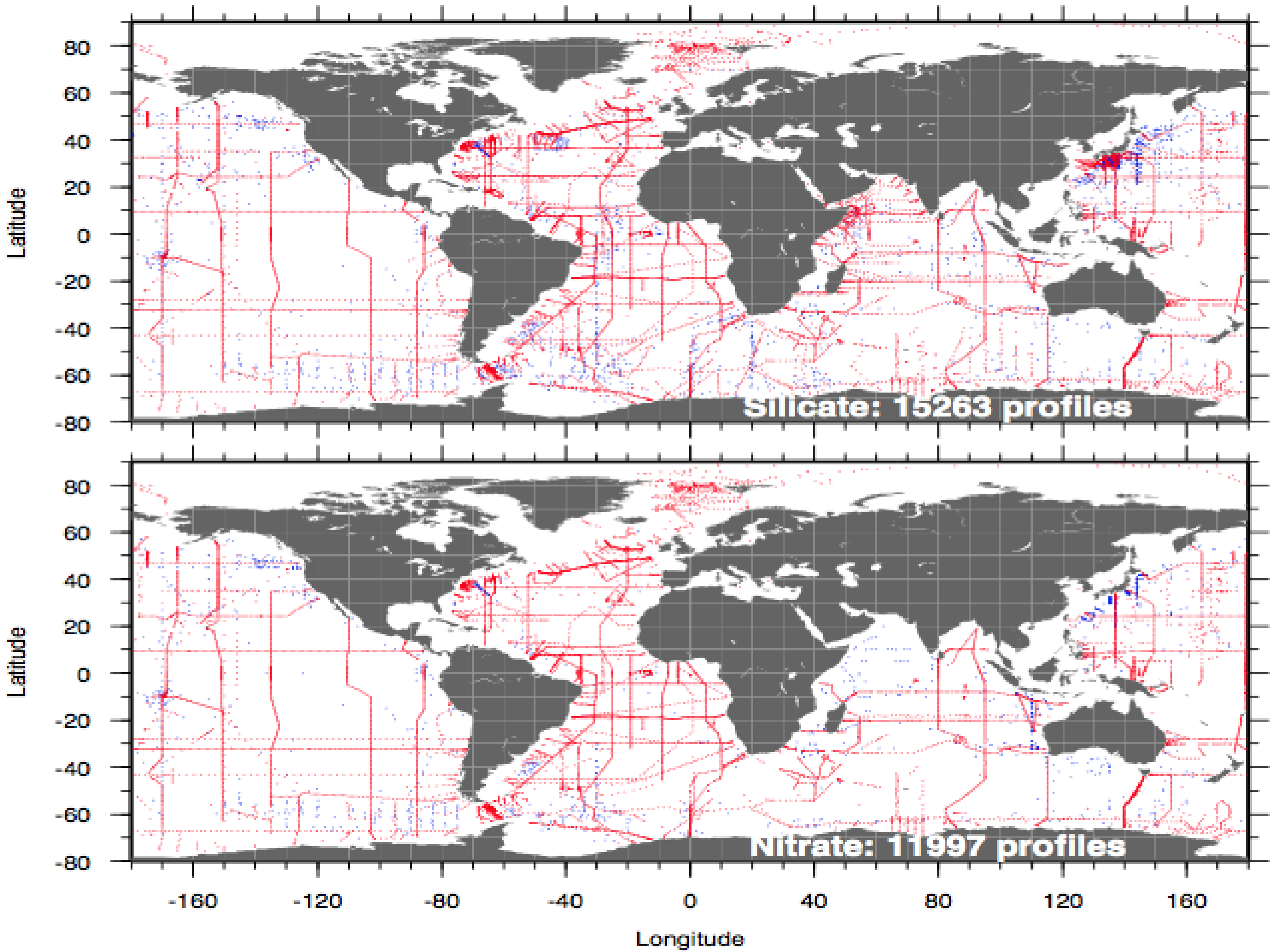
Specific Motivation

- Climate change modelling
- Visualization of oceanographic data
- Approximation of scalar fields using various scattered data interpolation methods
- Limitations of having extremely sparse datasets
- Flow-field-aware directional interpolation

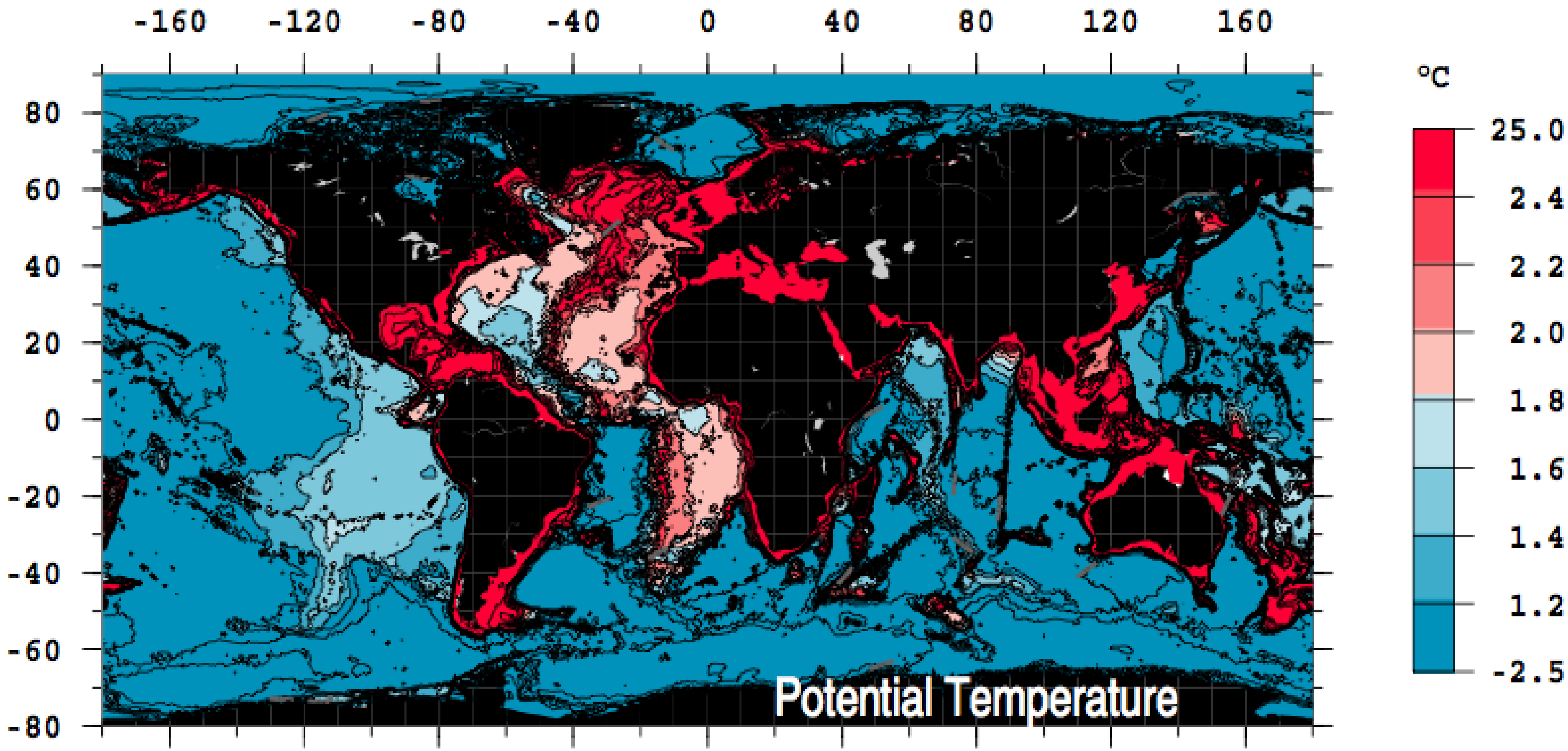

Latitude
Longitude



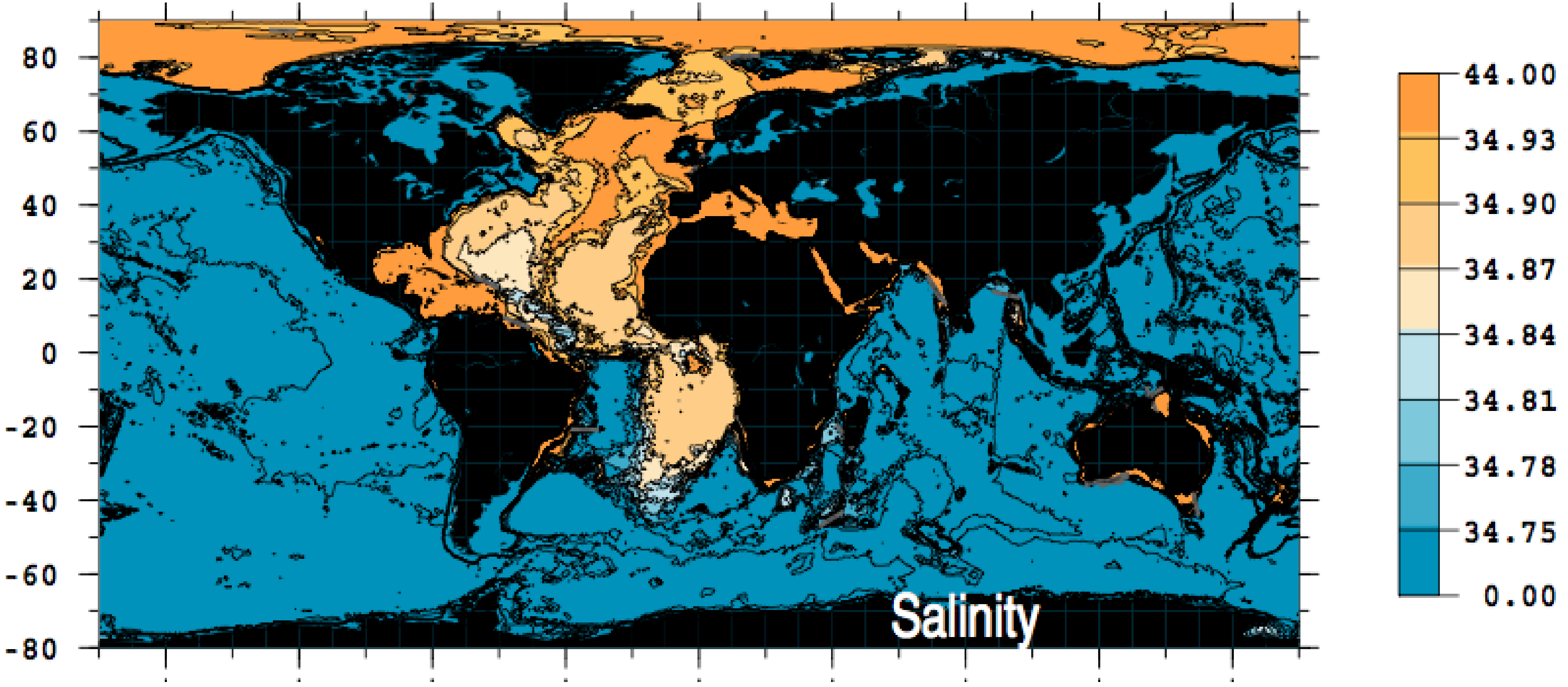
Longitude
-160
-120
-80
-40
0
40
80
120
160
Latitude
80
60
40
20
0
-20
-40
-60
-80
Salinity: 47063 profiles
Oxygen: 25786 profiles



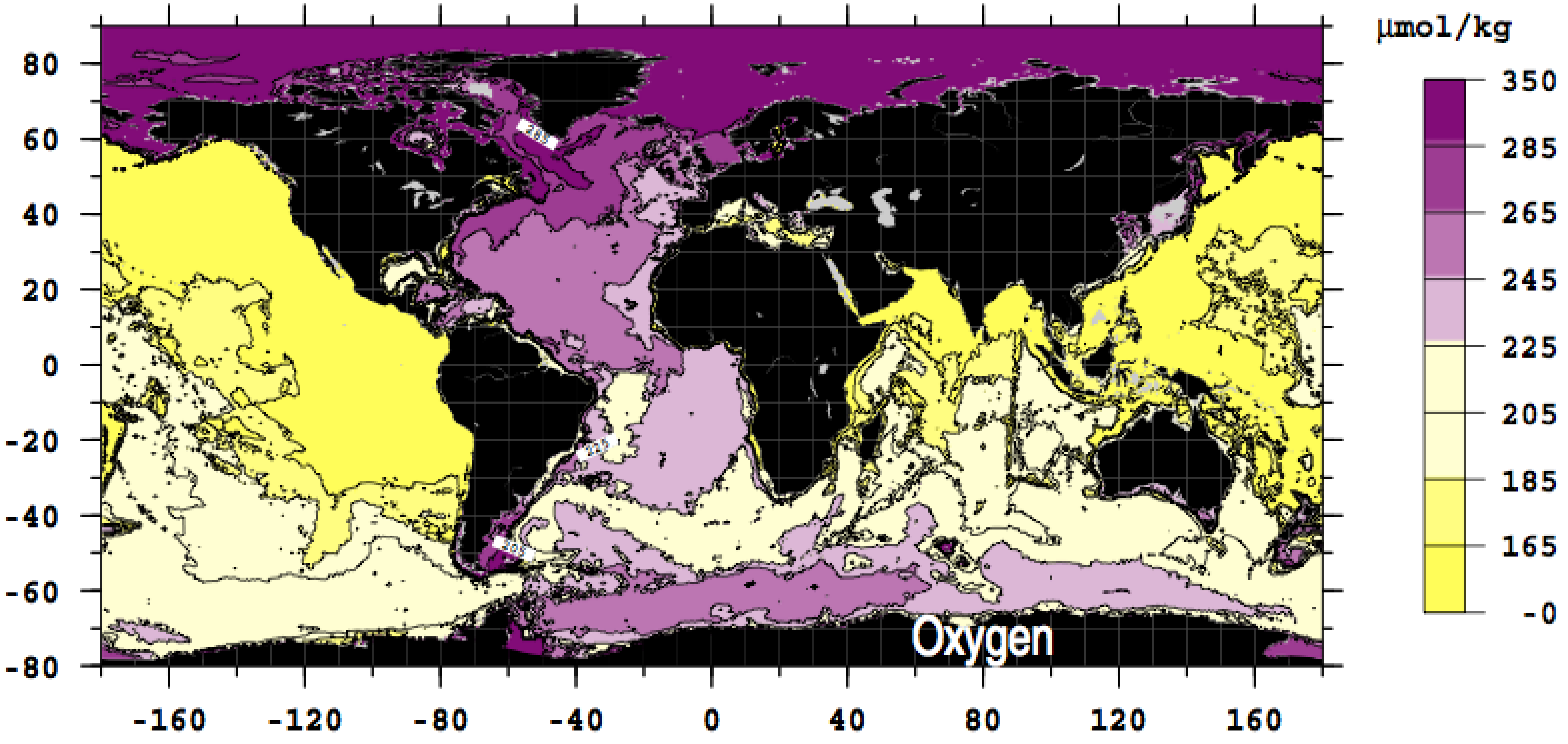
Silicate: 15263 profiles
Nitrate: 11997 profiles
Latitude
Latitude
80
60
40
20
0
-20
-40
-60
-80
-160
-120
-80
-40
0
40
80
120
160
Longitude



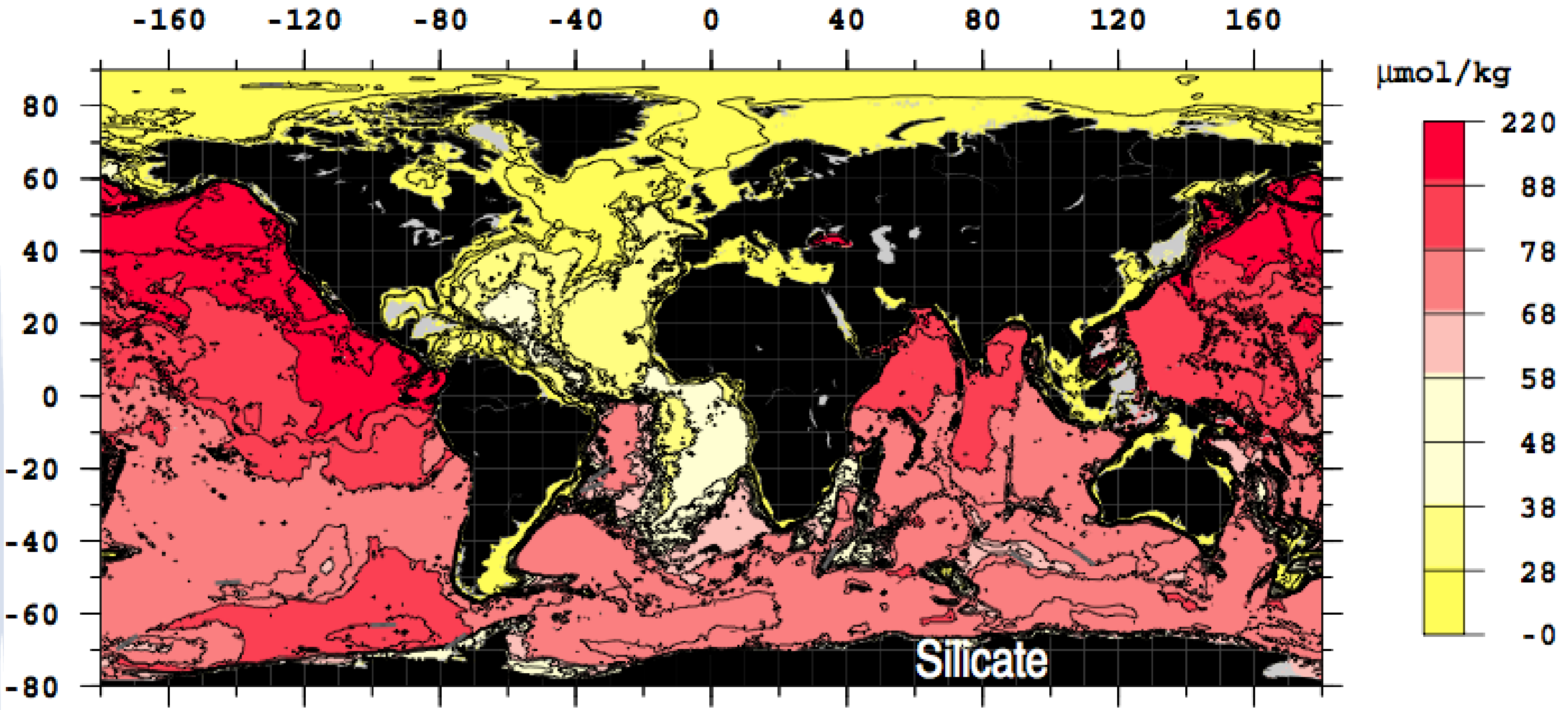
-160
-120
-80
-40
0
40
80
120
160
80
60
40
20
0
-20
-40
-60
-80
°C
25.0
2.4
2.2
2.0
1.8
1.6
1.4
1.2
-2.5
Potential Temperature



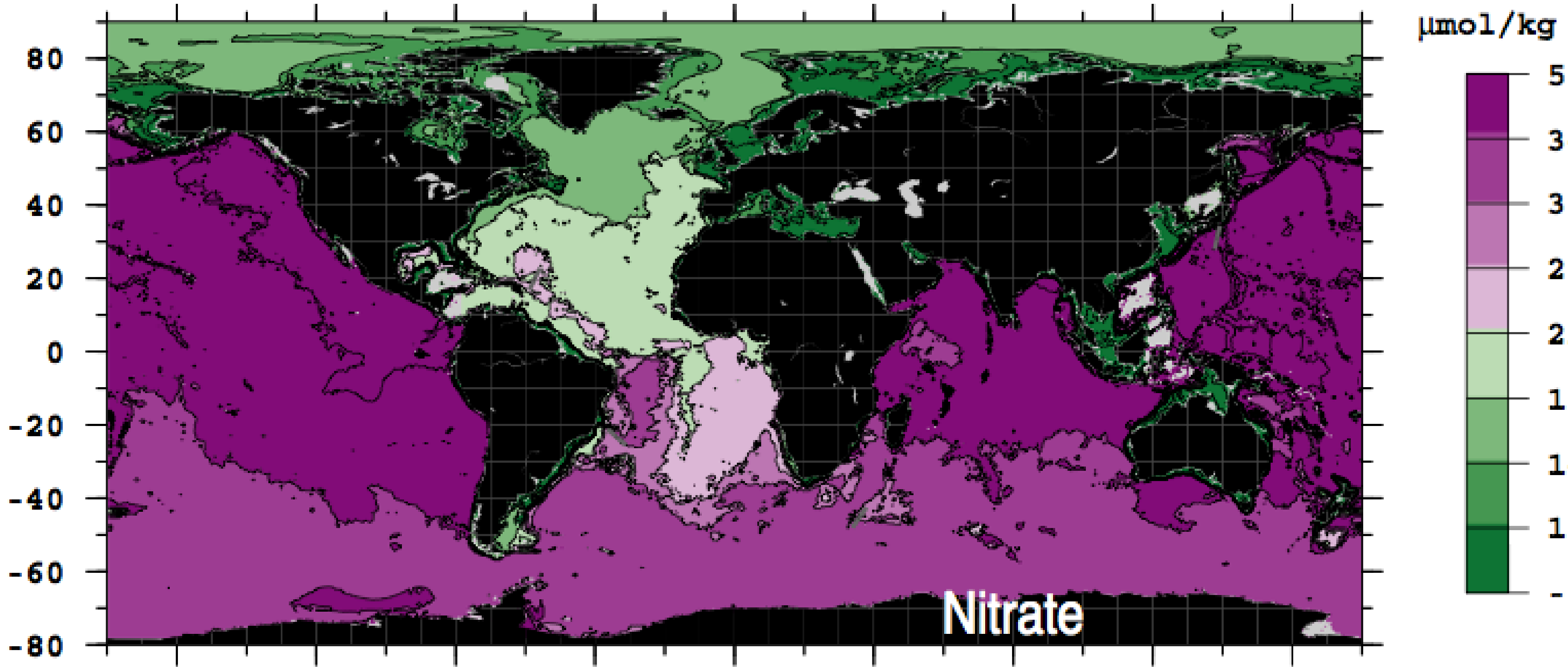
80
60
40
20
0
-20
-40
-60
-80
Salinity
44.00
34.93
34.90
34.87
34.84
34.81
34.78
34.75
0.00



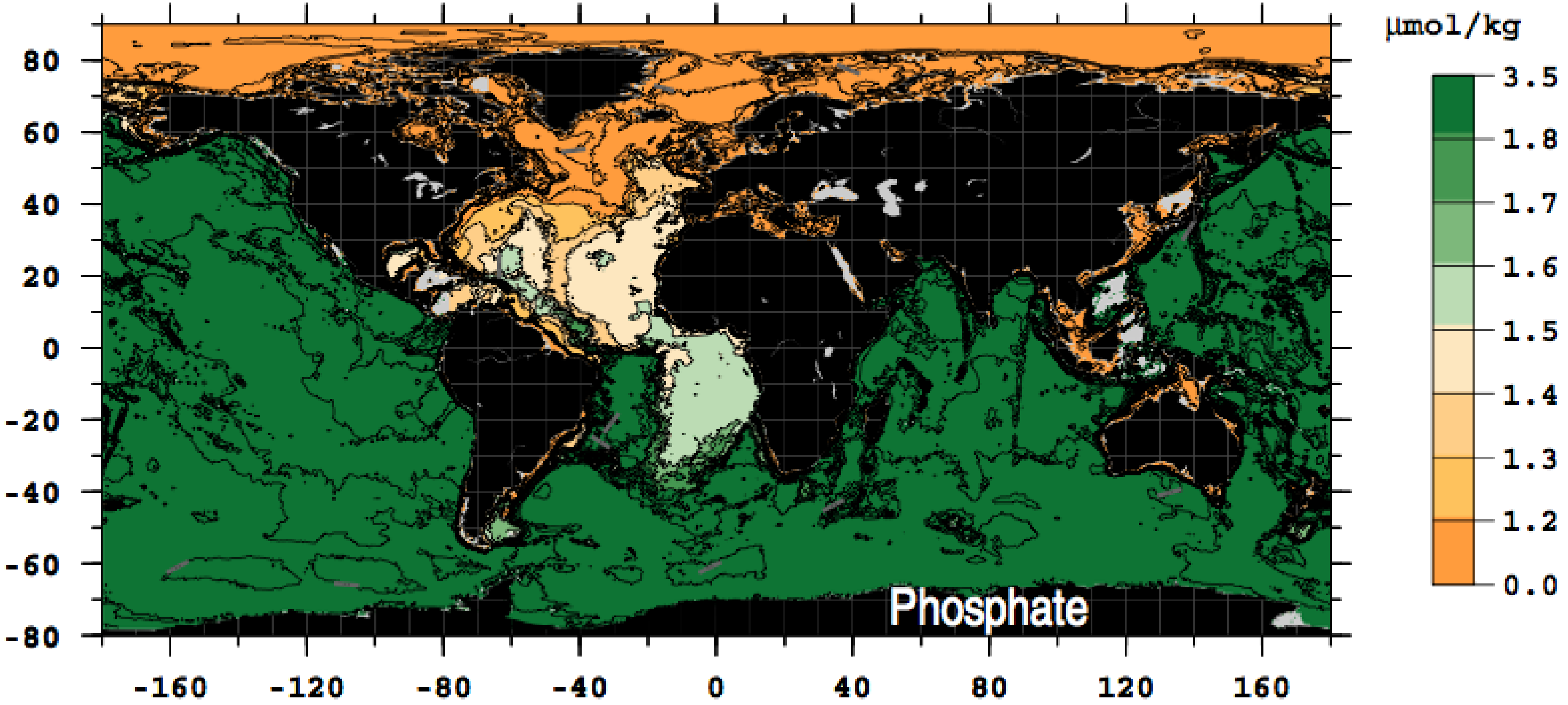
µmol/kg
350
285
265
245
225
205
185
165
-0
Oxygen
80
60
40
20
0
-20
-40
-60
-80
-160
-120
-80
-40
0
40
80
120
160



-160
-120
-80
-40
0
40
80
120
160
80
60
40
20
0
-20
-40
-60
-80
µmol/kg
220
88
78
68
58
48
38
28
-0
Silicate

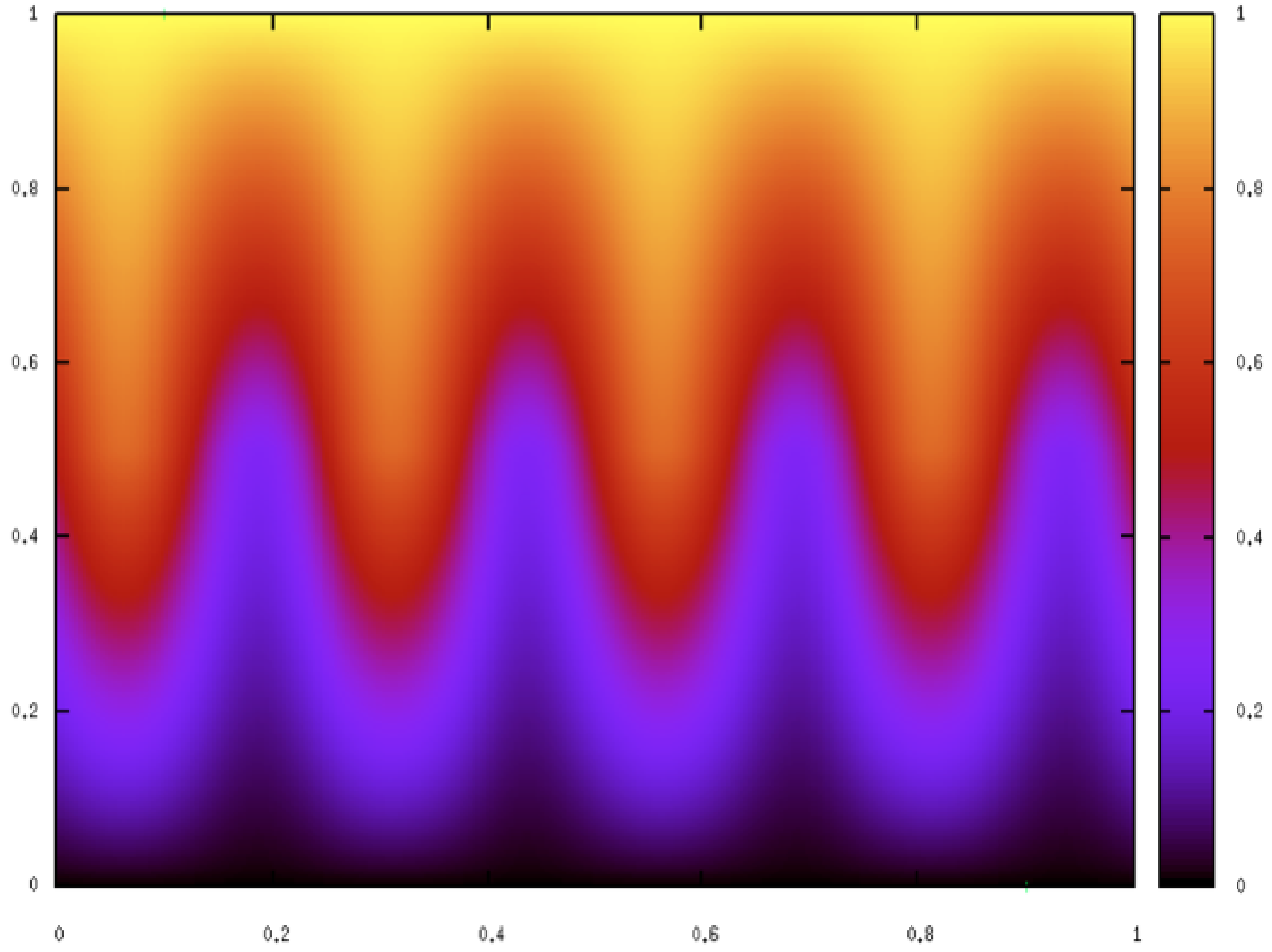


80
60
40
20
0
-20
-40
-60
-80
Nitrate
μmol/kg
50
34
30
26
22
18
14
10
-0

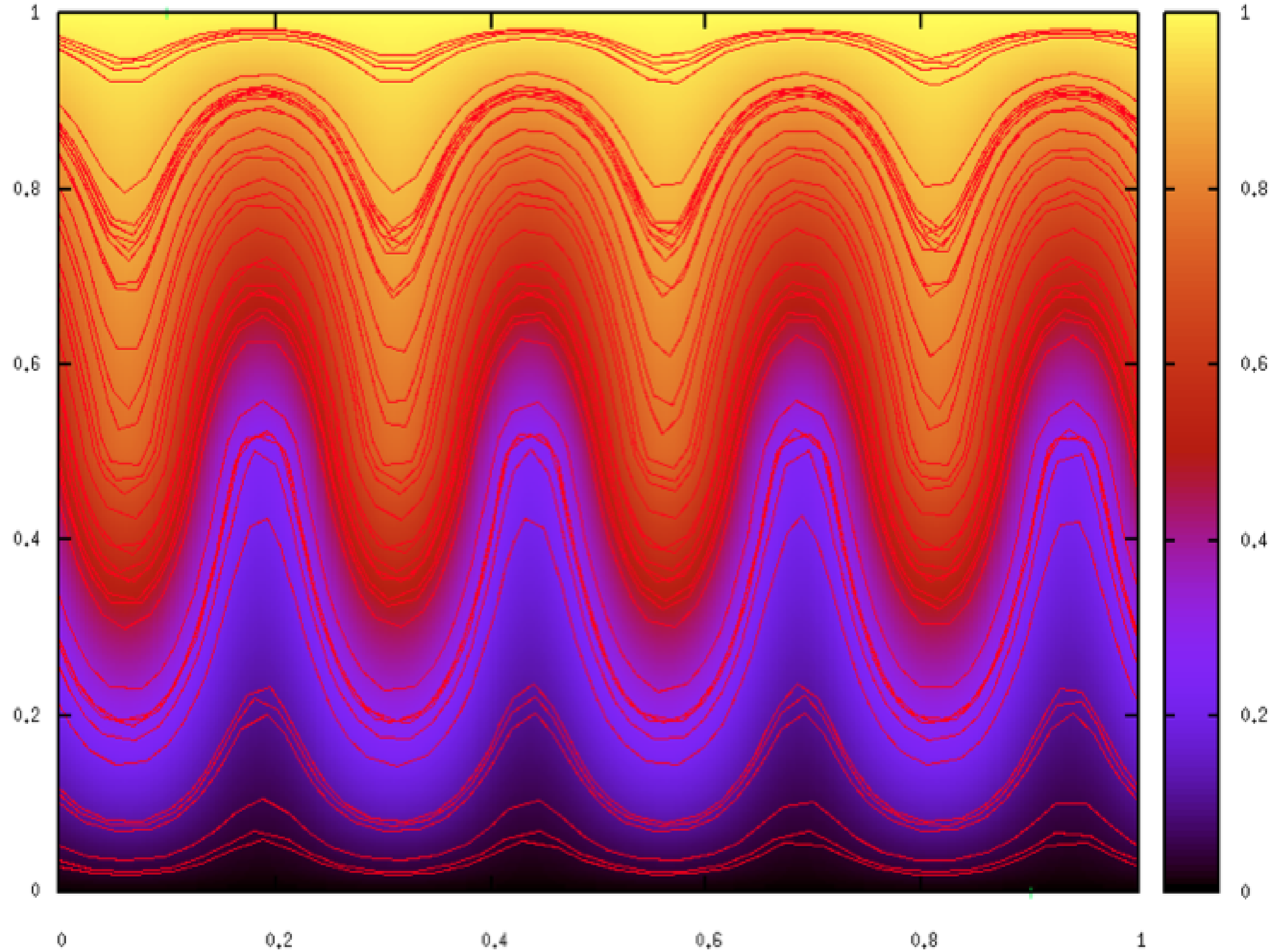


µmol/kg
3.5
1.8
1.7
1.6
1.5
1.4
1.3
1.2
0.0
80
60
40
20
0
-20
-40
-60
-80
-160
-120
-80
-40
0
40
80
120
160
Phosphate

Directional Interpolation

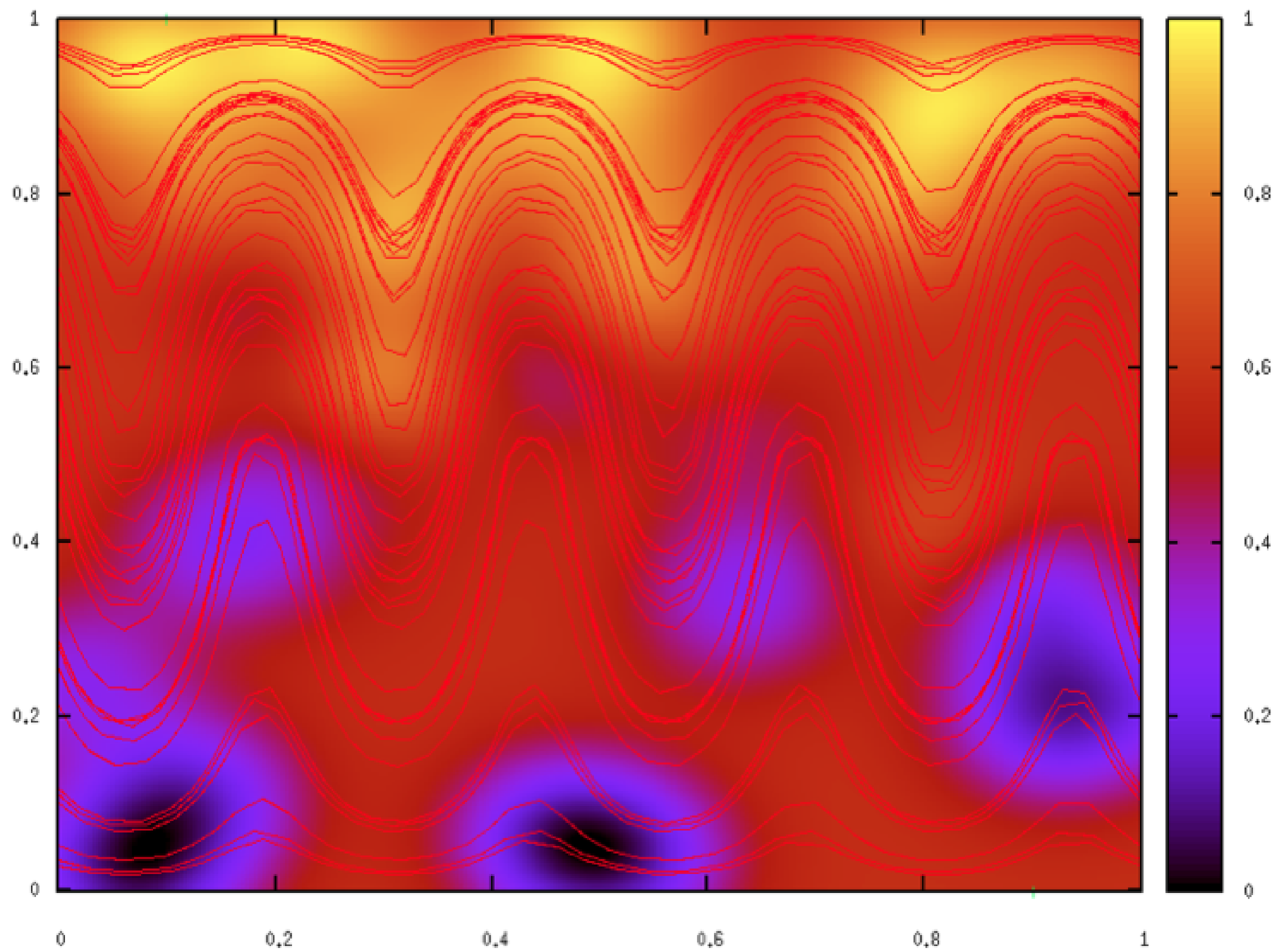


Directional Interpolation



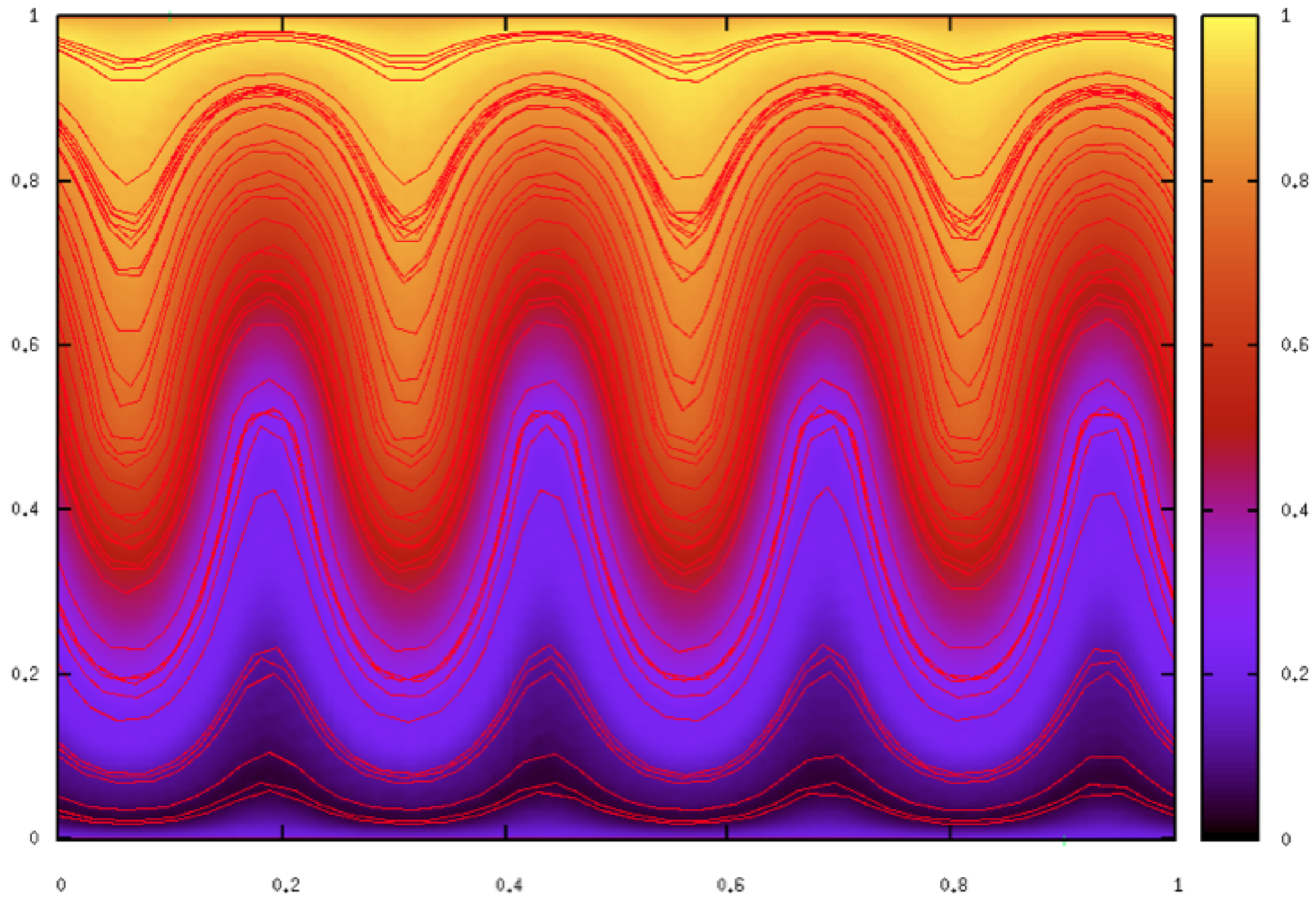
Directional Interpolation

Nondirectional interpolation (OI) (CORRLEN = 0.1)

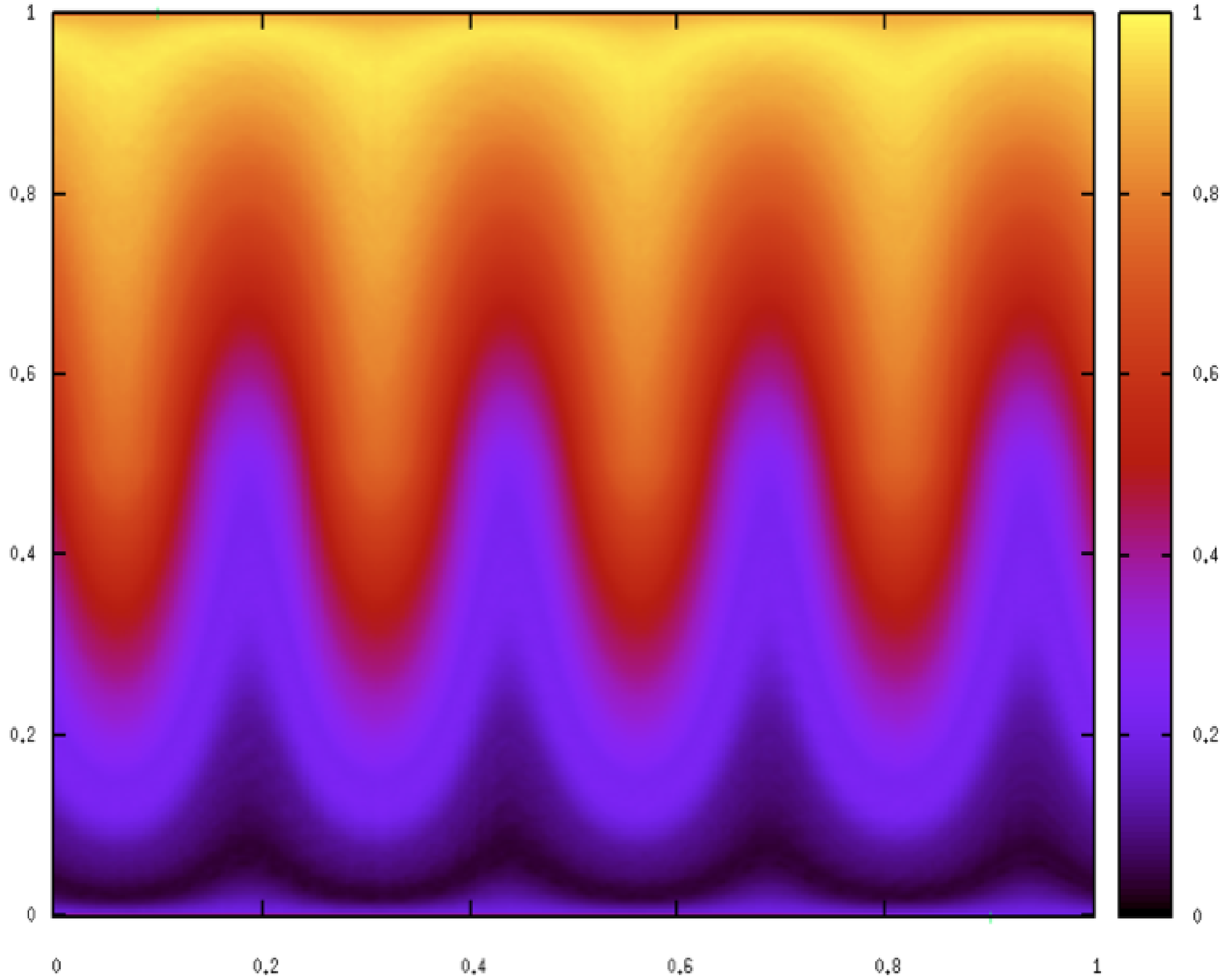


Directional Interpolation

Modified Hausdorff distance method (CORRLEN = 0.1)



Directional Interpolation



Scalar Field Approximation in a Flow Field

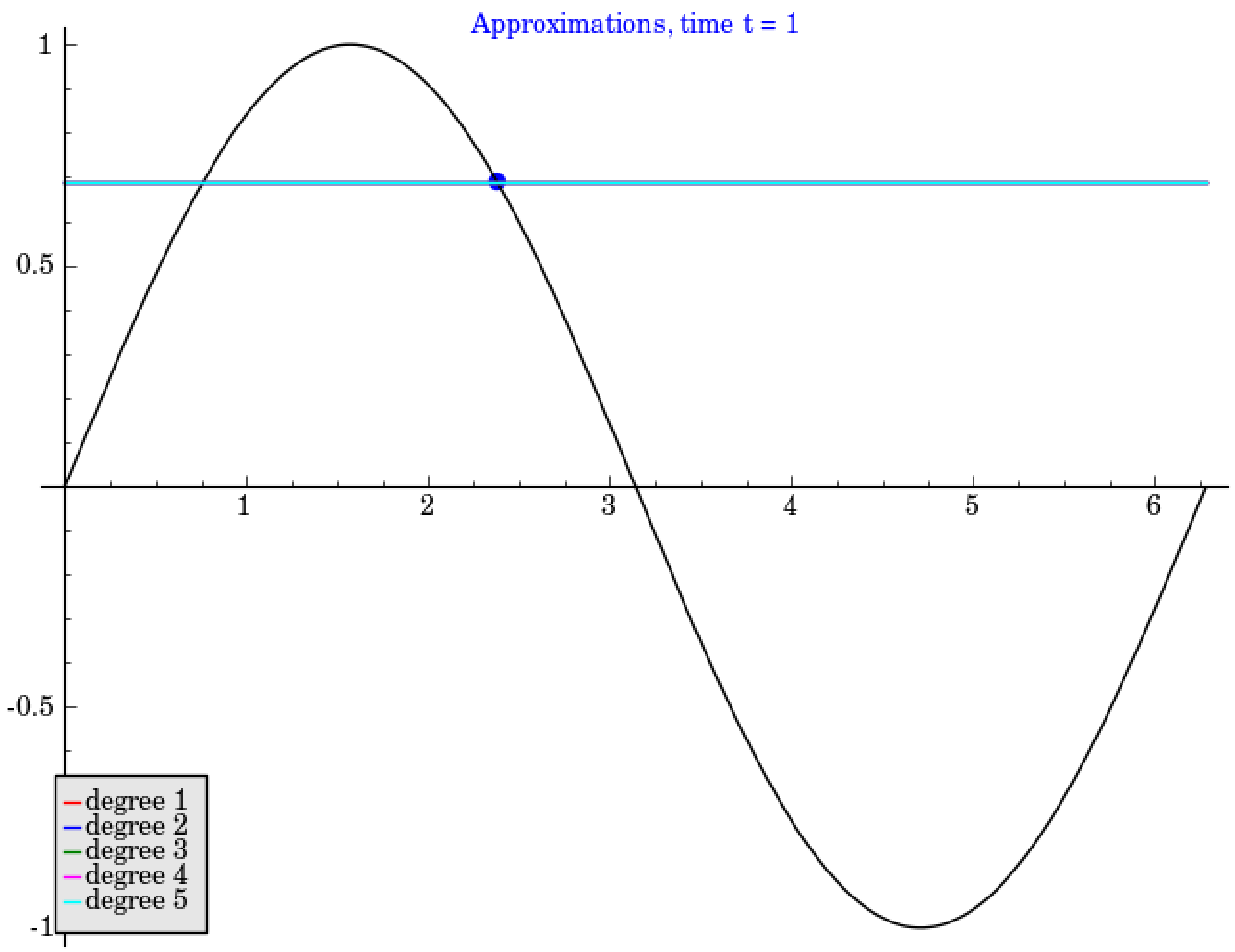
- Can we do something analogous to flow-field-aware directional interpolation, but without a priori knowledge of the flow field?
- Might want to sample scalar fields at nearby points in order to get flow field approximations
- In general, a spatial statistics problem
- Can we determine where additional data points are needed?

Simple Test Problems

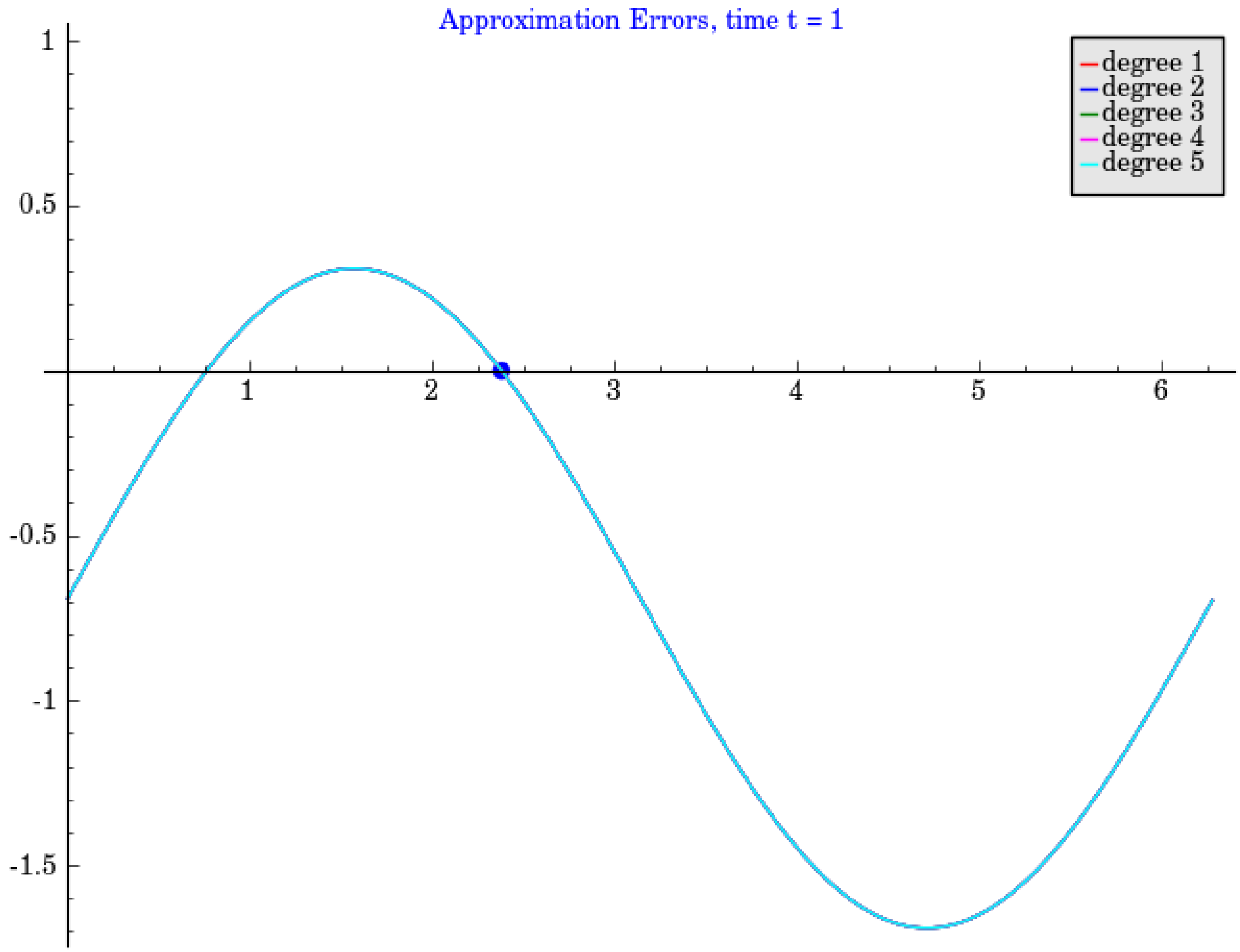
- Approximation methods in 1D and 2D
  - Spline interpolation
  - Regression
  - Scattered data interpolation

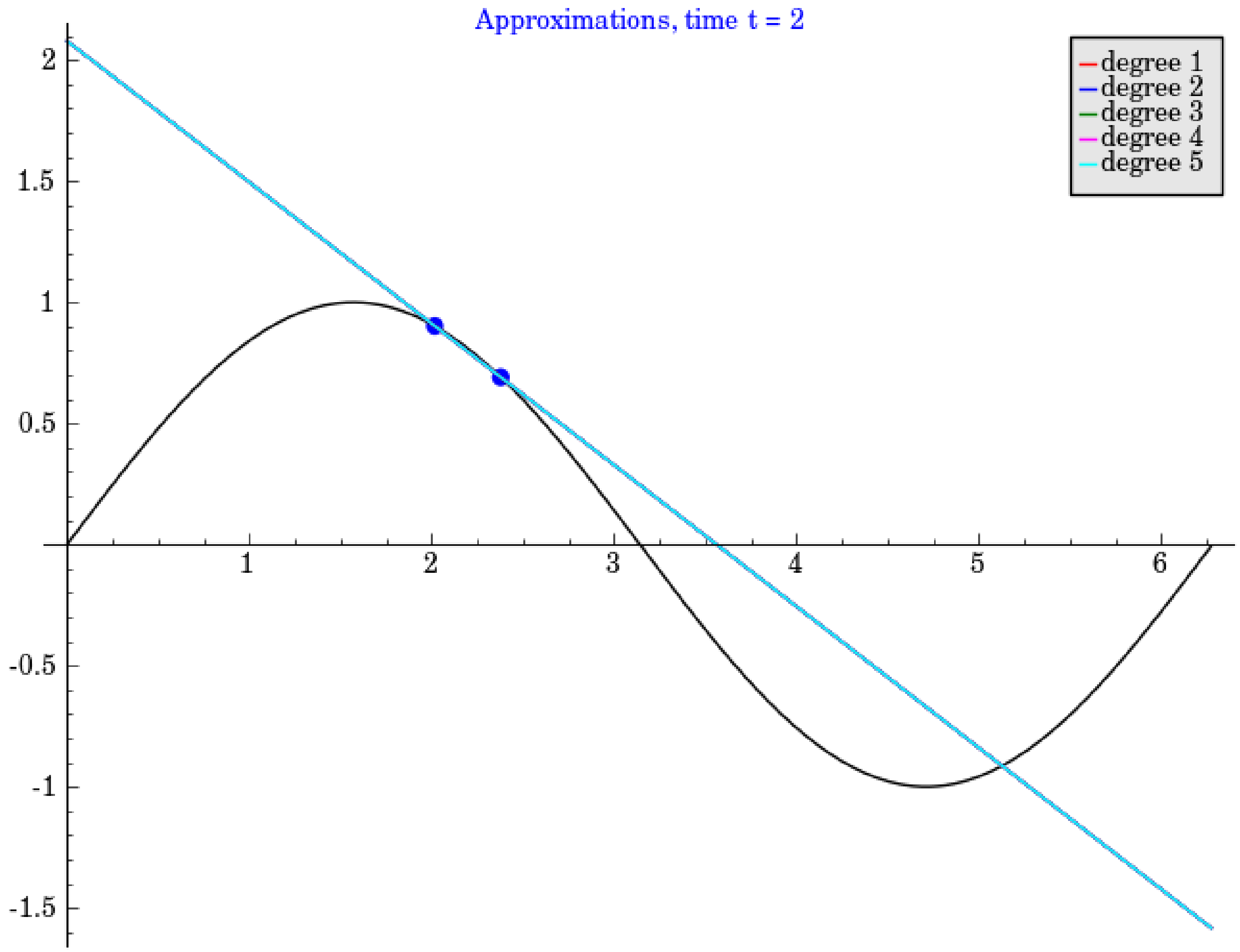
Using Computational Mechanics

- How to formulate a mathematical approximation problem as a dynamical system?
- First approach : sequential sampling of a given (unknown) function
- As with computational mechanics, a primary goal of our approximation problem is **prediction**

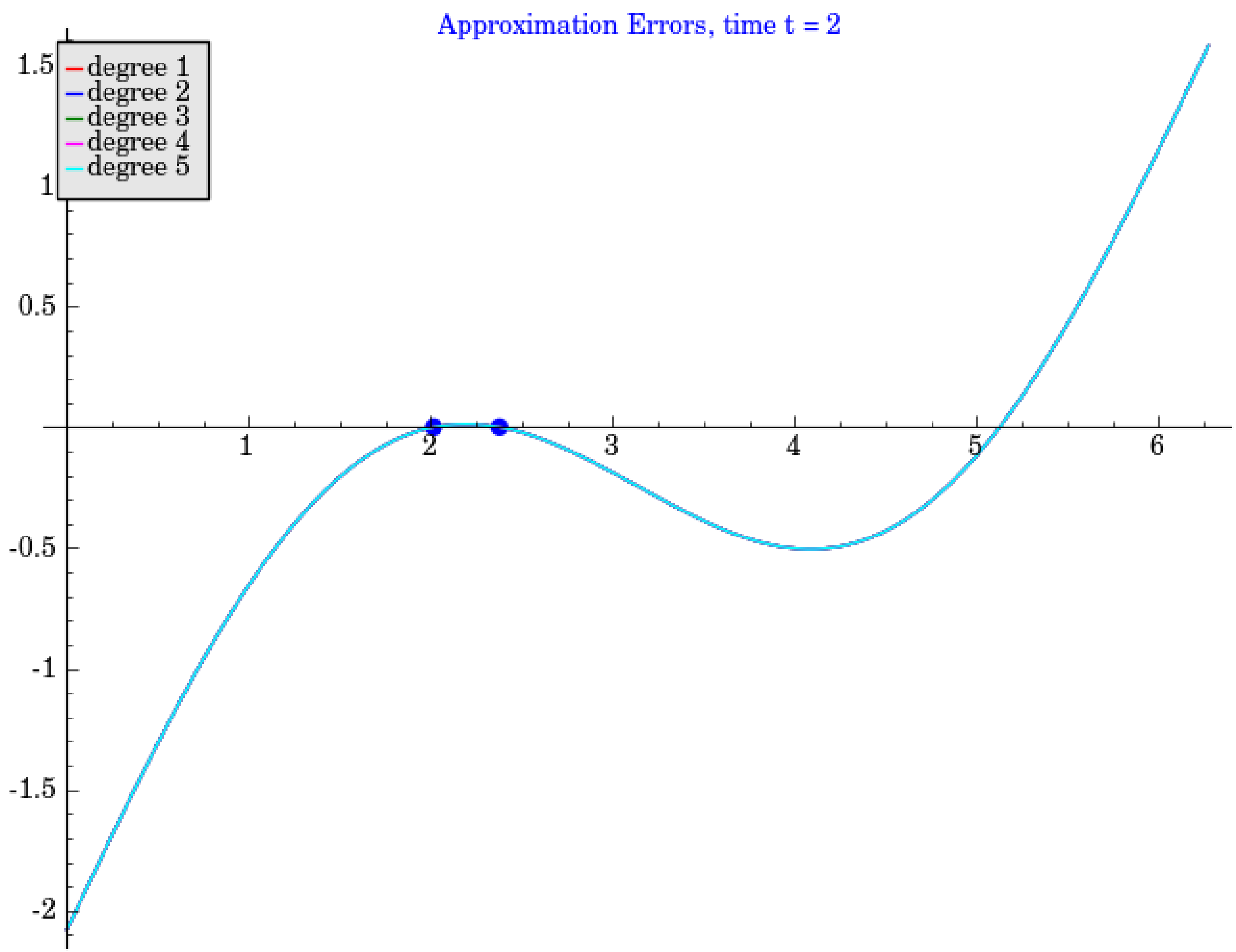


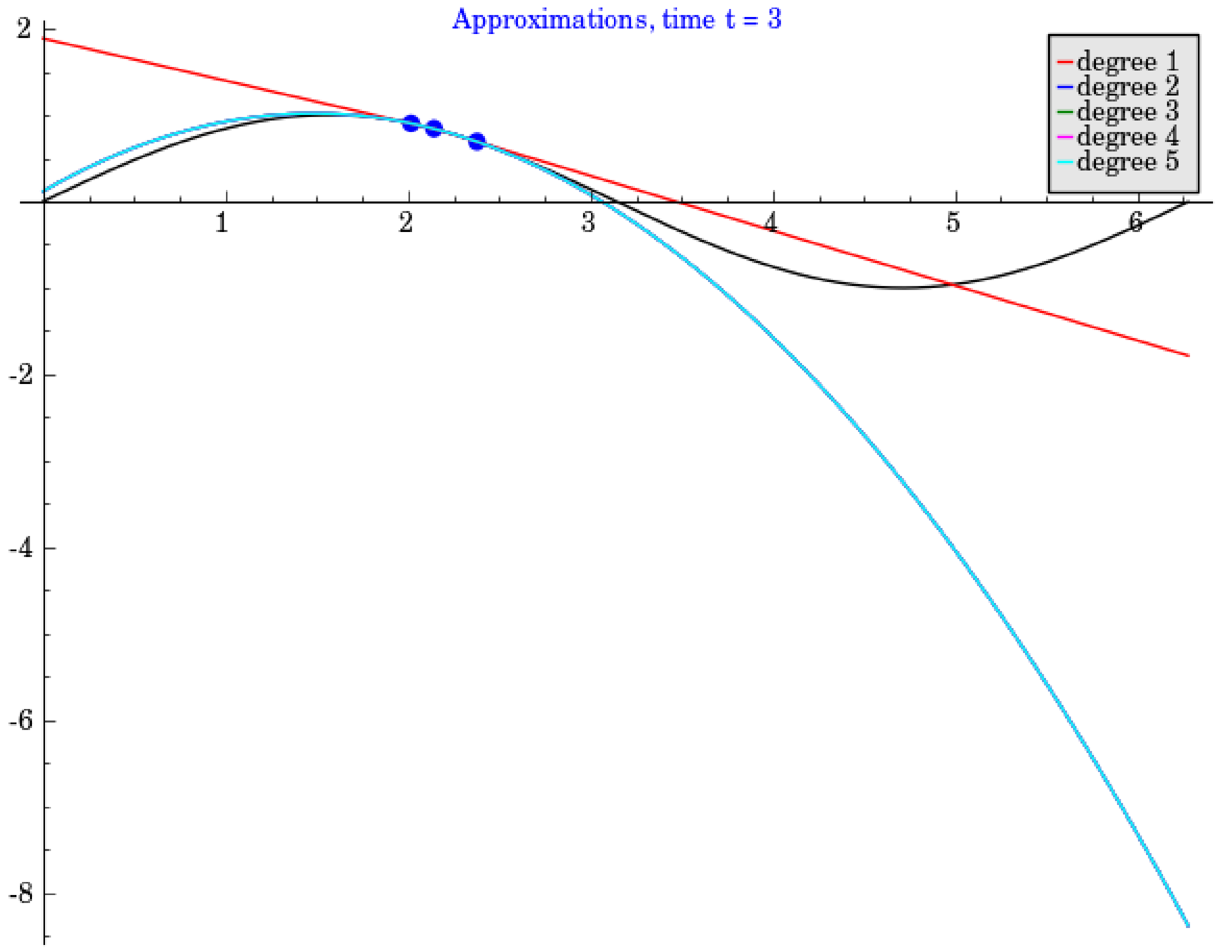
Approximations, time t = 1
1
0.5
-0.5
-1
1
2
3
4
5
6
degree 1
degree 2
degree 3
degree 4
degree 5


Approximation Errors, time t = 1
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
-1.5
1
2
3
4
5
6

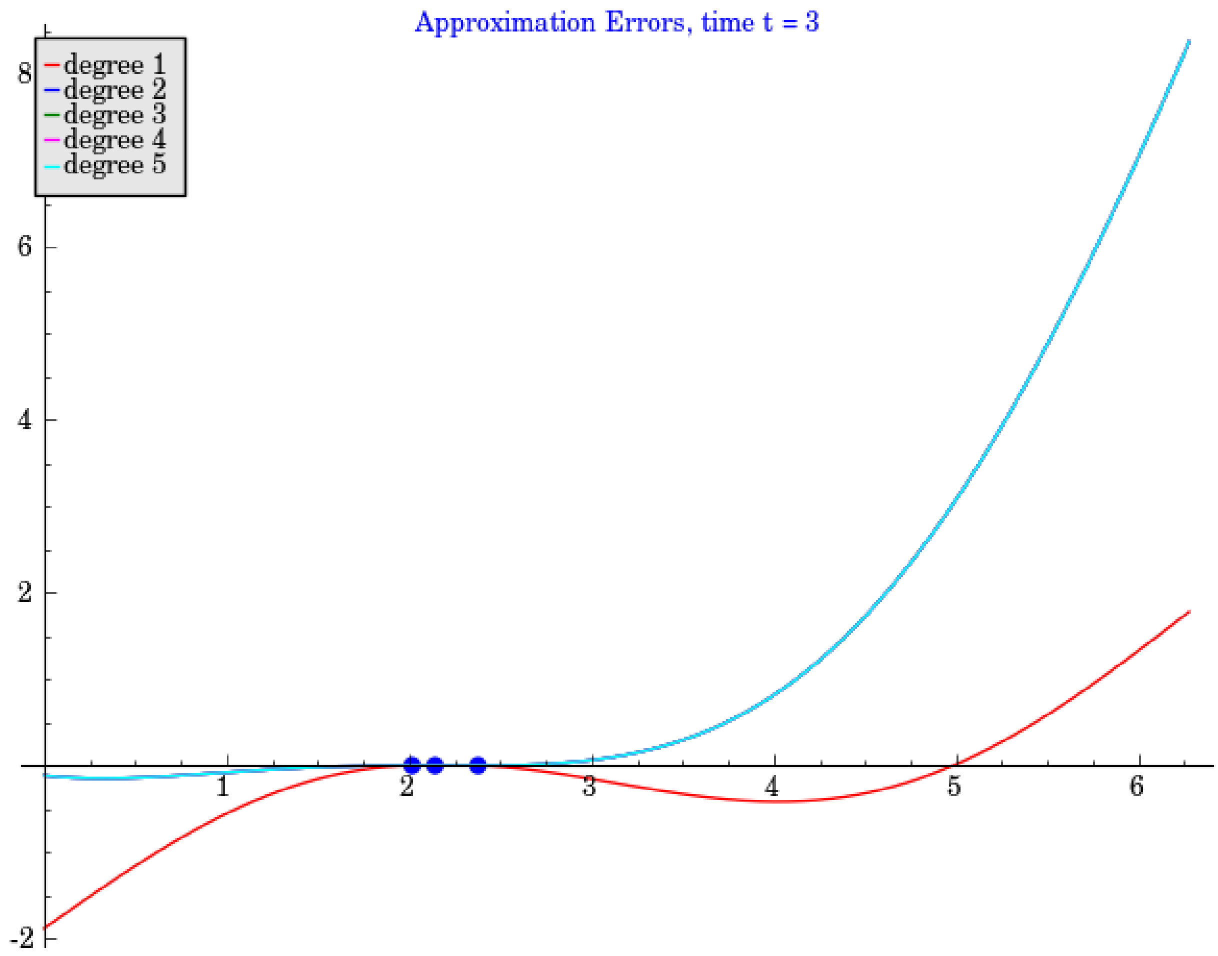


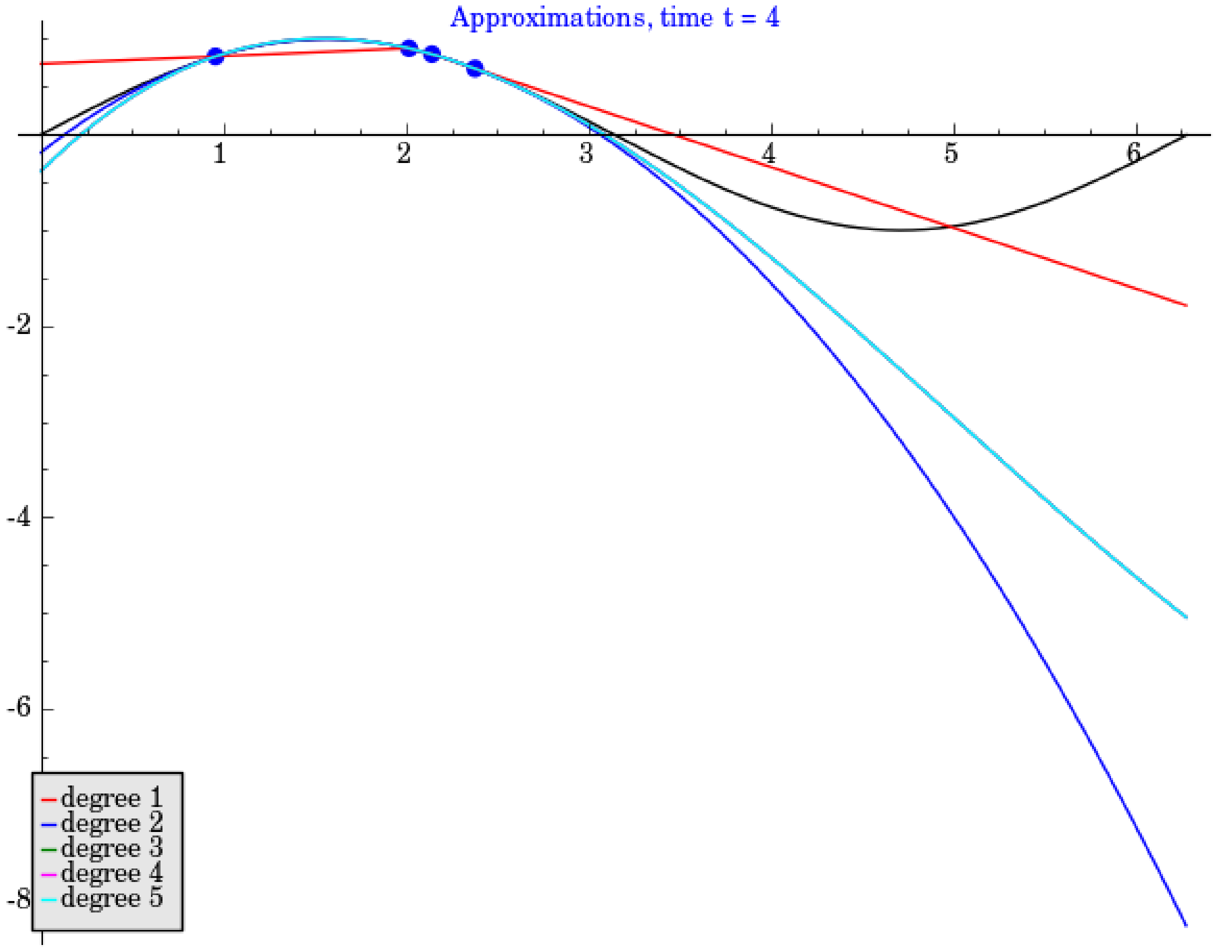
Approximations, time t = 2
degree 1
degree 2
degree 3
degree 4
degree 5
2
1.5
1
0.5
-0.5
-1
-1.5
1
2
3
4
5
6


Approximation Errors, time t = 2
degree 1
degree 2
degree 3
degree 4
degree 5
1.5
1
0.5
-0.5
-1
-1.5
-2
1
2
3
4
5
6

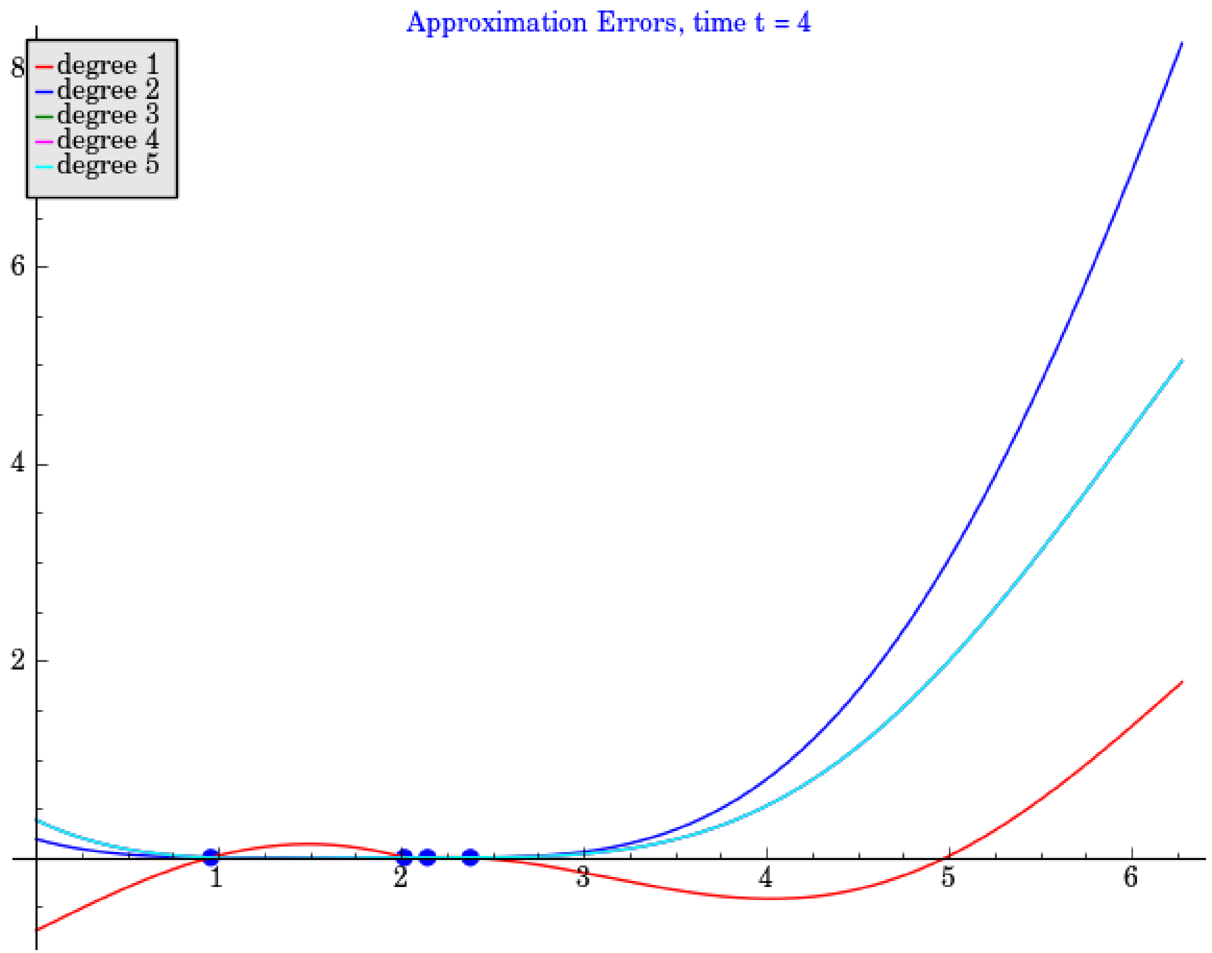


Approximations, time t = 3
degree 1
degree 2
degree 3
degree 4
degree 5
2
-2
-4
-6
-8
1
2
3
4
5
6

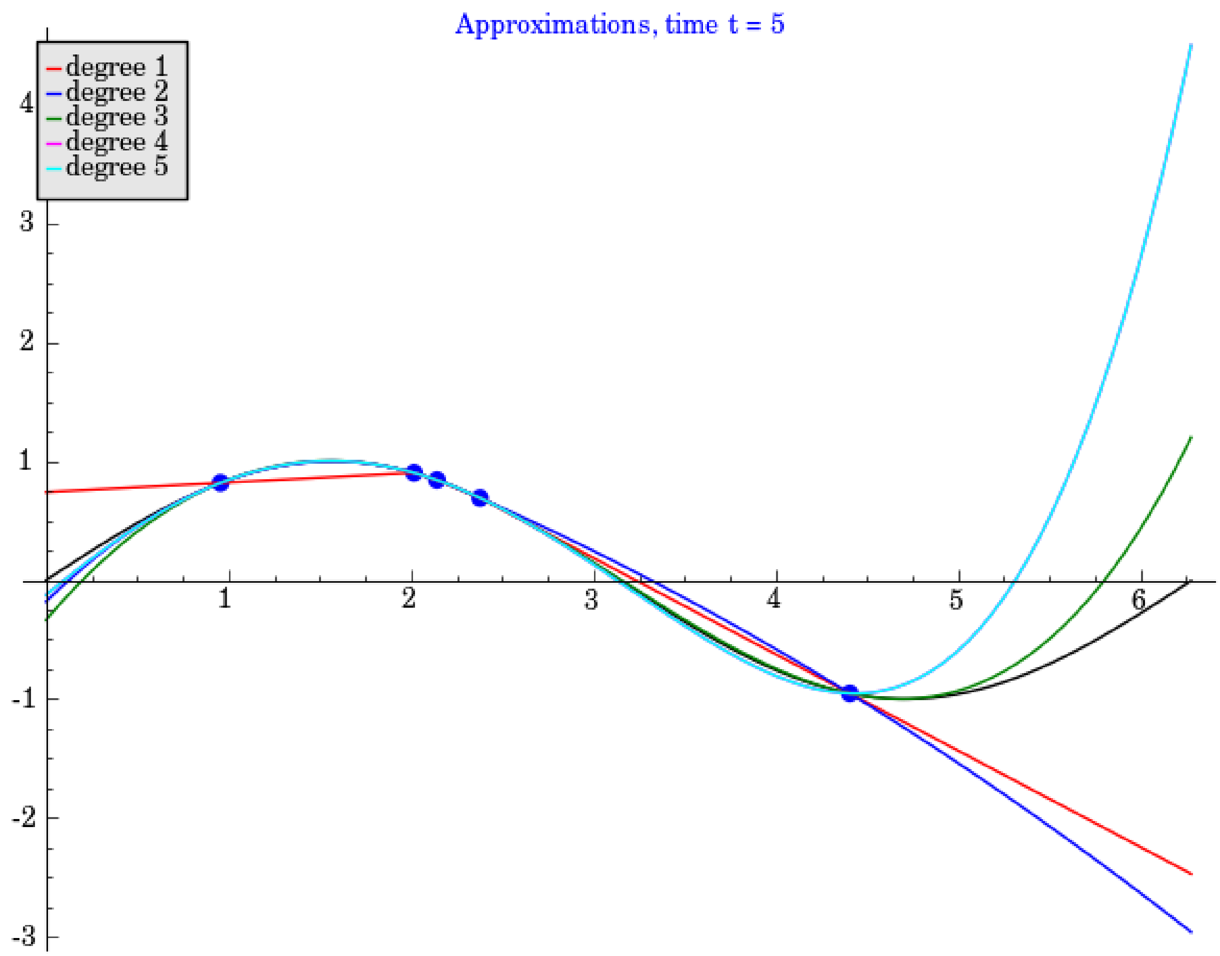

Approximation Errors, time t = 3
degree 1
degree 2
degree 3
degree 4
degree 5
8
6
4
2
-2
1
2
3
4
5
6



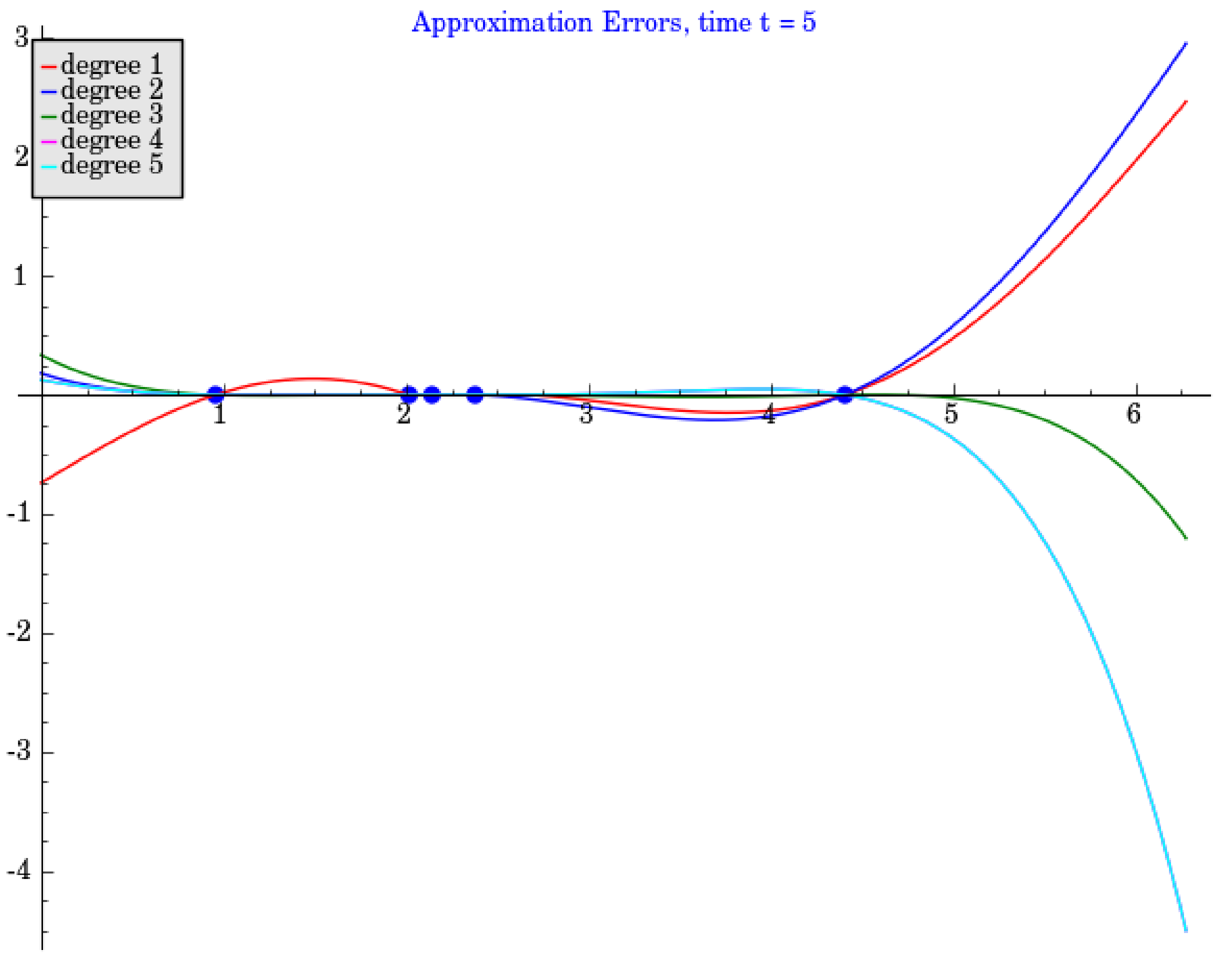
Approximations, time t = 4
1
2
3
4
5
6
-2
-4
-6
-8
degree 1
degree 2
degree 3
degree 4
degree 5

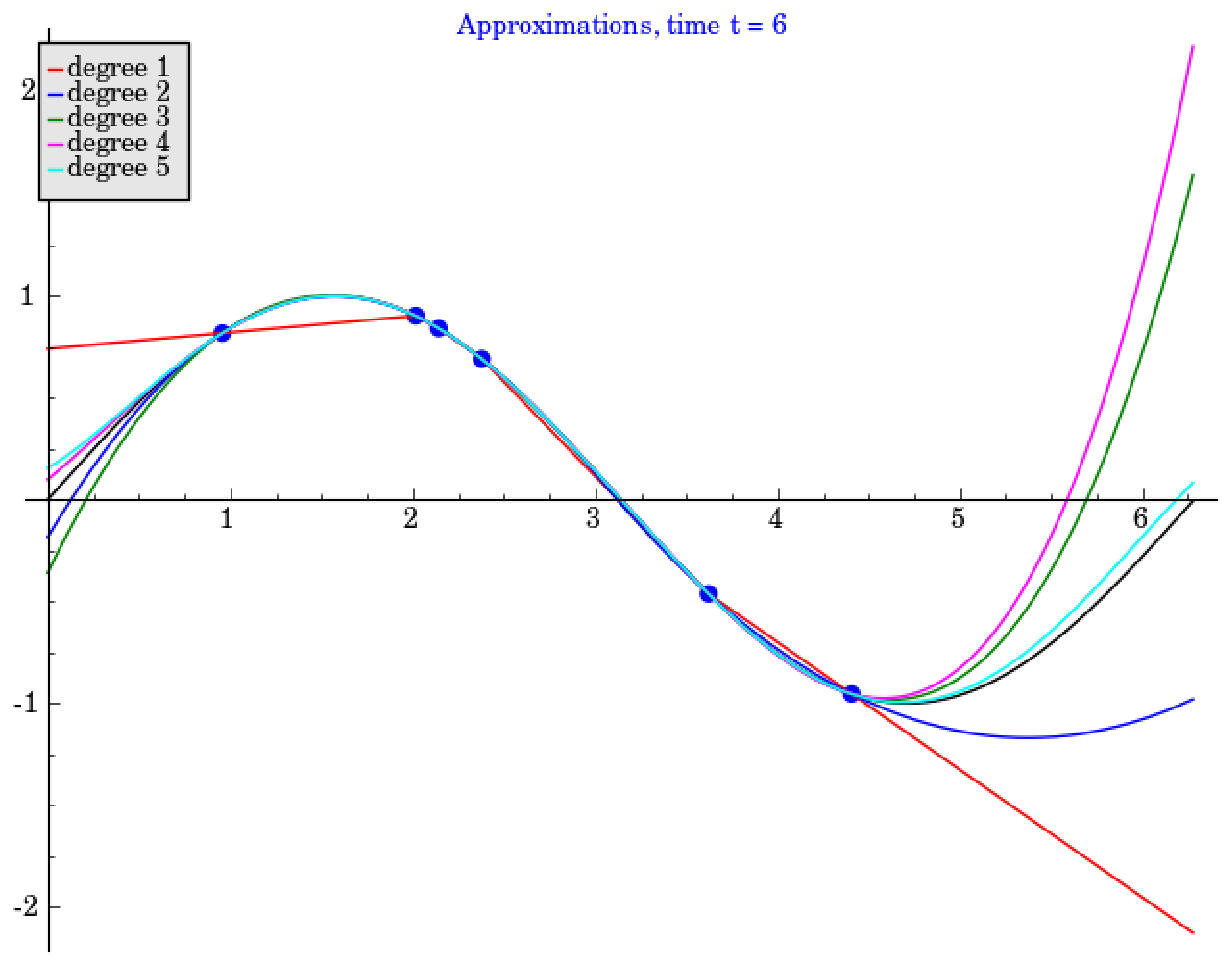


Approximation Errors, time t = 4
degree 1
degree 2
degree 3
degree 4
degree 5
8
6
4
2
1
2
3
4
5
6

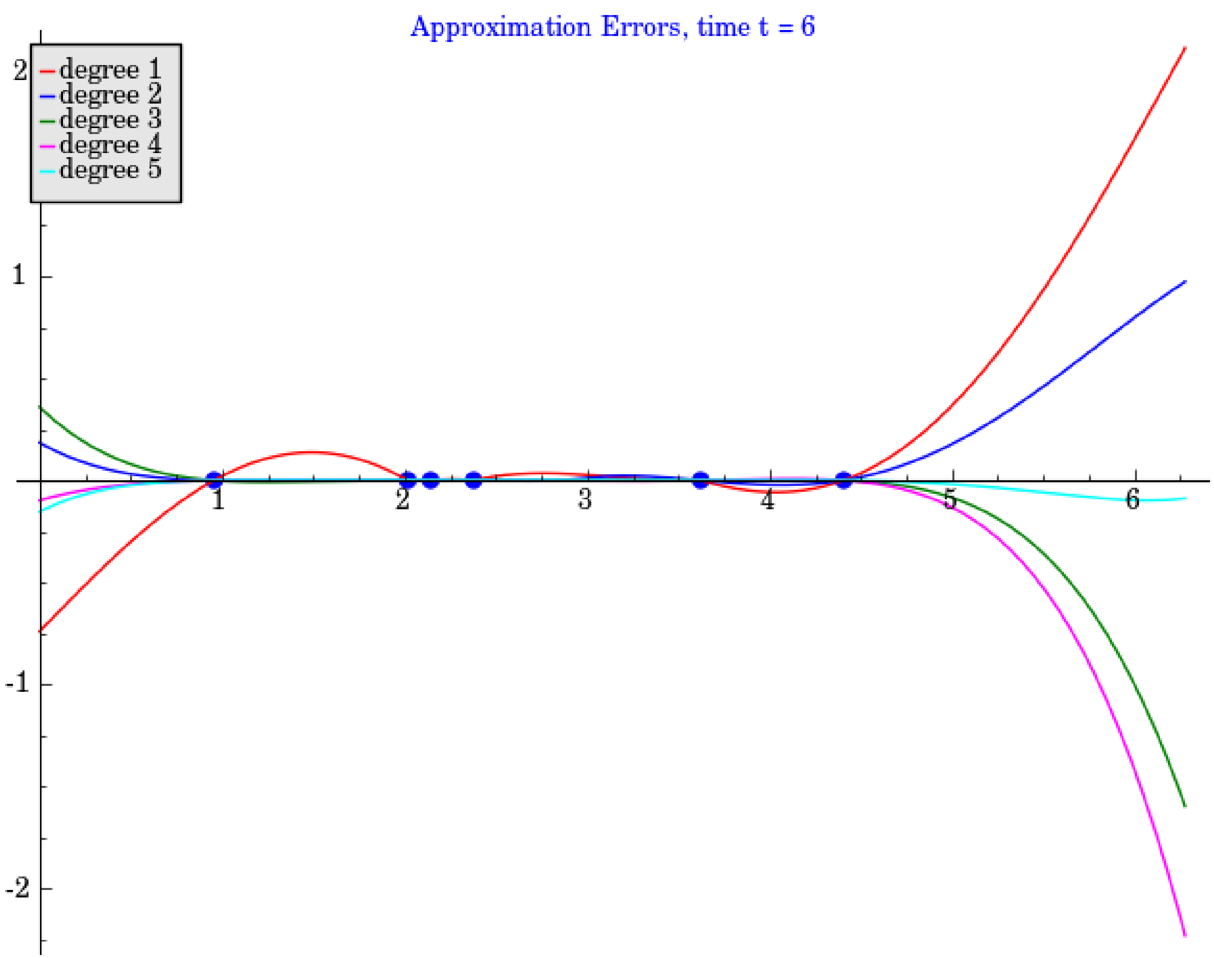


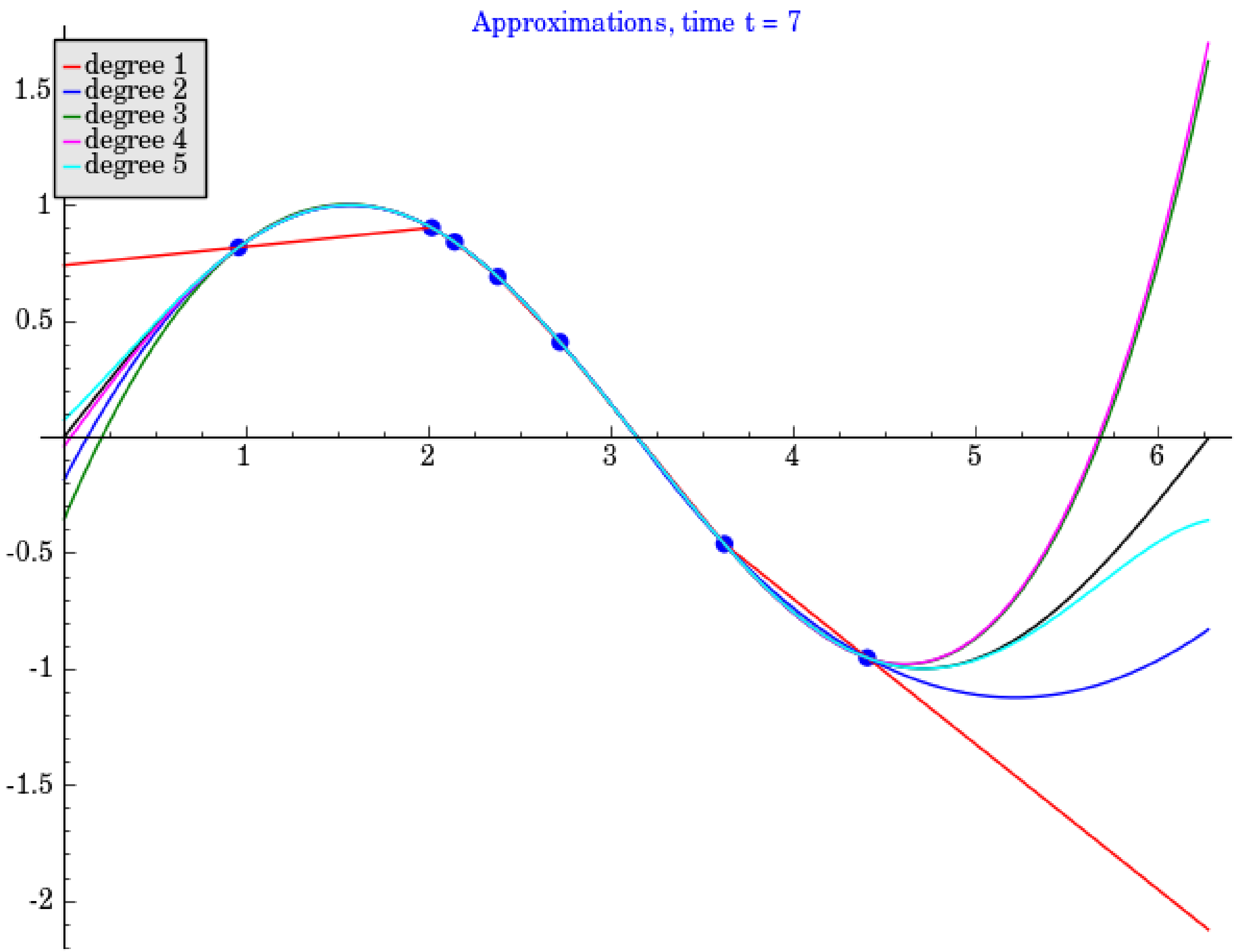
Approximations, time t = 5
degree 1
degree 2
degree 3
degree 4
degree 5
4
3
2
1
-1
-2
-3
1
2
3
4
5
6


Approximation Errors, time t = 5
degree 1
degree 2
degree 3
degree 4
degree 5
3
2
1
-1
-2
-3
-4
1
2
3
4
5
6

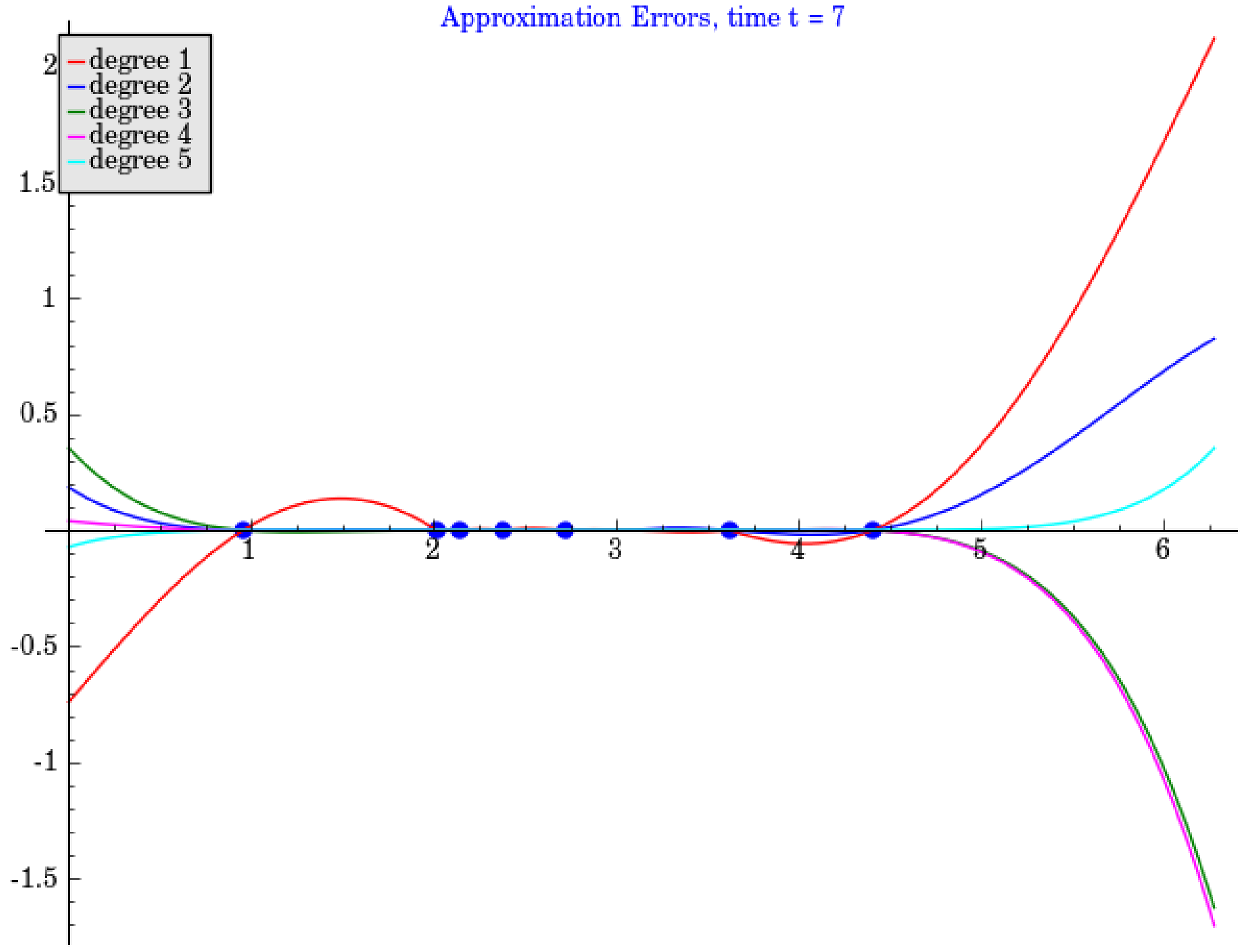


Approximations, time t = 6
degree 1
degree 2
degree 3
degree 4
degree 5
2
1
-1
-2
1
2
3
4
5
6

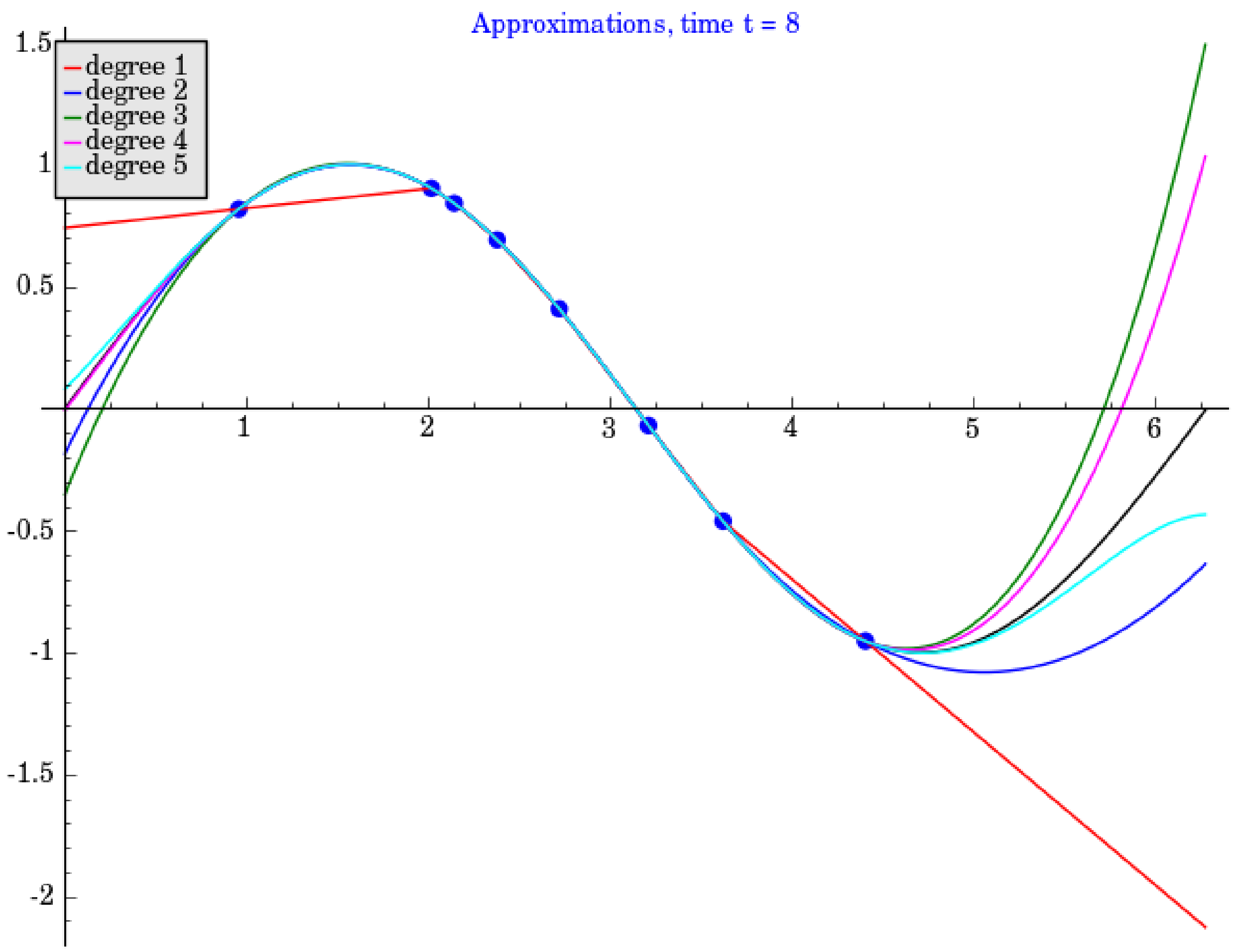

Approximation Errors, time t = 6
degree 1
degree 2
degree 3
degree 4
degree 5
2
1
-1
-2
1
2
3
4
5
6



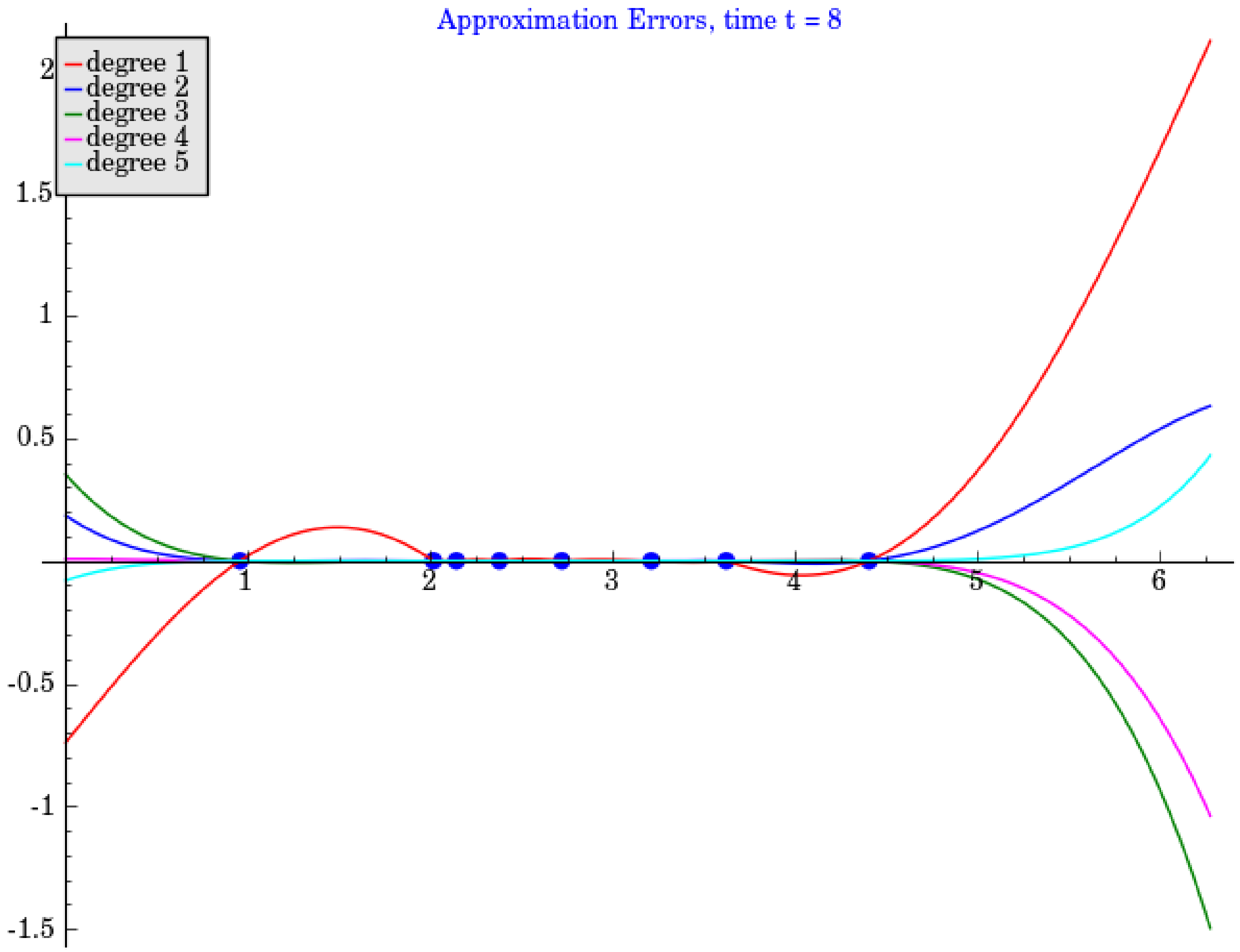
Approximations, time t = 7
degree 1
degree 2
degree 3
degree 4
degree 5
1.5
1
0.5
-0.5
-1
-1.5
-2
1
2
3
4
5
6



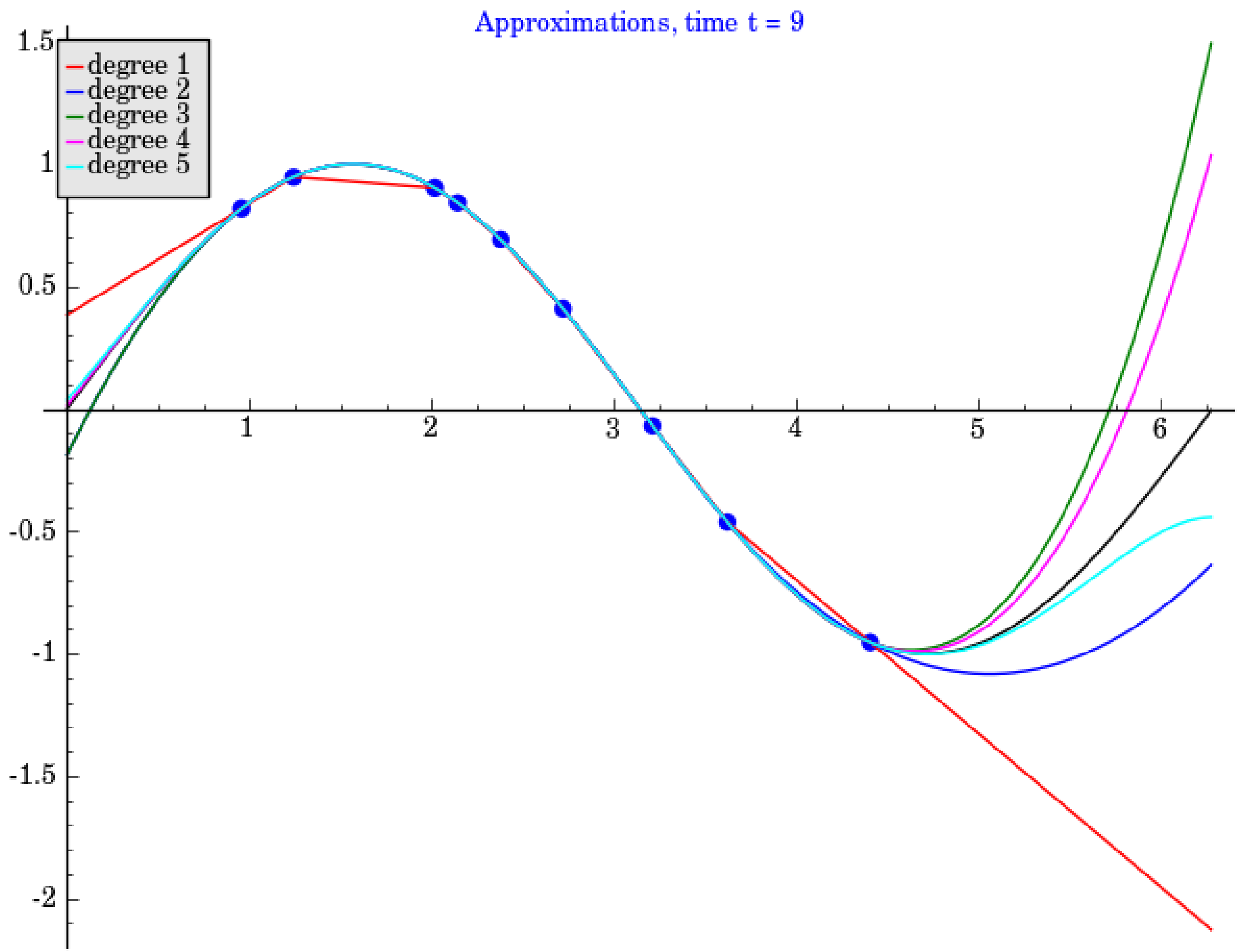
Approximation Errors, time t = 7
degree 1
degree 2
degree 3
degree 4
degree 5
2
1.5
1
0.5
-0.5
-1
-1.5
1
2
3
4
5
6



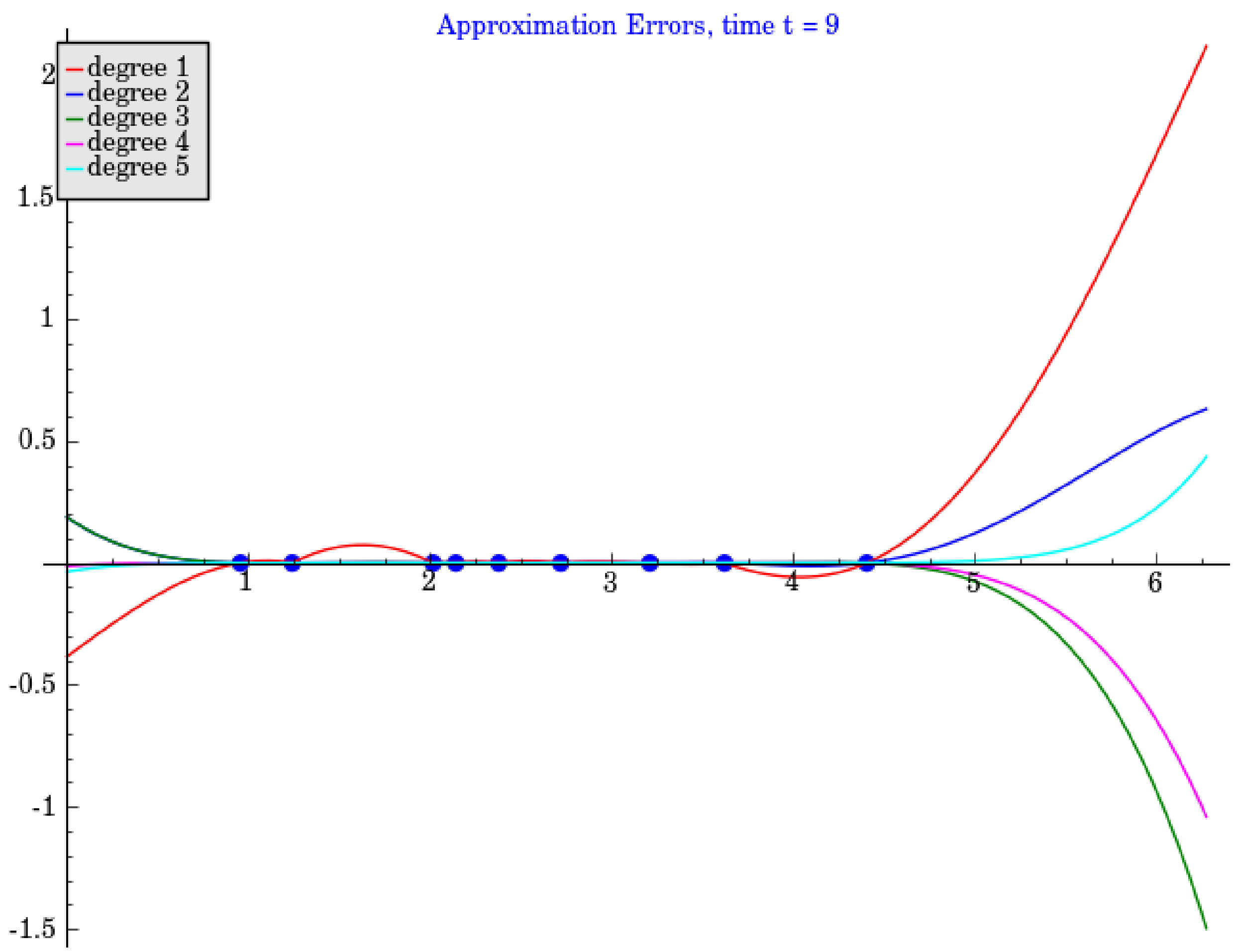
Approximations, time t = 8
degree 1
degree 2
degree 3
degree 4
degree 5
1.5
1
0.5
-0.5
-1
-1.5
-2
1
2
3
4
5
6



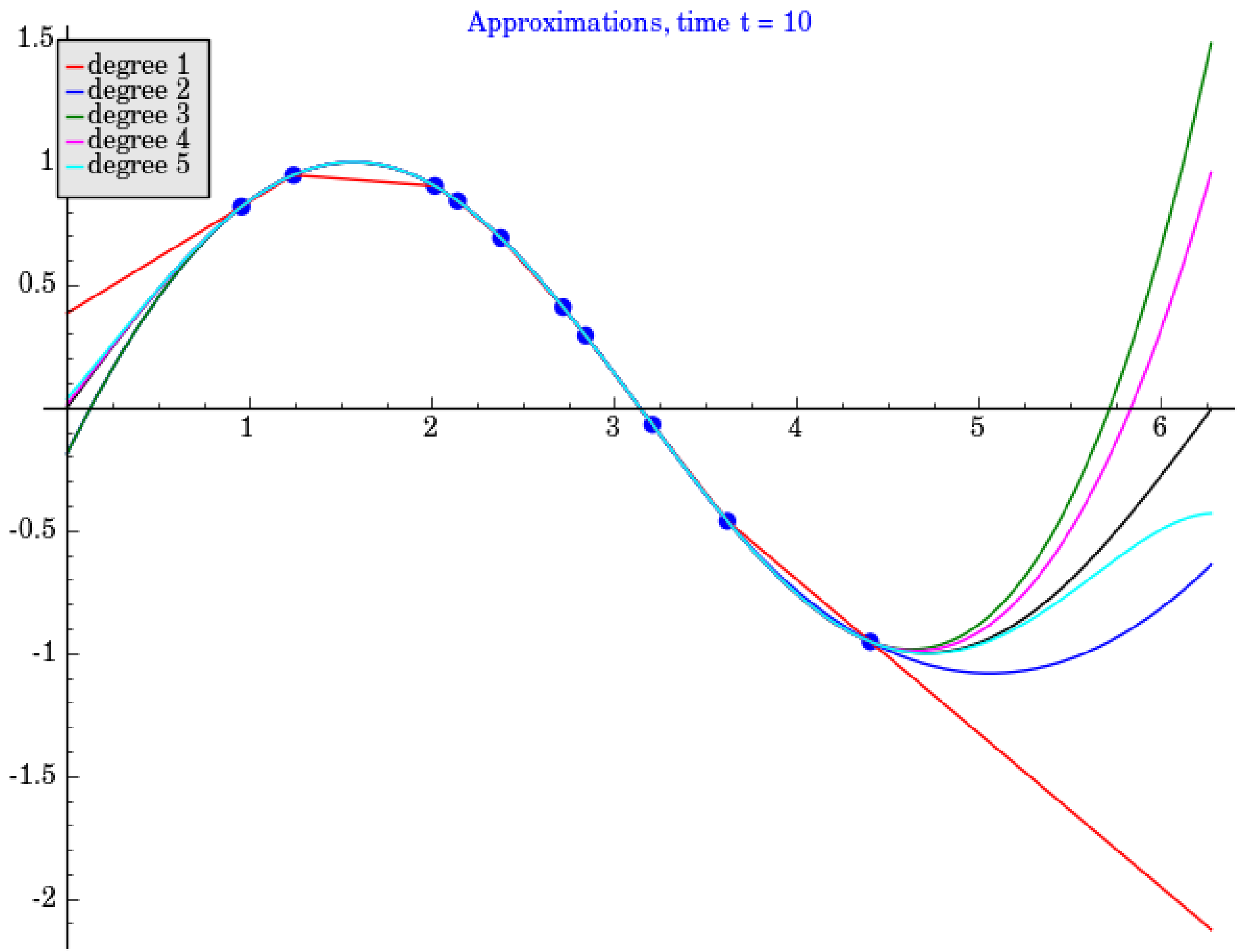
Approximation Errors, time t = 8
degree 1
degree 2
degree 3
degree 4
degree 5
2
1.5
1
0.5
-0.5
-1
-1.5
1
2
3
4
5
6



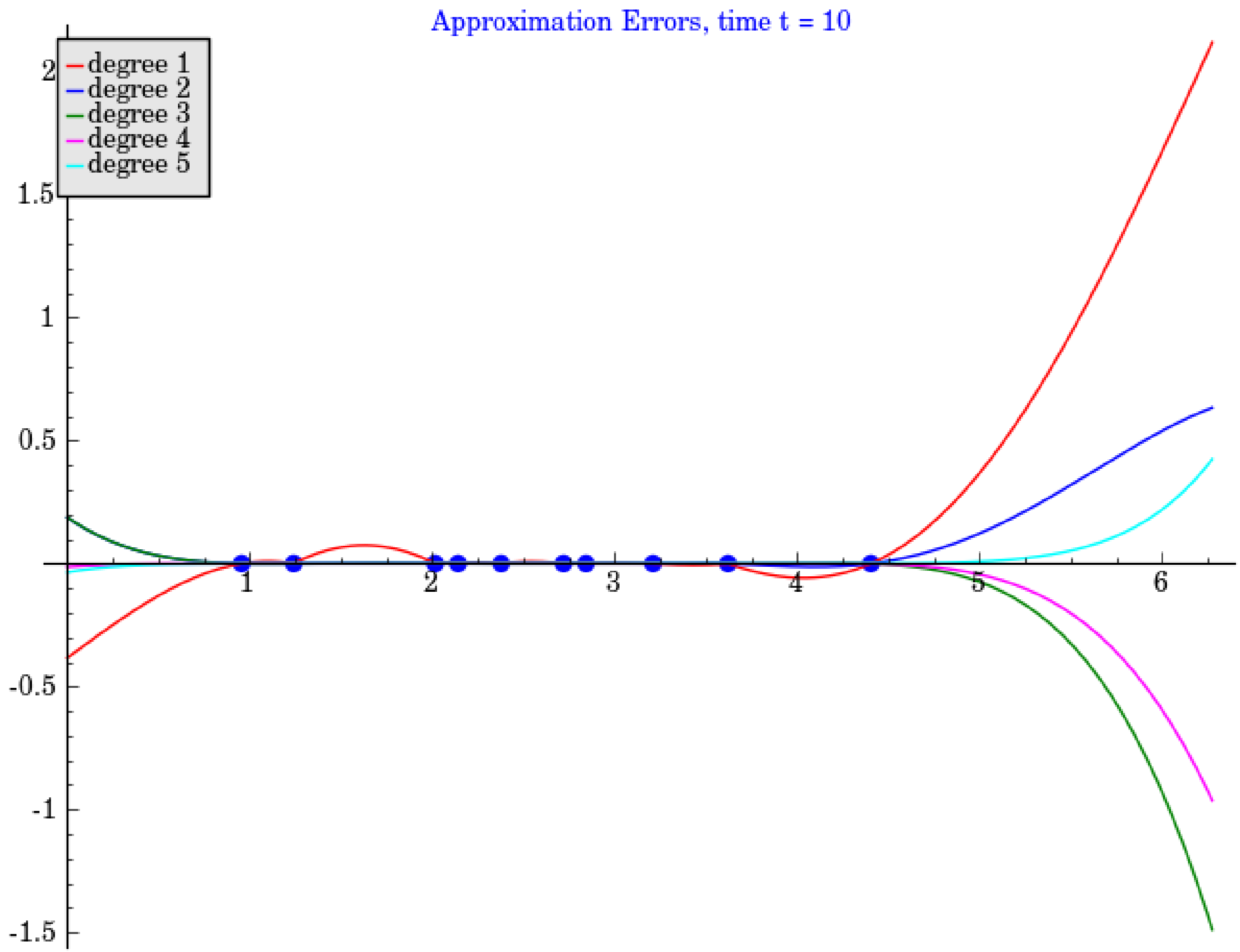
Approximations, time t = 9
degree 1
degree 2
degree 3
degree 4
degree 5
1.5
1
0.5
-0.5
-1
-1.5
-2
1
2
3
4
5
6



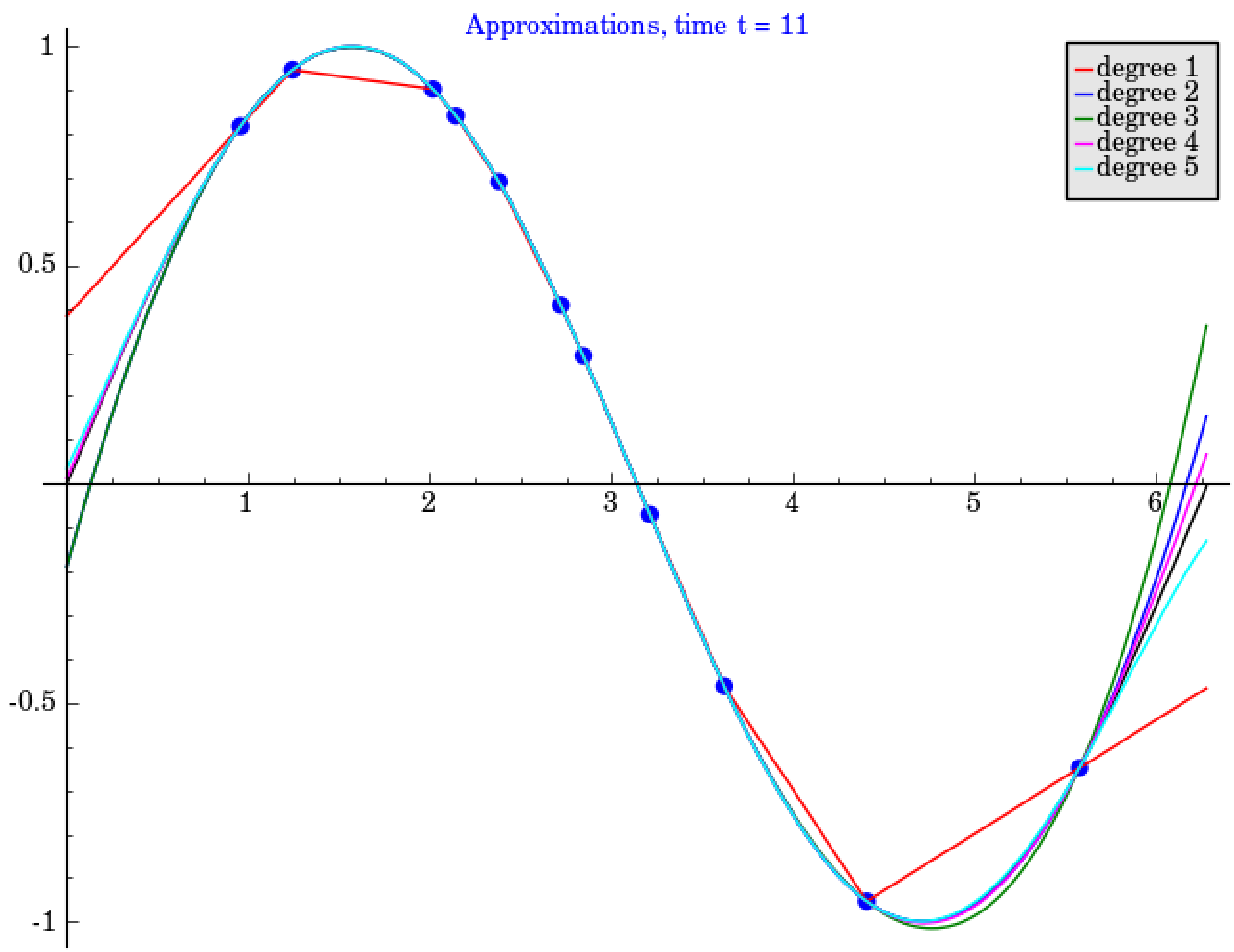
Approximation Errors, time t = 9
degree 1
degree 2
degree 3
degree 4
degree 5
2
1.5
1
0.5
-0.5
-1
-1.5
1
2
3
4
5
6



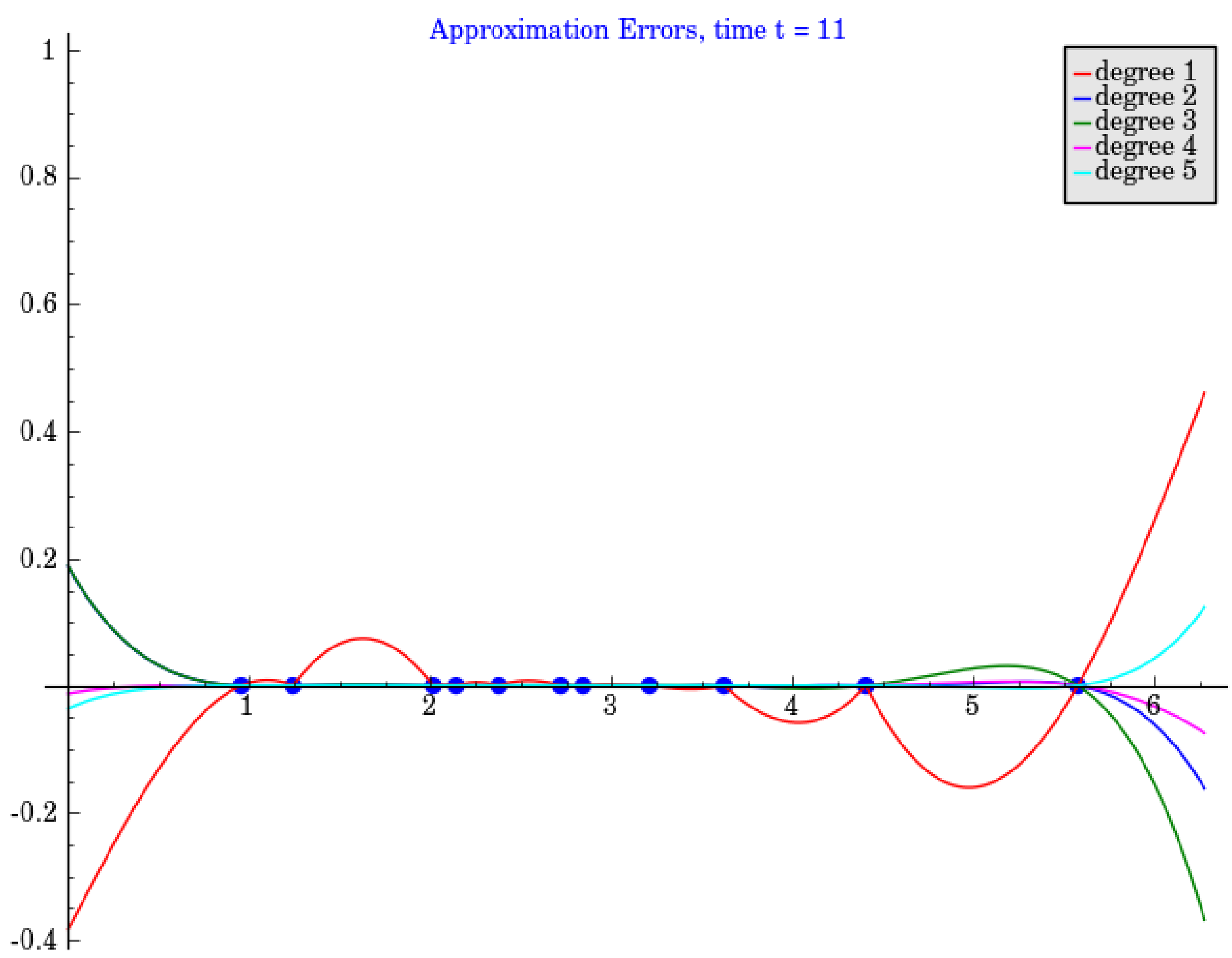
Approximations, time t = 10
degree 1
degree 2
degree 3
degree 4
degree 5
1.5
1
0.5
-0.5
-1
-1.5
-2
1
2
3
4
5
6

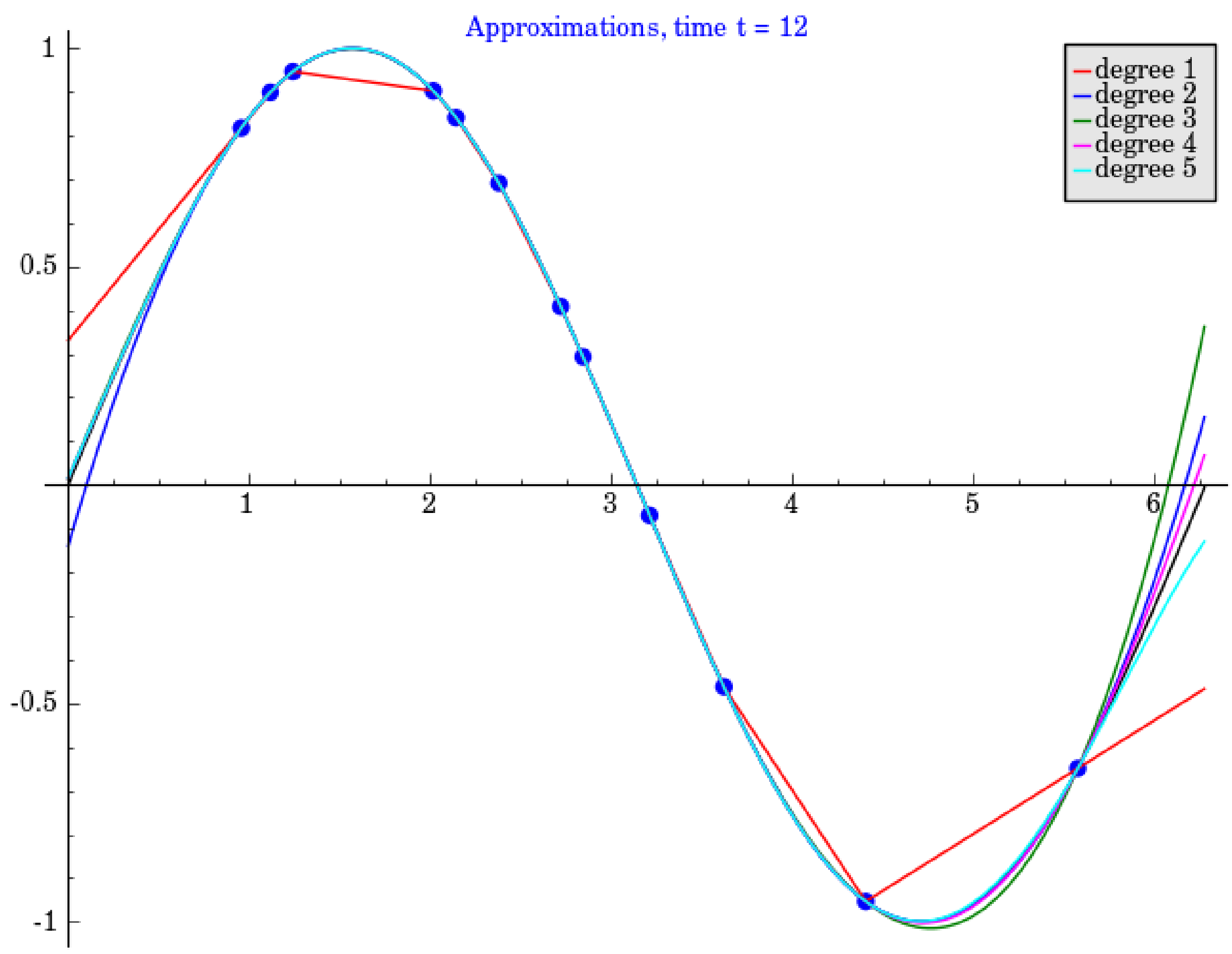


Approximation Errors, time t = 10
degree 1
degree 2
degree 3
degree 4
degree 5
2
1.5
1
0.5
-0.5
-1
-1.5
1
2
3
4
5
6

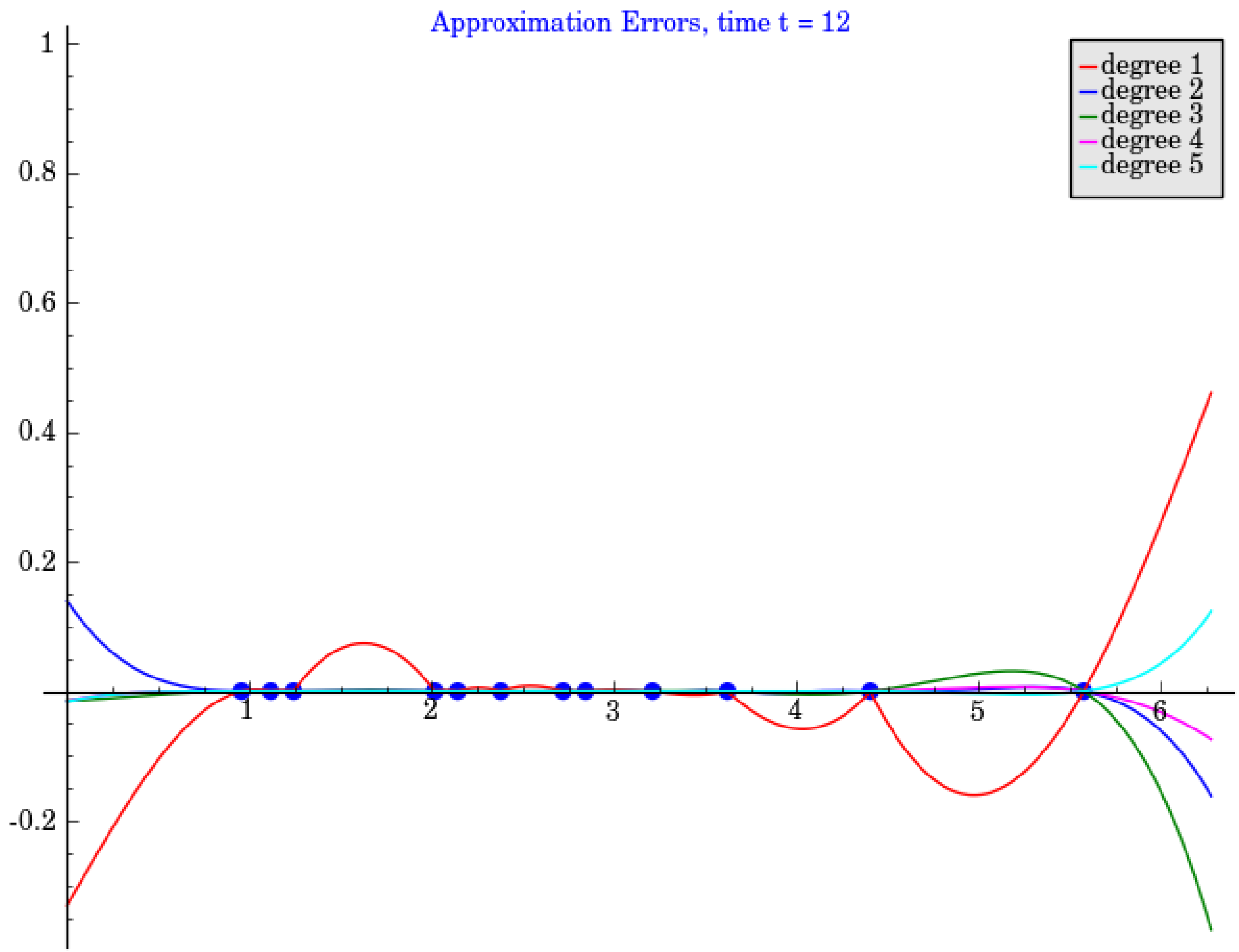


Approximations, time t = 11
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6

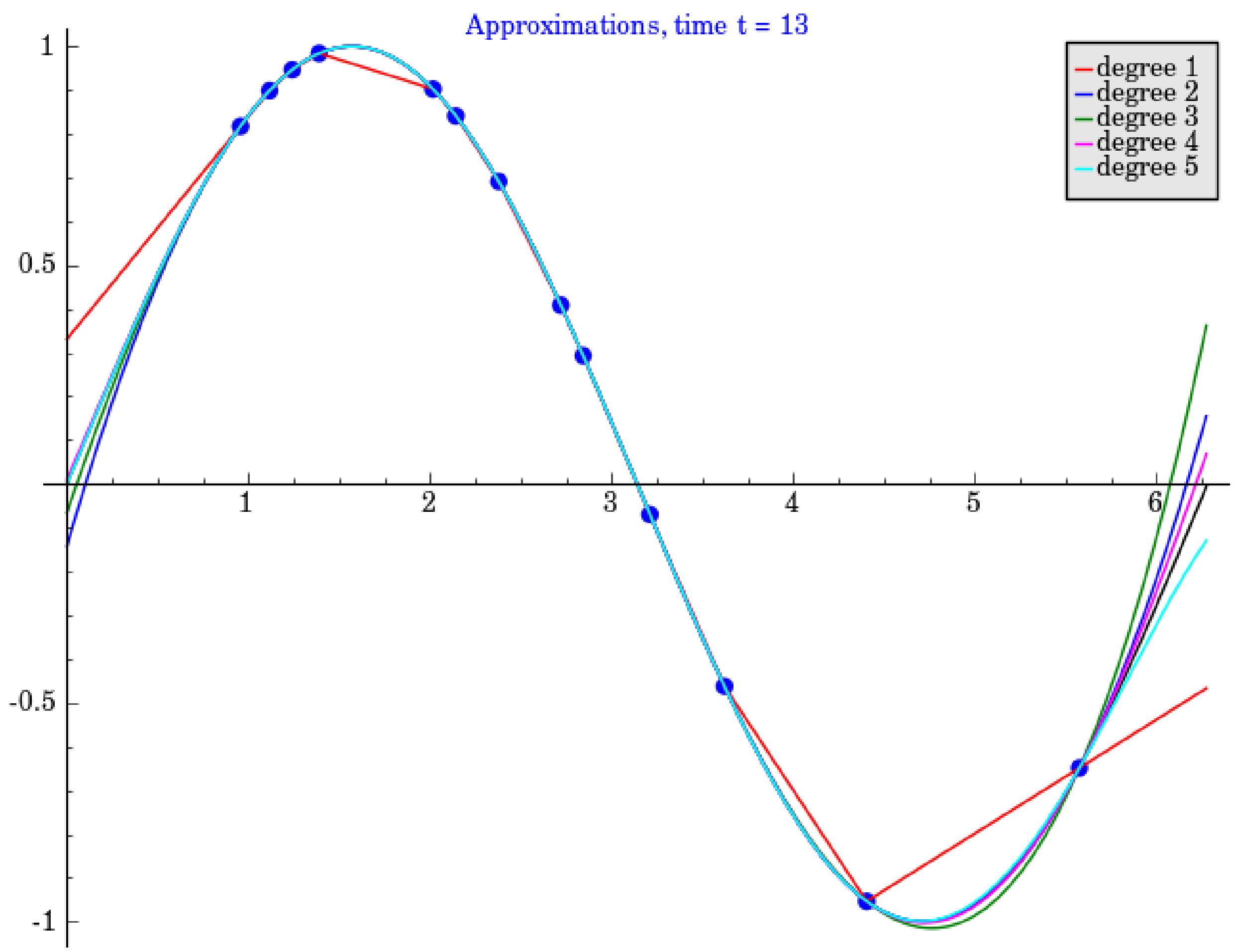

Approximation Errors, time t = 11
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
-0.4
1
2
3
4
5
6



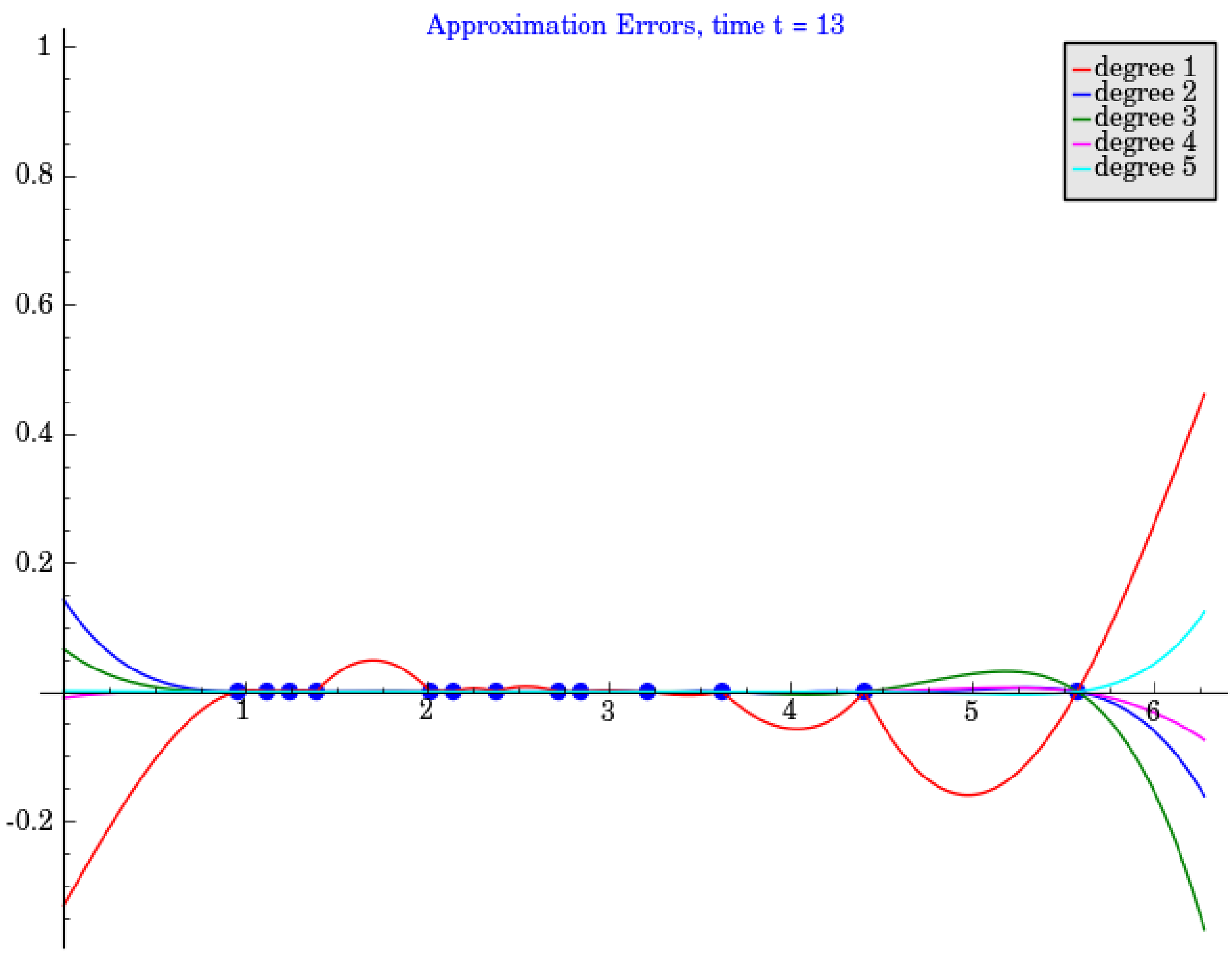
Approximations, time t = 12
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6

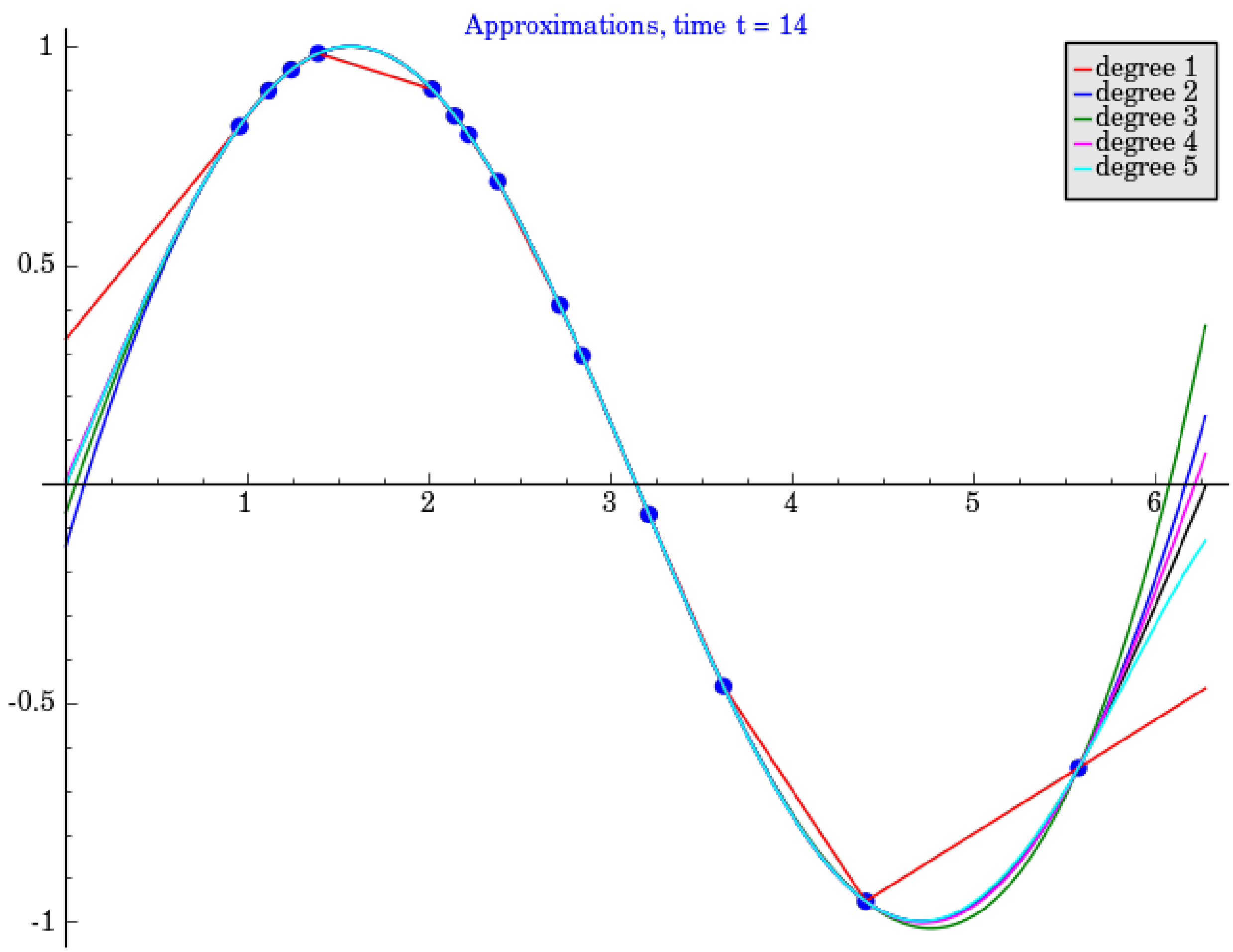


Approximation Errors, time t = 12
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
1
2
3
4
5
6

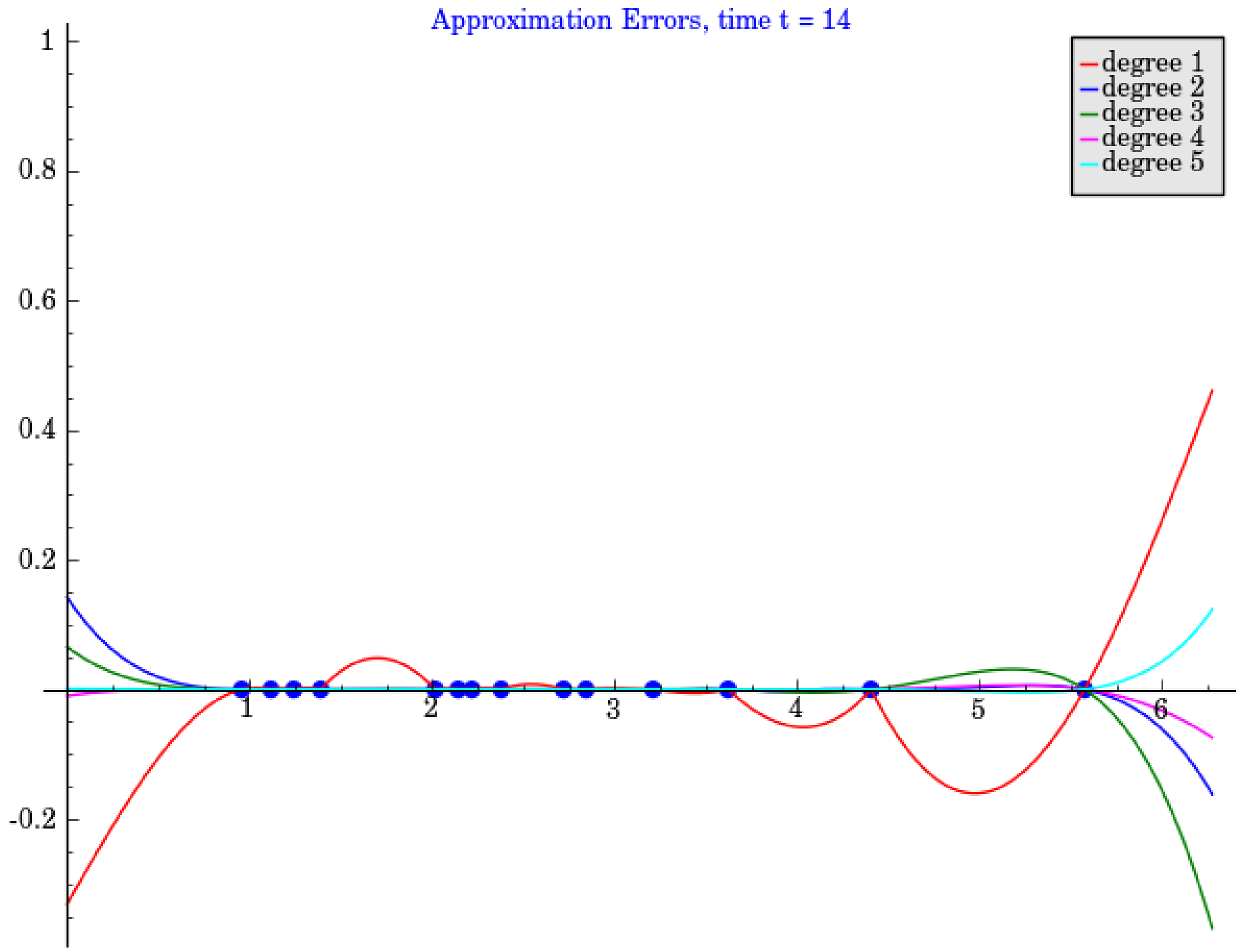


Approximations, time t = 13
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6

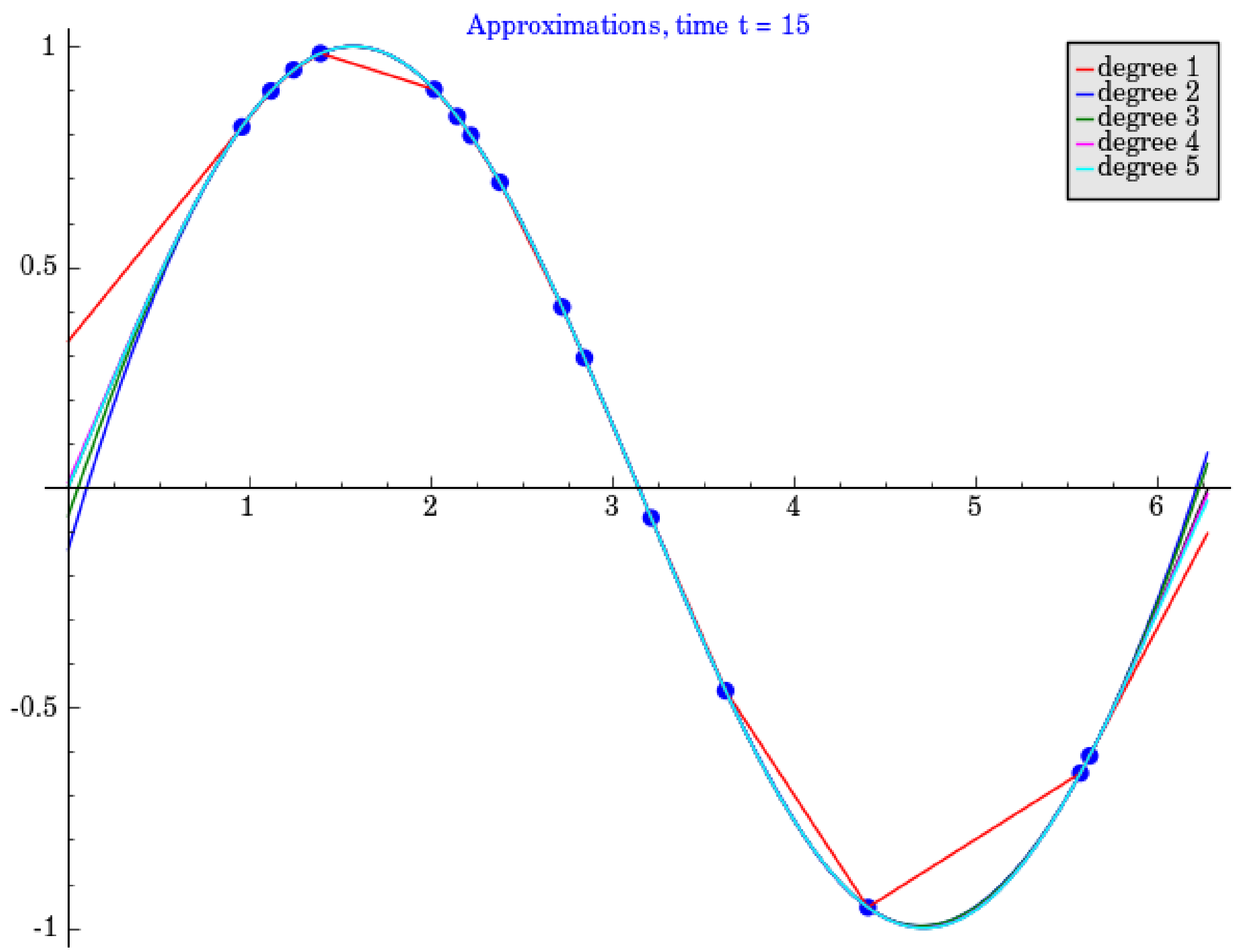

Approximation Errors, time t = 13
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
1
2
3
4
5
6

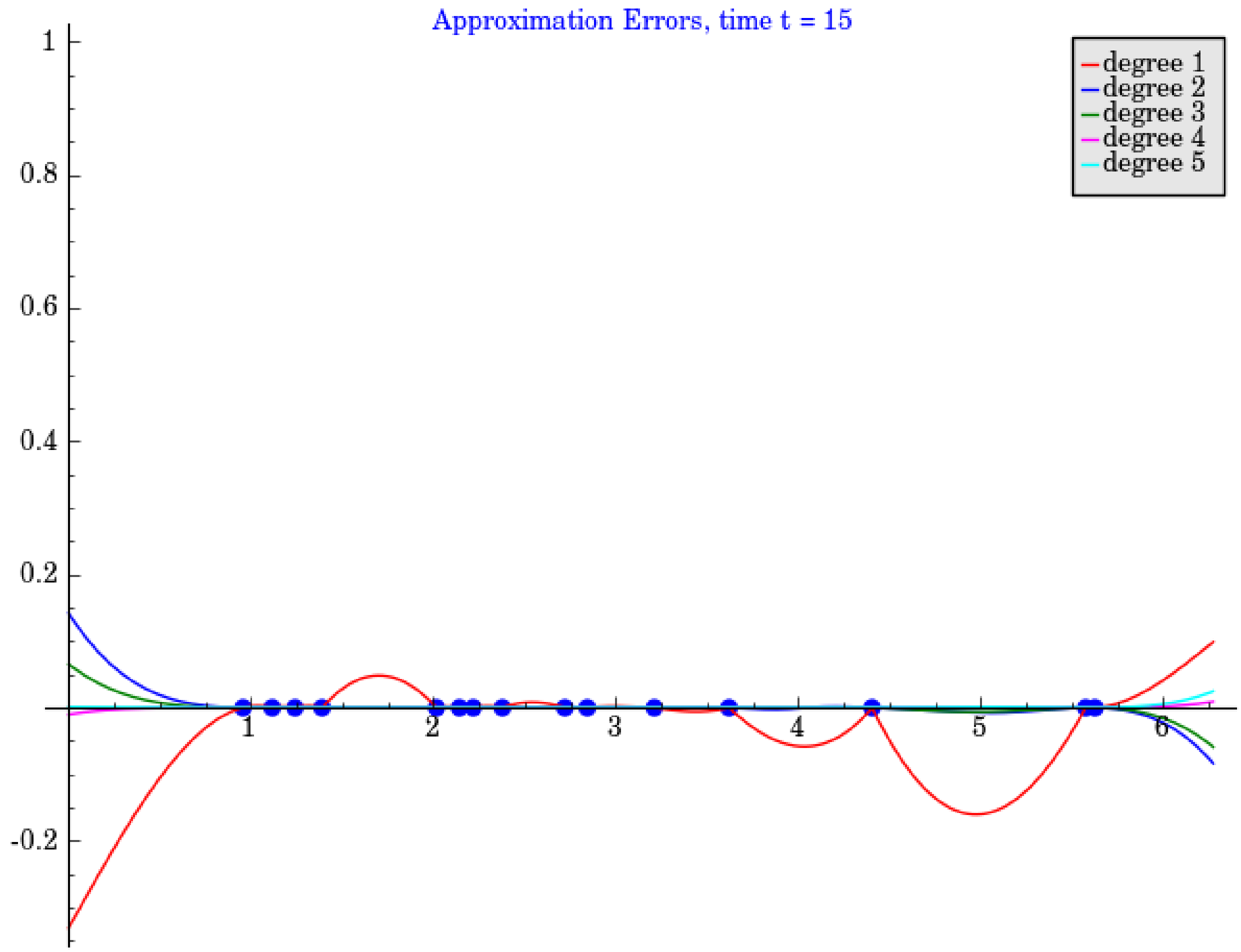


Approximations, time t = 14
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6

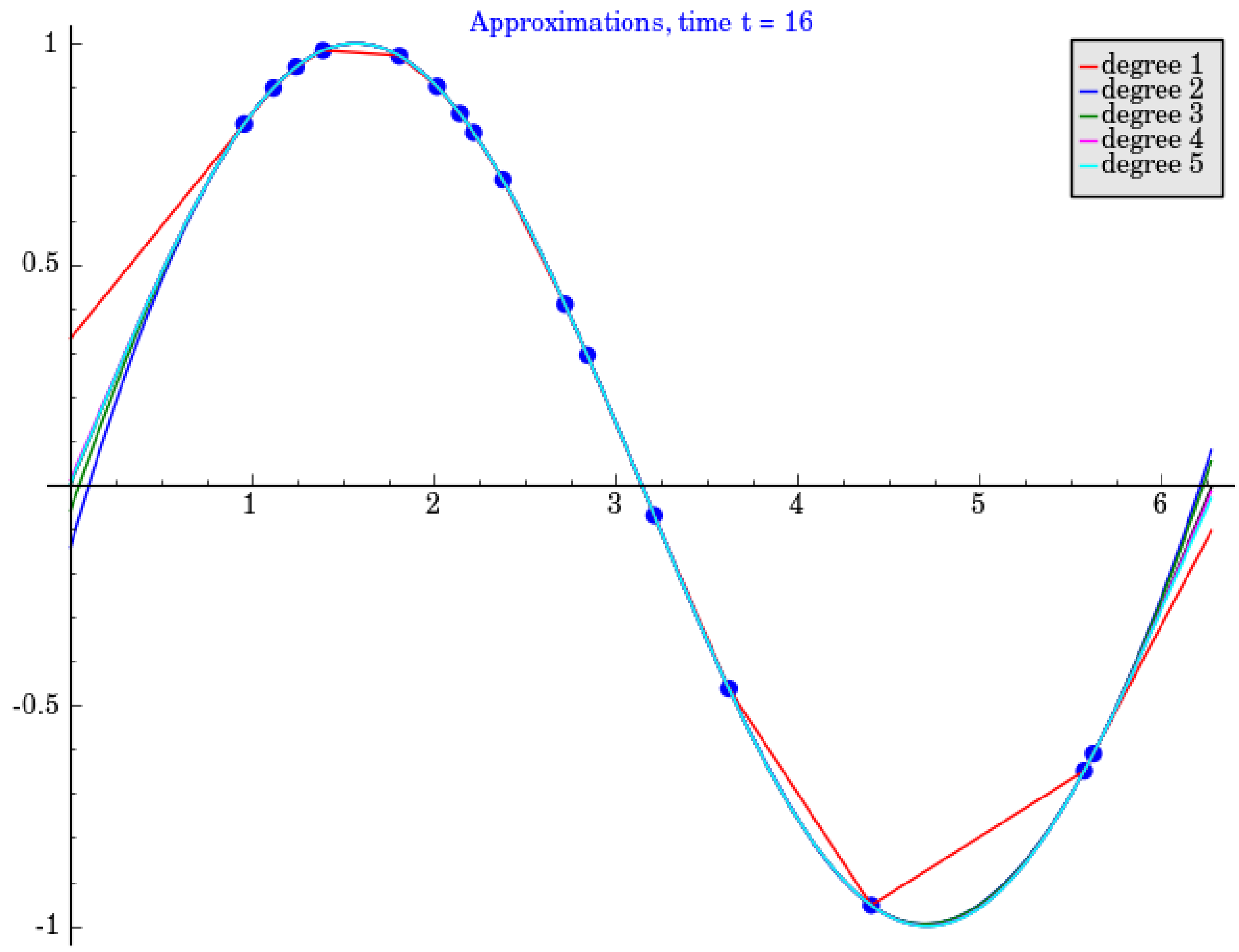


Approximation Errors, time t = 14
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
1
2
3
4
5
6

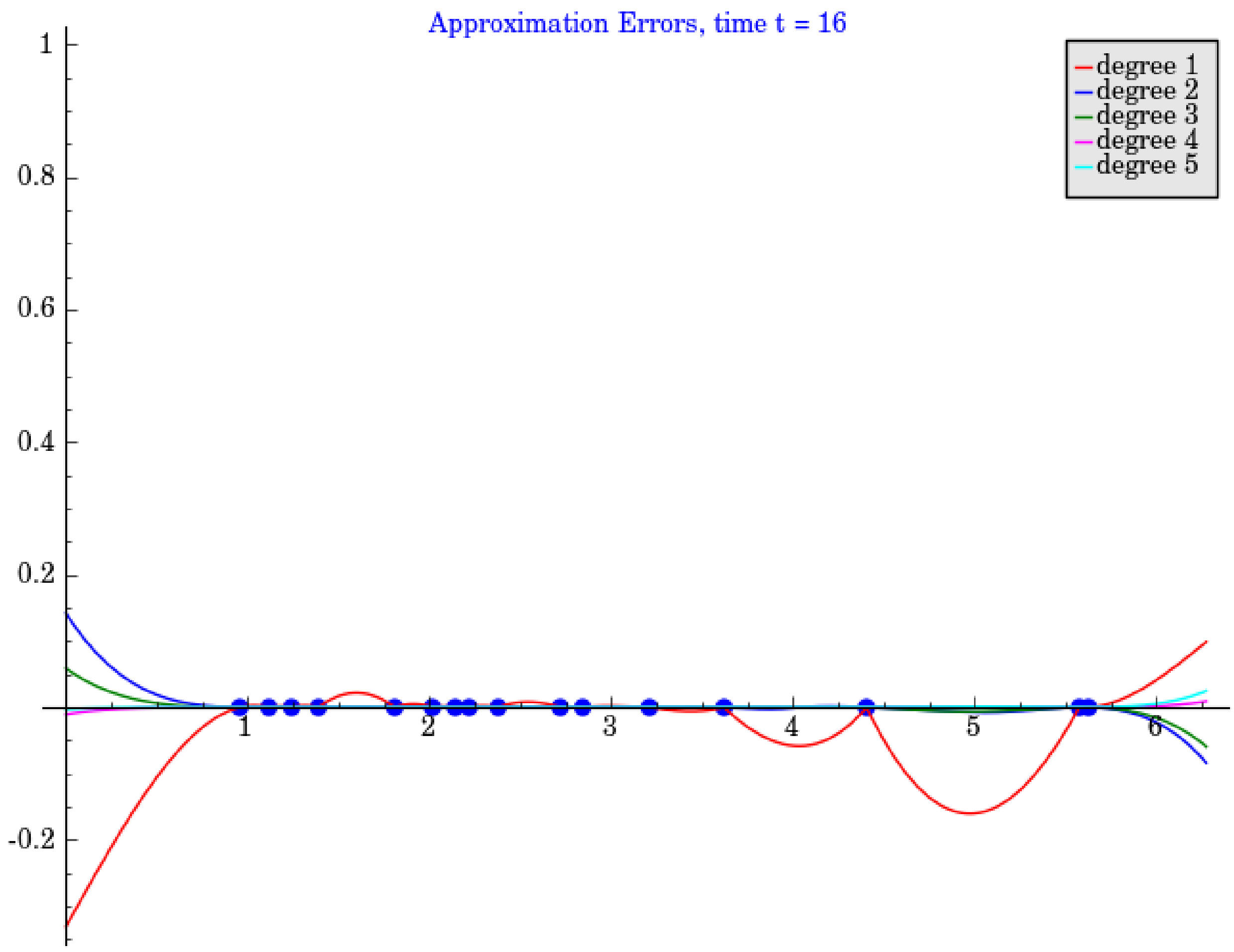

Approximations, time t = 15
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



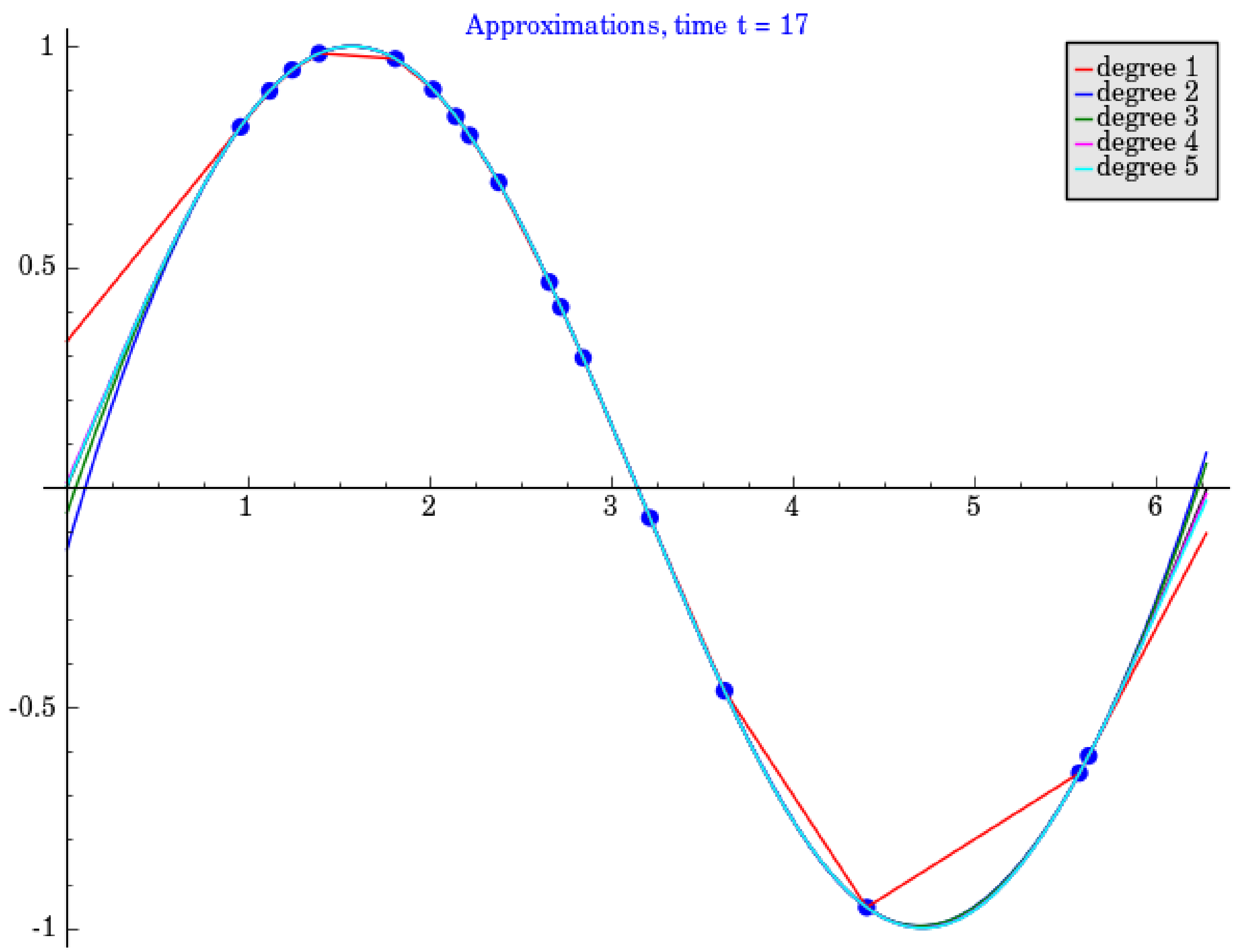
Approximation Errors, time t = 15
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
1
2
3
4
5
6



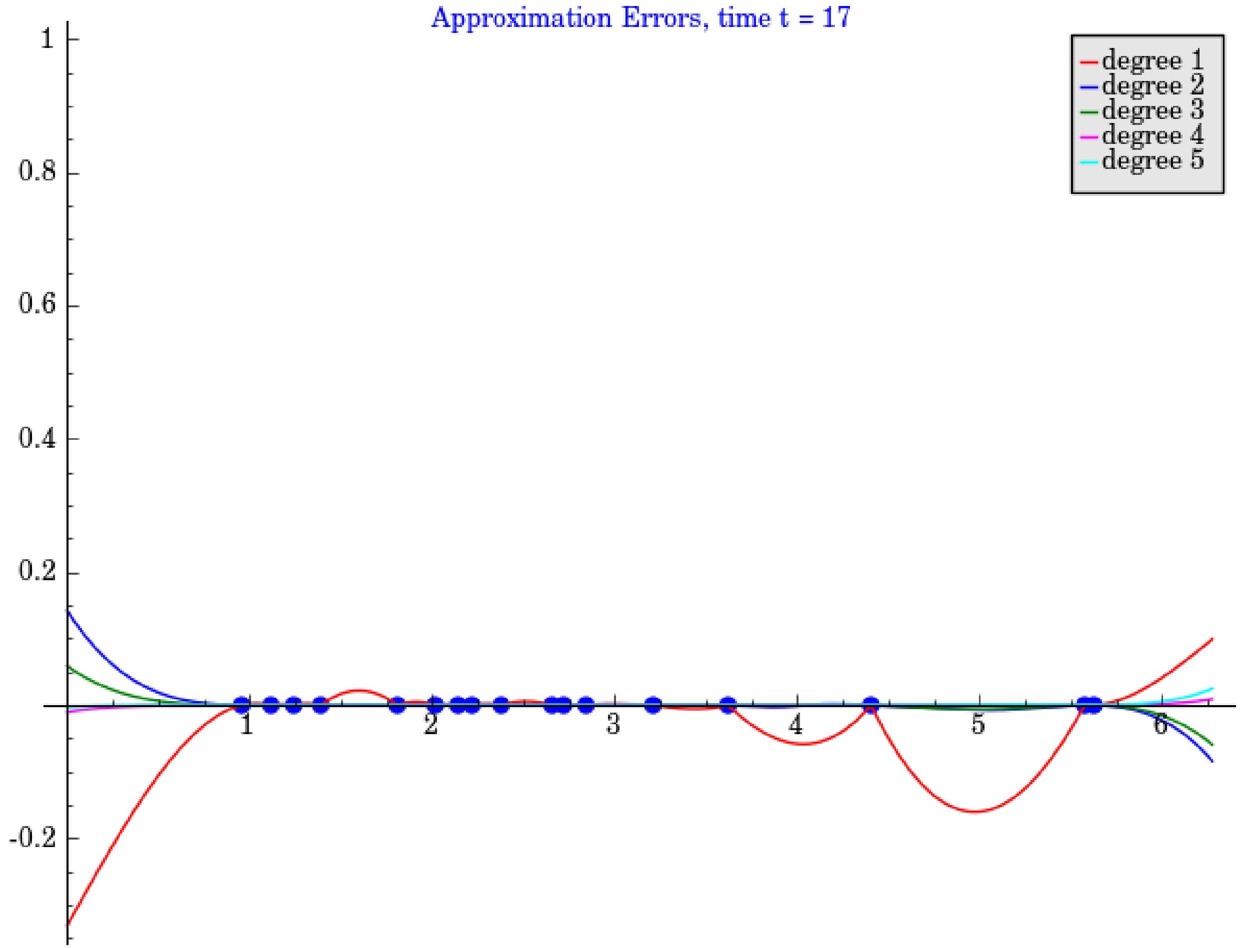
Approximations, time t = 16
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



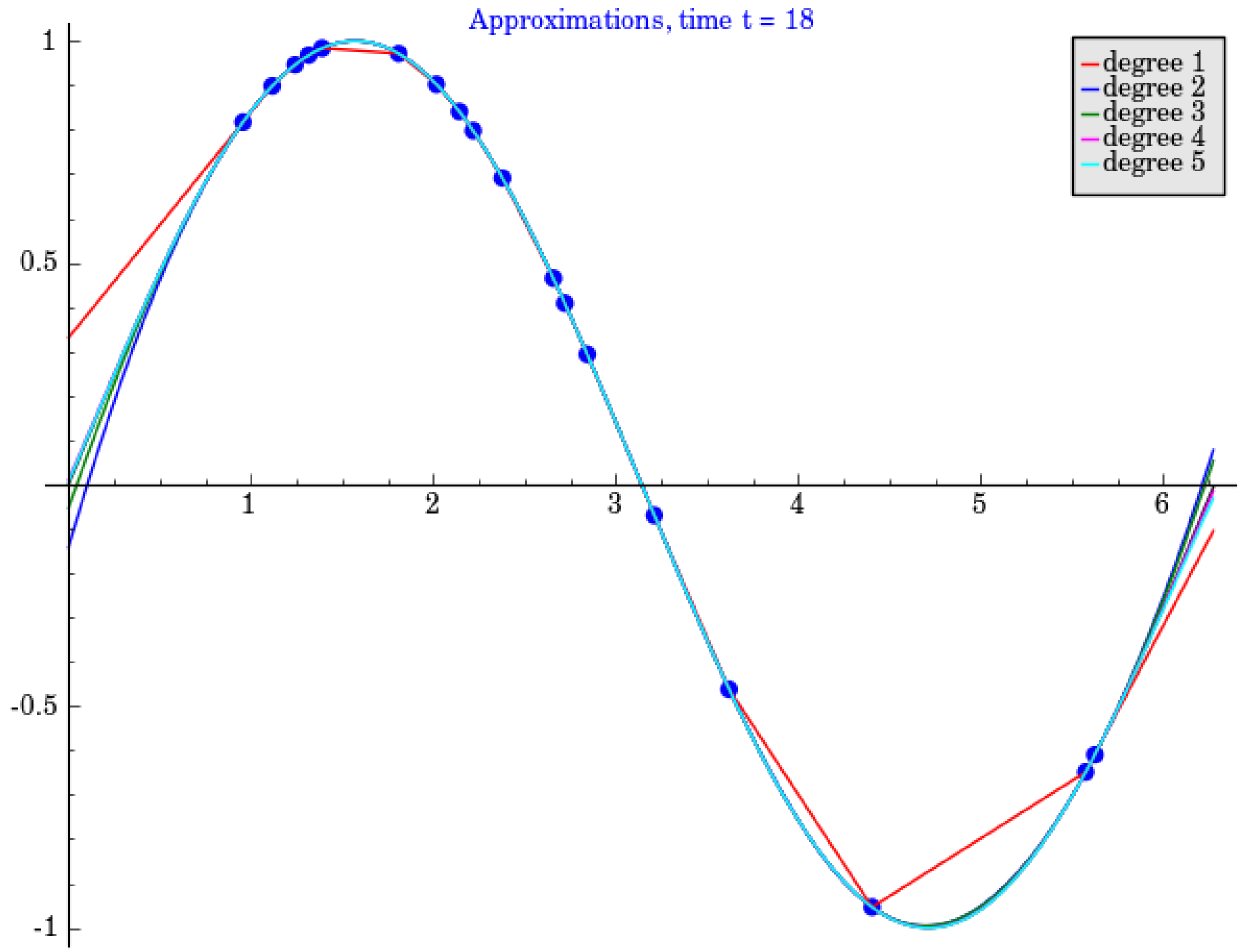
Approximation Errors, time t = 16
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
1
2
3
4
5
6

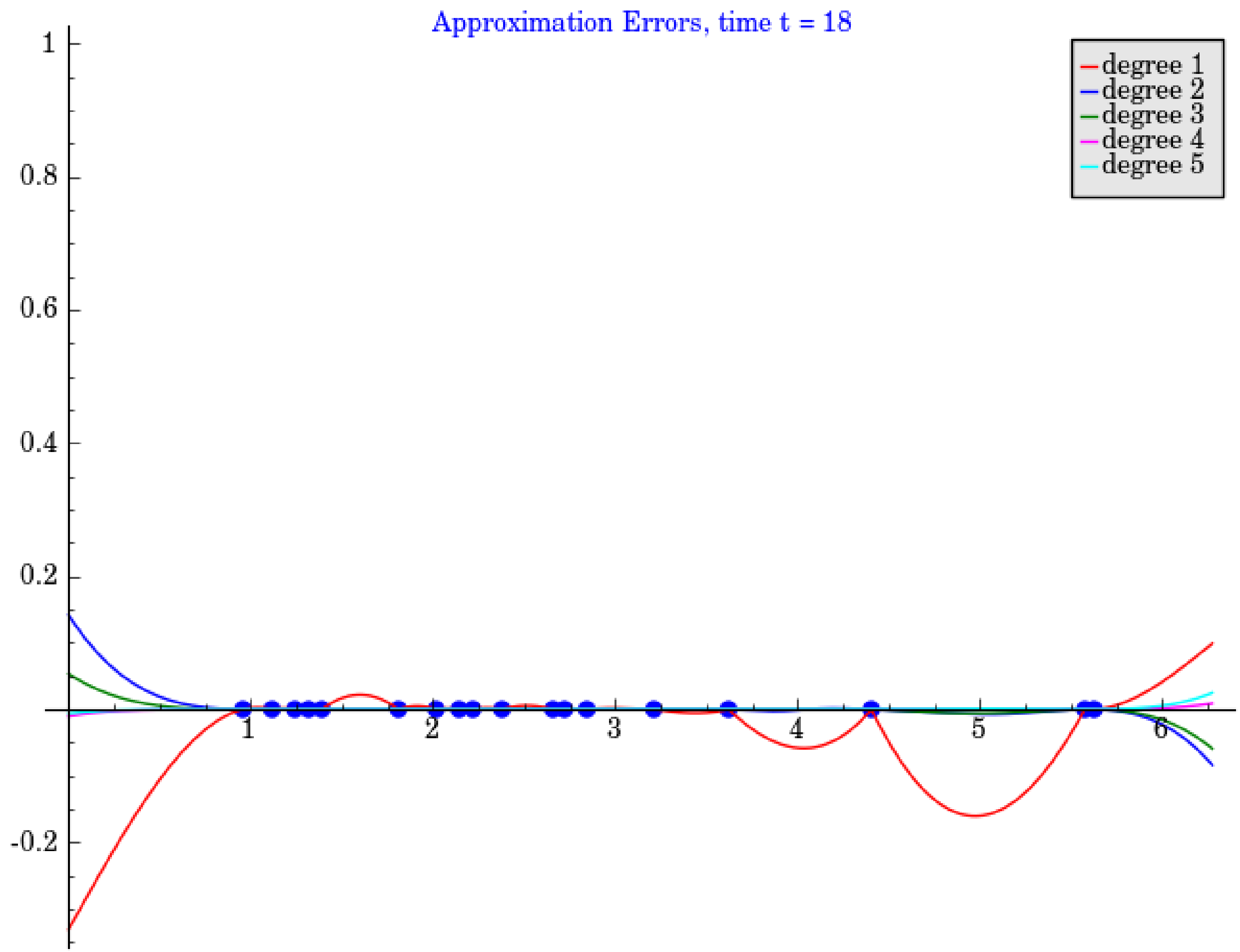


Approximations, time t = 17
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6

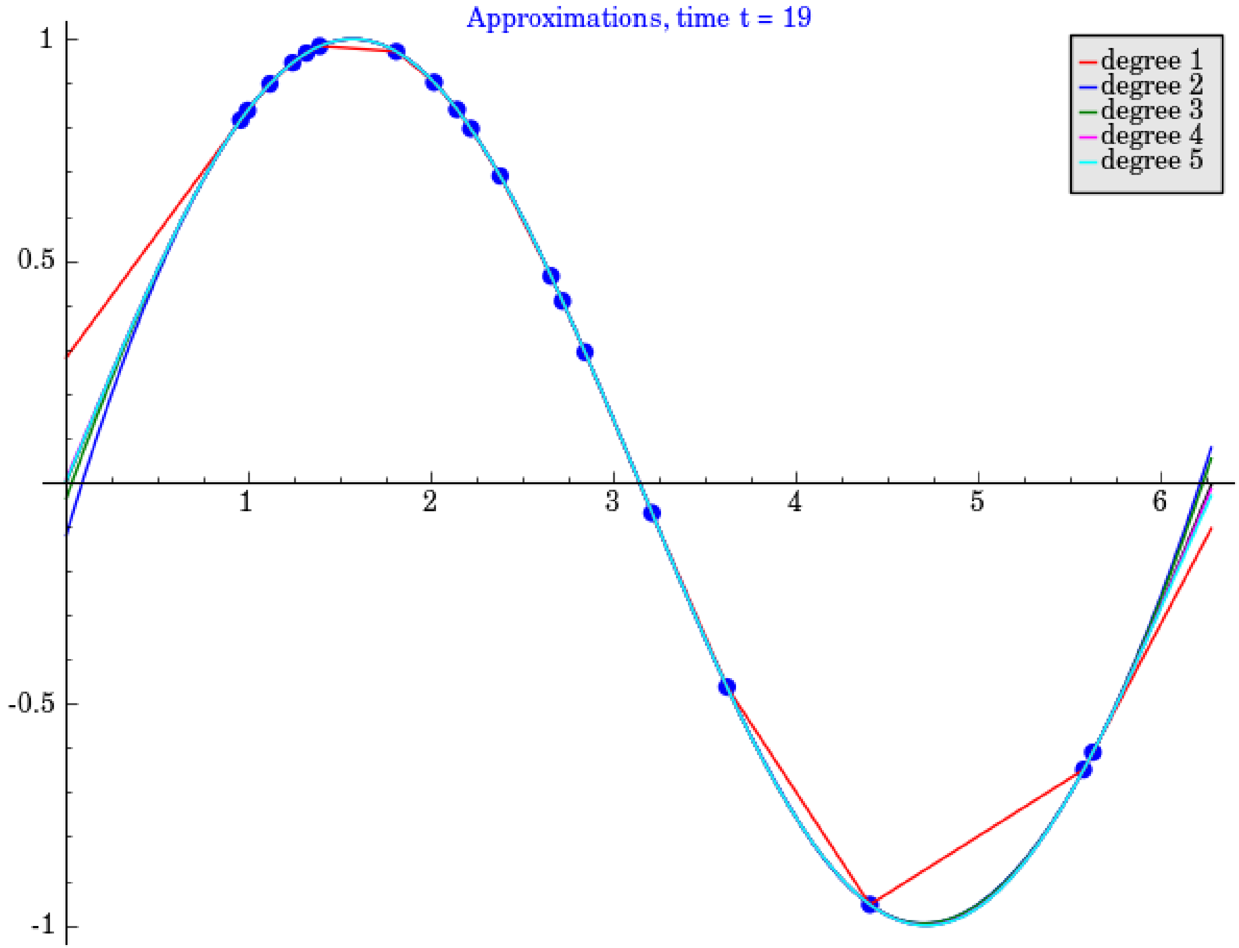


Approximation Errors, time t = 17
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
1
2
3
4
5
6

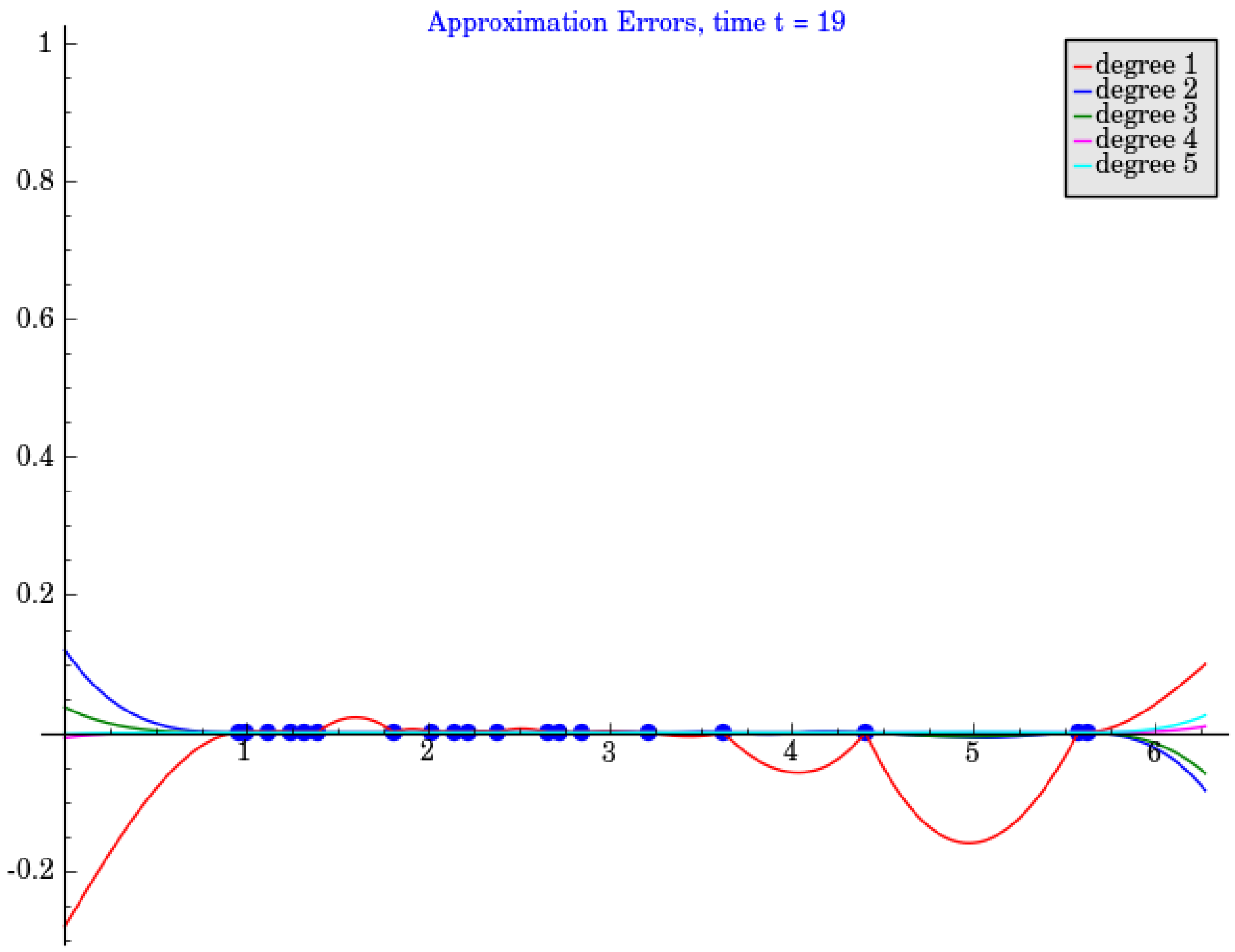

Approximations, time t = 18
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



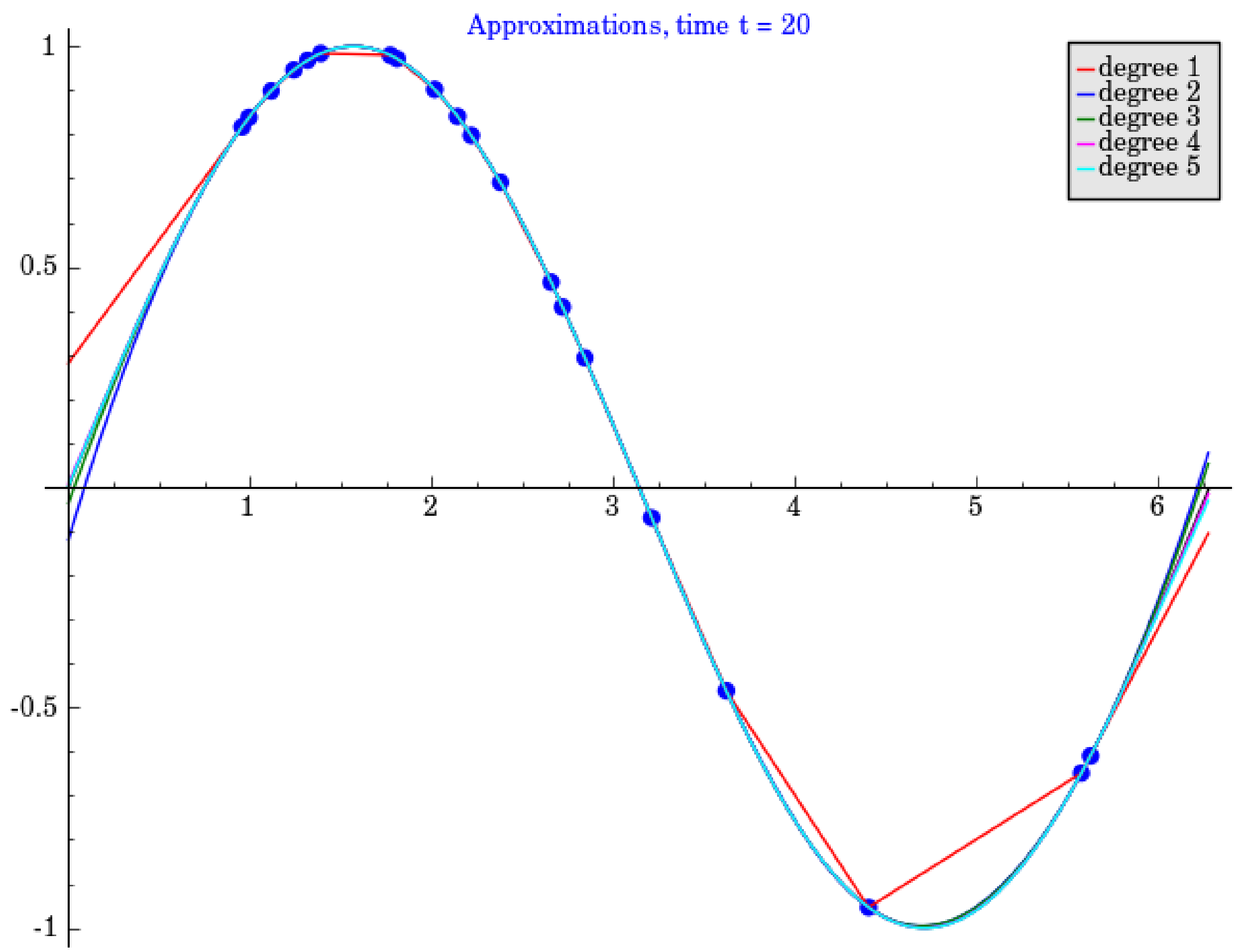
Approximation Errors, time t = 18
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
1
2
3
4
5
6



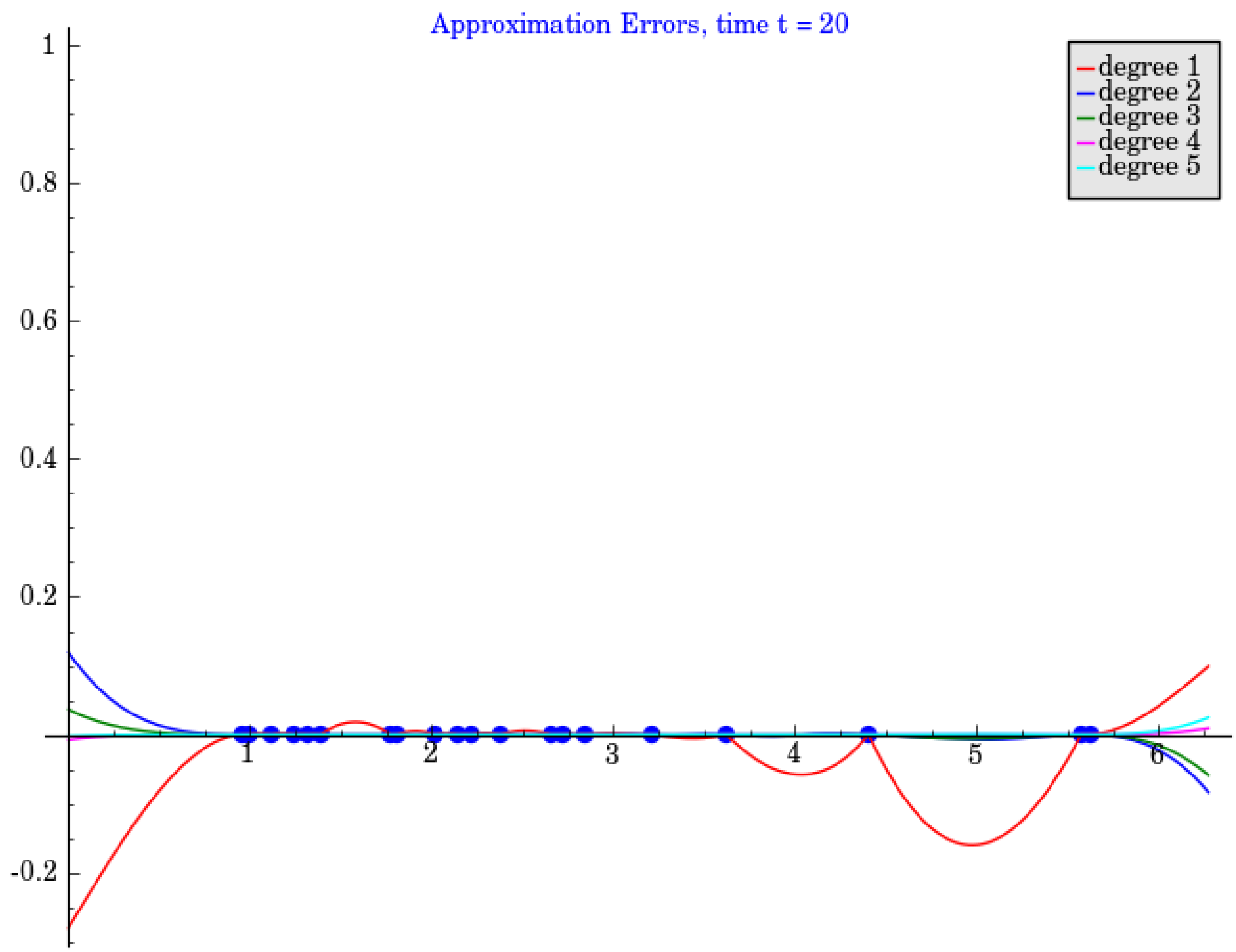
Approximations, time t = 19
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



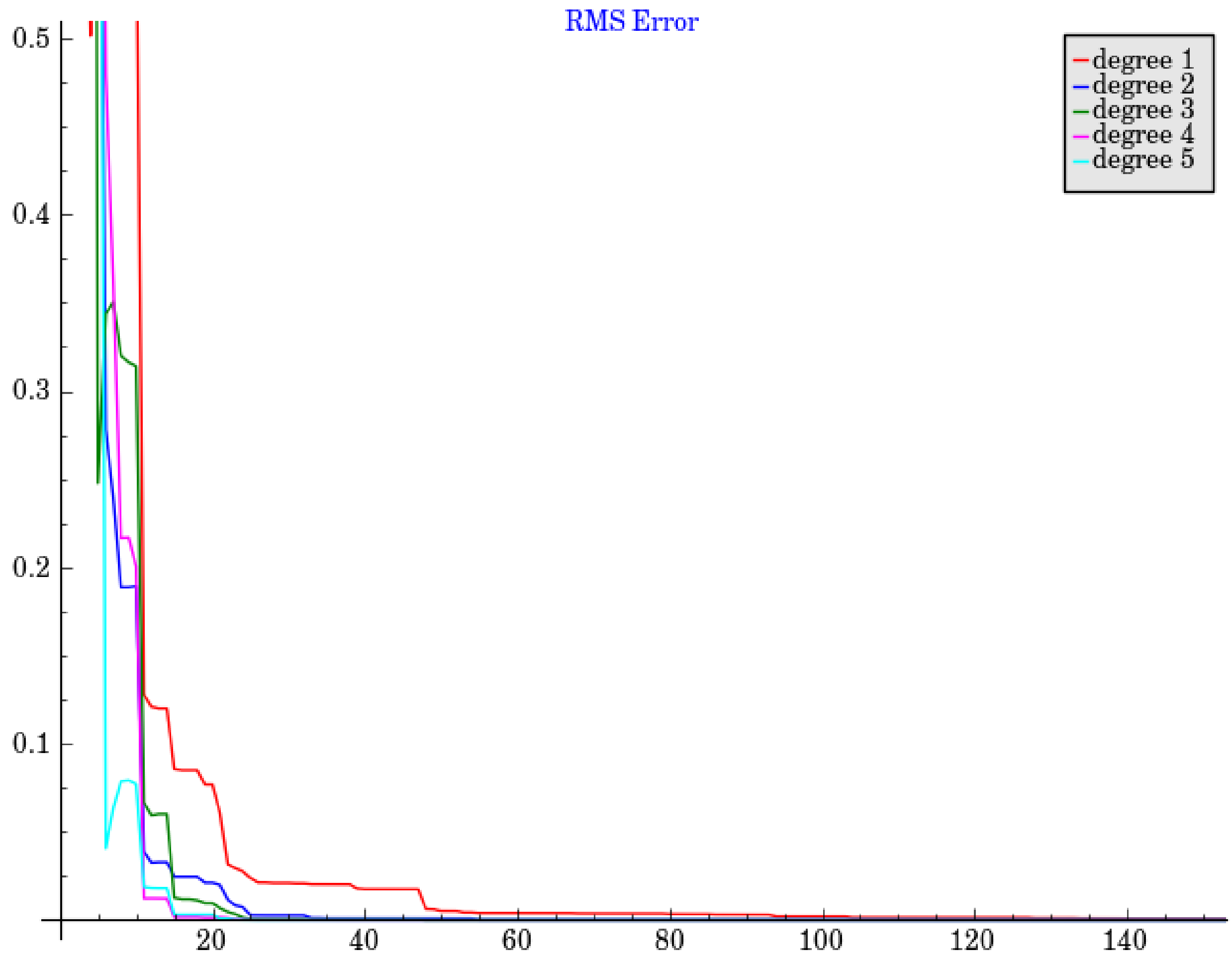
Approximation Errors, time t = 19
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
1
2
3
4
5
6

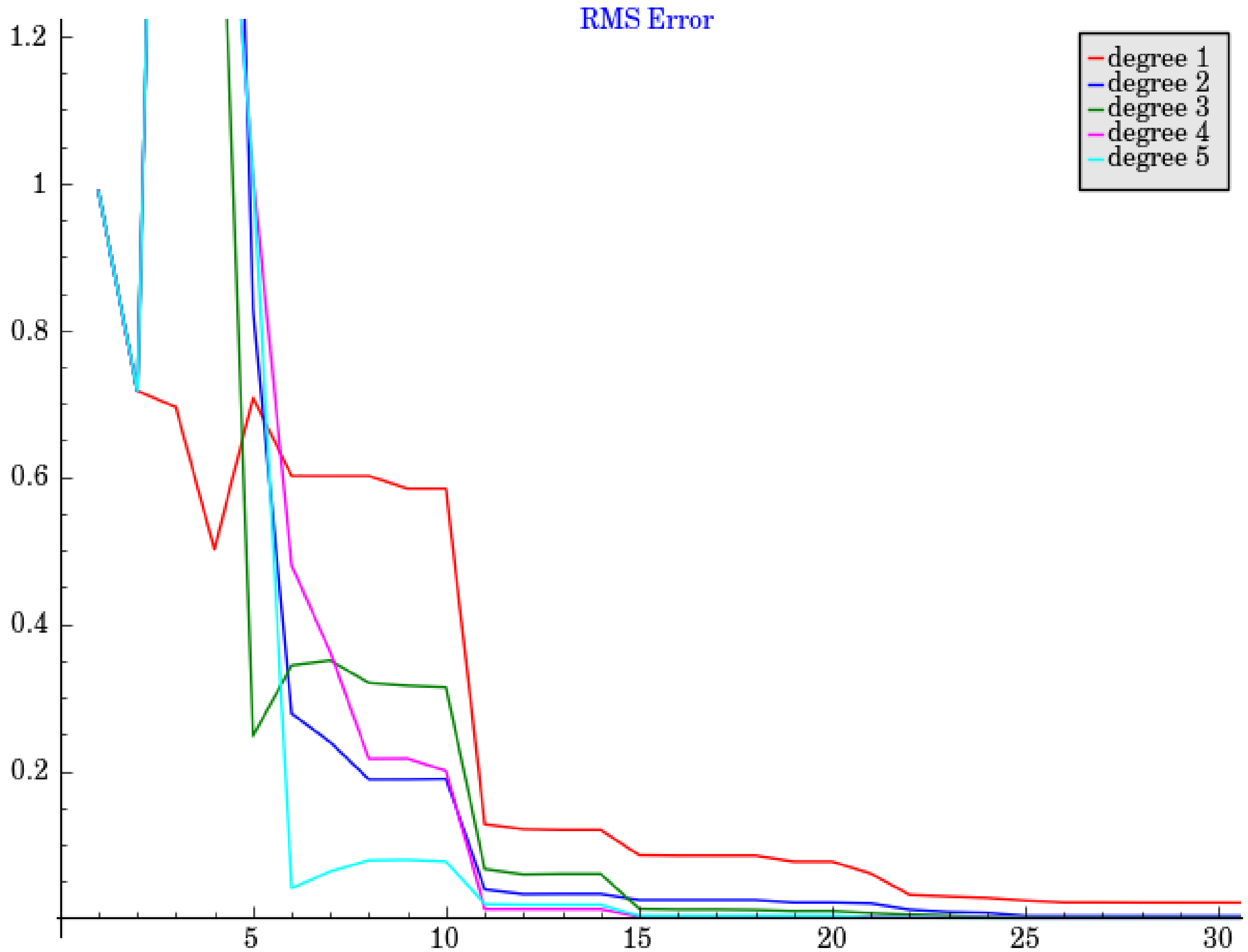


Approximations, time t = 20
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



Approximation Errors, time t = 20
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
1
2
3
4
5
6

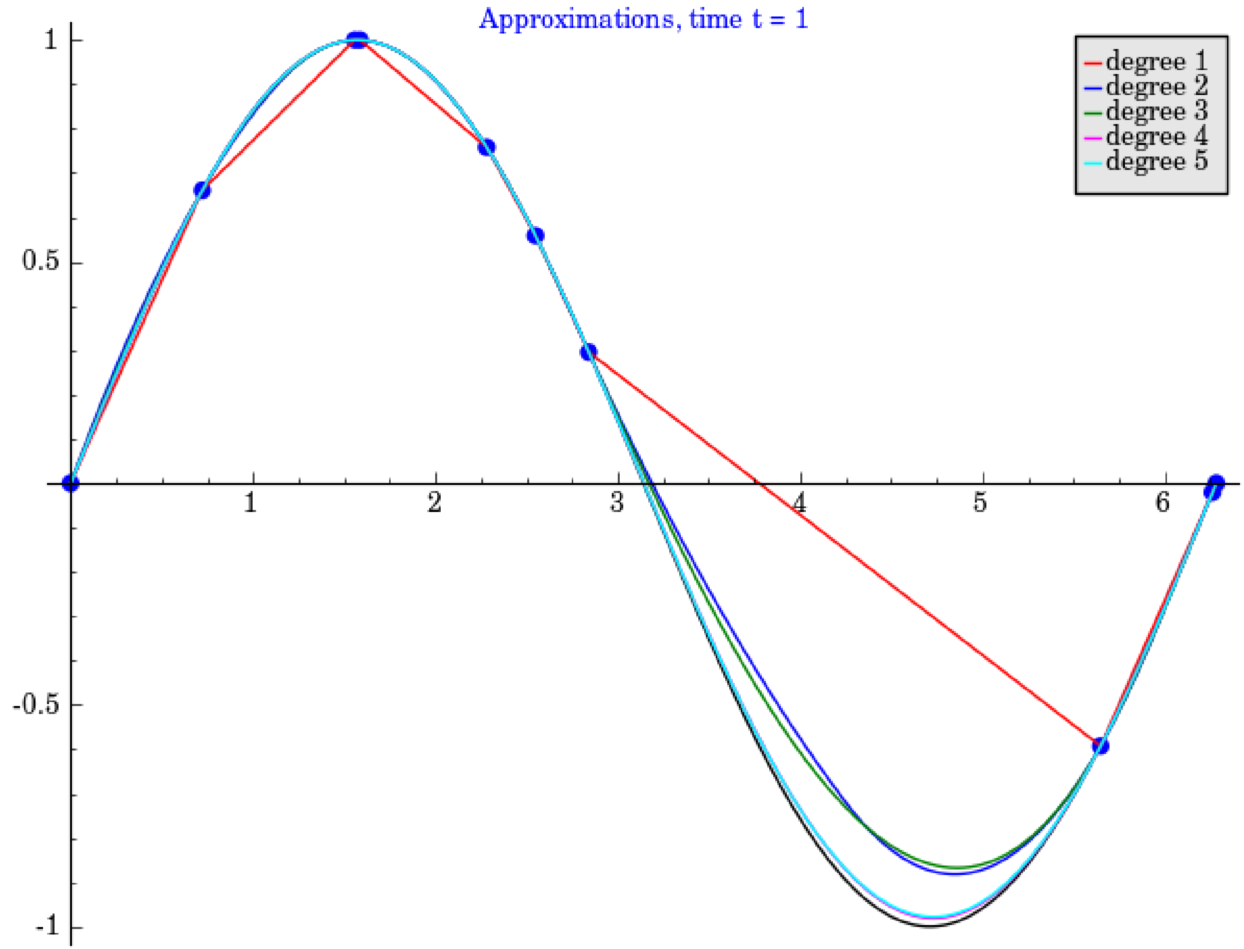

RMS Error
degree 1
degree 2
degree 3
degree 4
degree 5
0.5
0.4
0.3
0.2
0.1
20
40
60
80
100
120
140



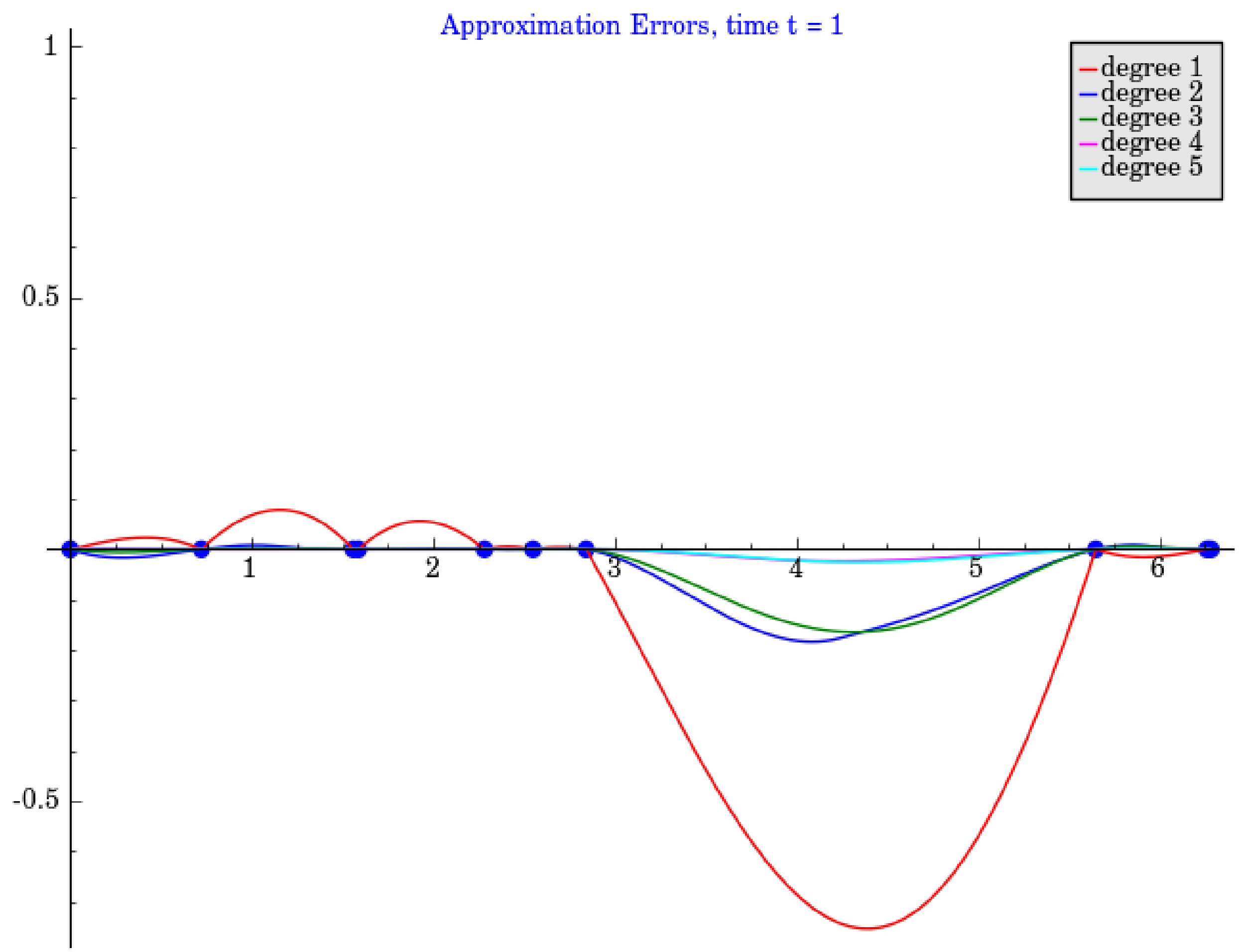
RMS Error
degree 1
degree 2
degree 3
degree 4
degree 5
1.2
1
0.8
0.6
0.4
0.2
5
10
15
20
25
30

Applying Computational Mechanics

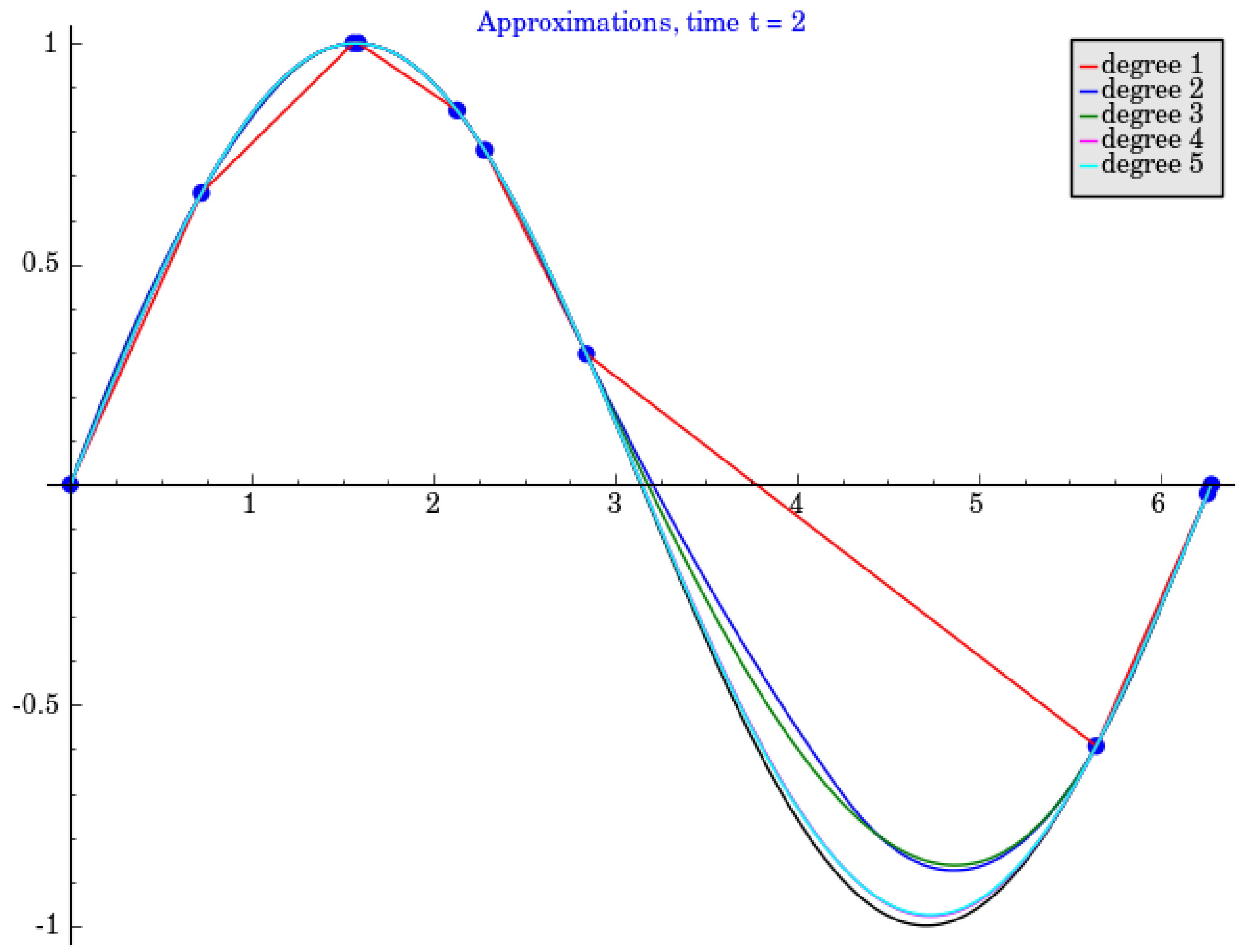
- Because of the quick decay of the RMS error, the system becomes uninteresting
- Also, not enough statistics to do (epsilon machine) inference
- Need a new dynamic system definition
  - Perturbation of sample points



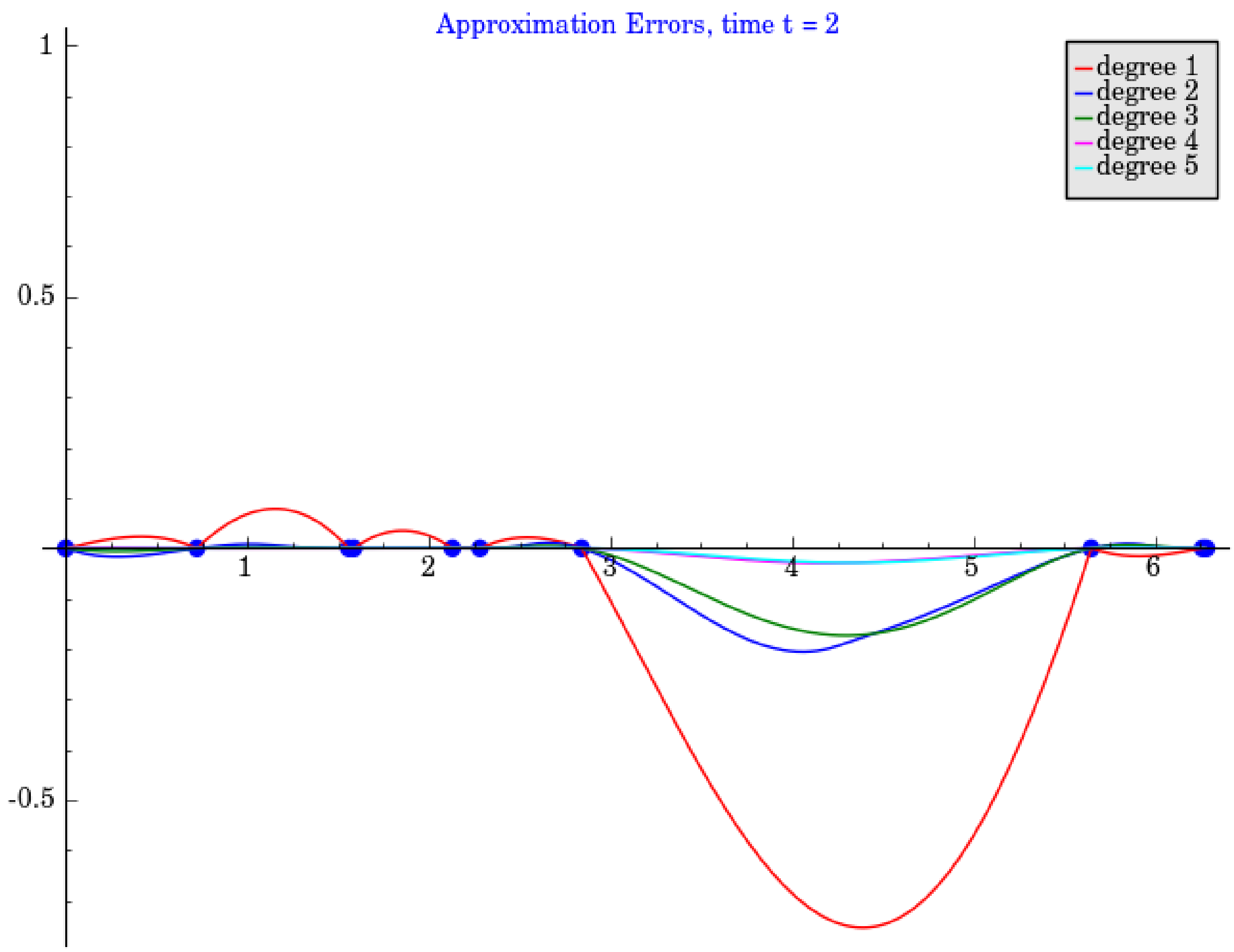
Approximations, time t = 1
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6

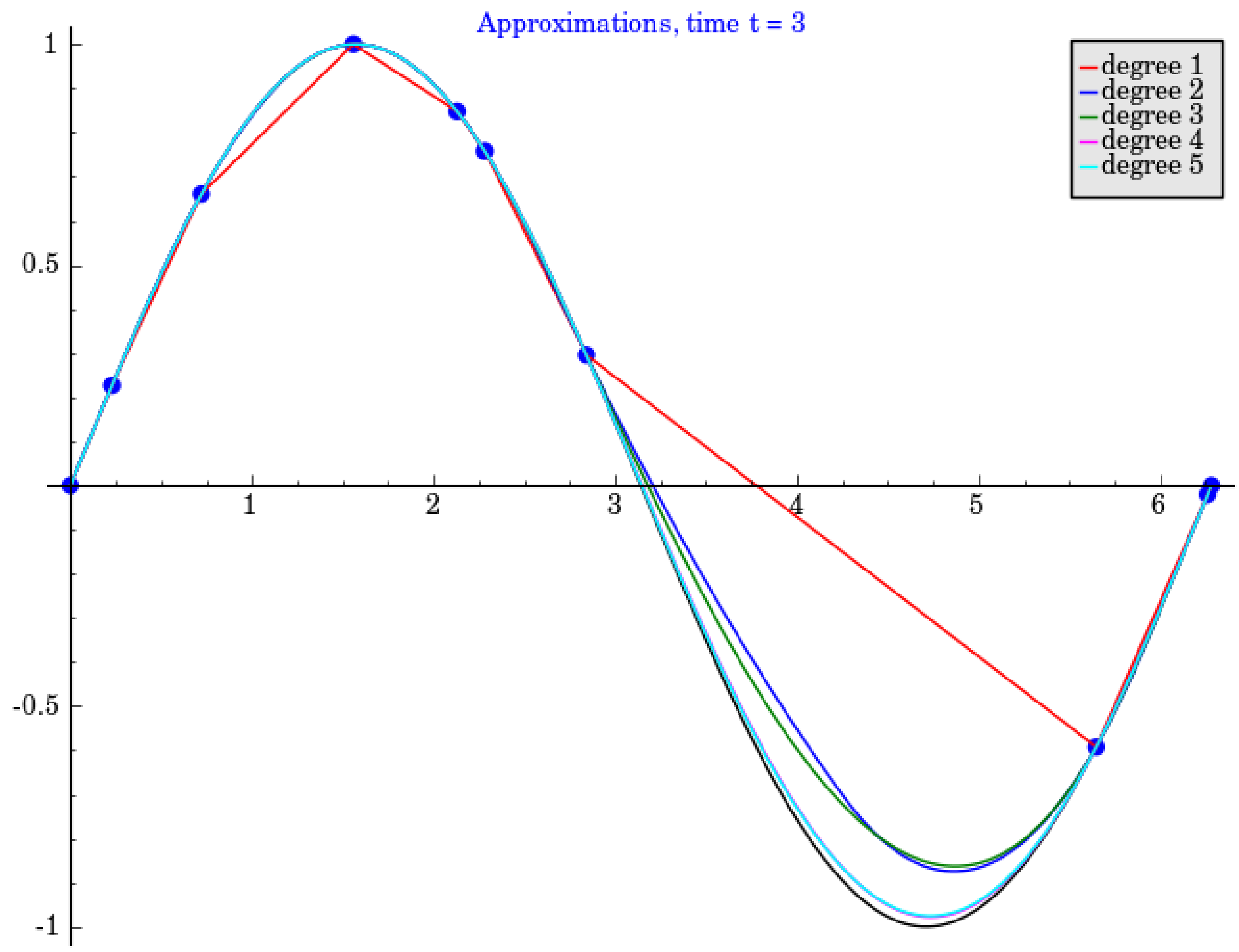


Approximation Errors, time t = 1
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
1
2
3
4
5
6

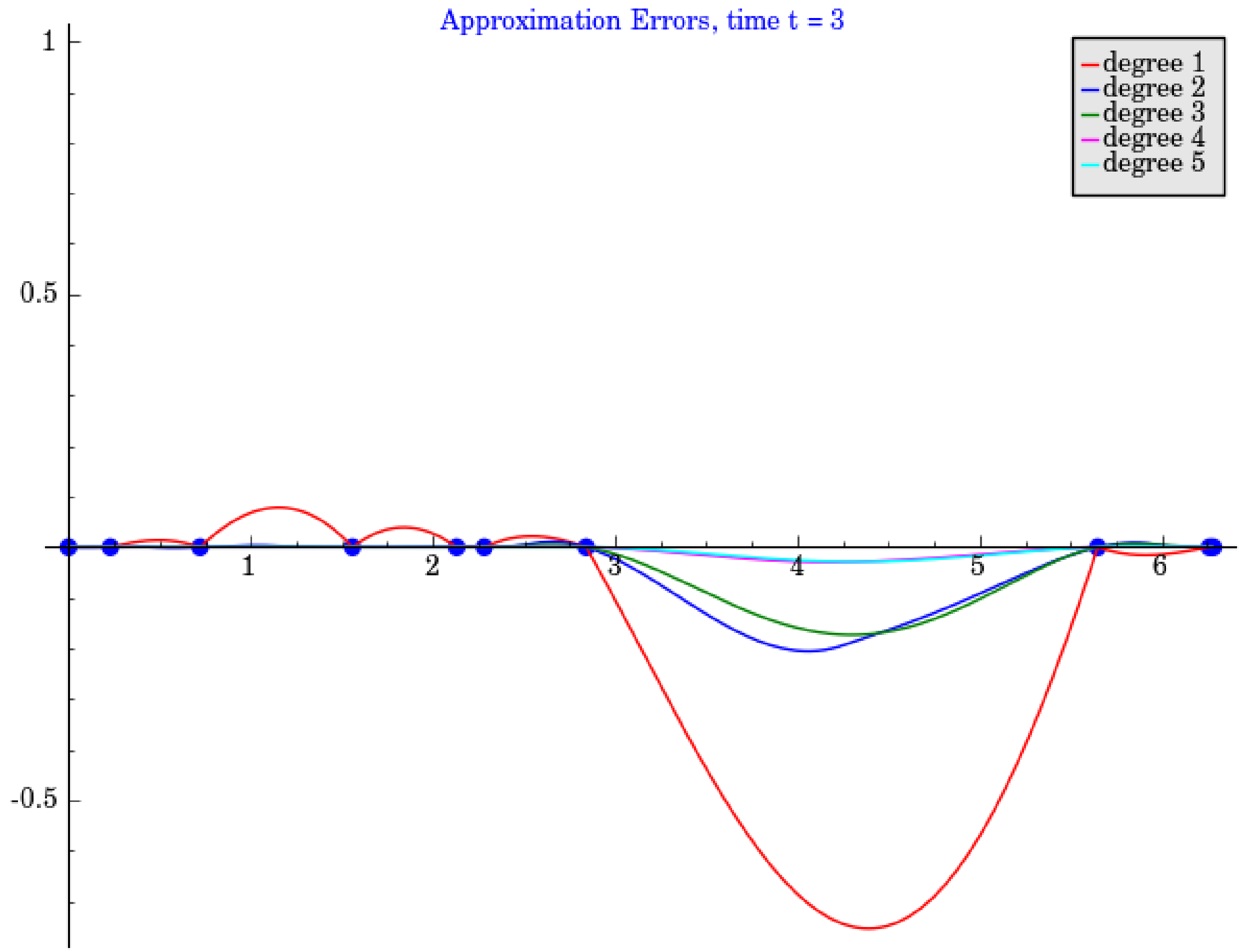


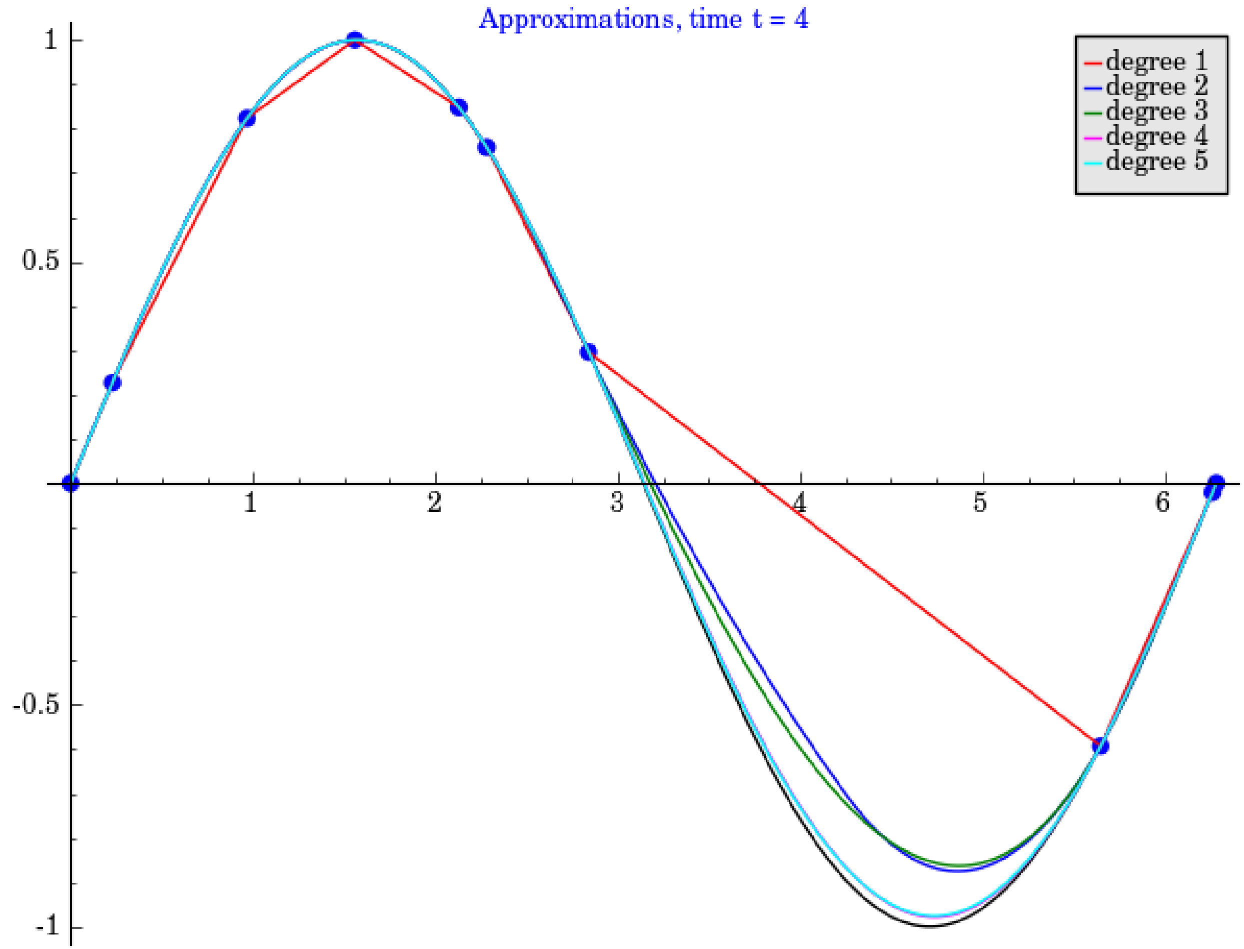
Approximations, time t = 2
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6


Approximation Errors, time t = 2
1
0.5
-0.5
1
2
3
4
5
6
degree 1
degree 2
degree 3
degree 4
degree 5

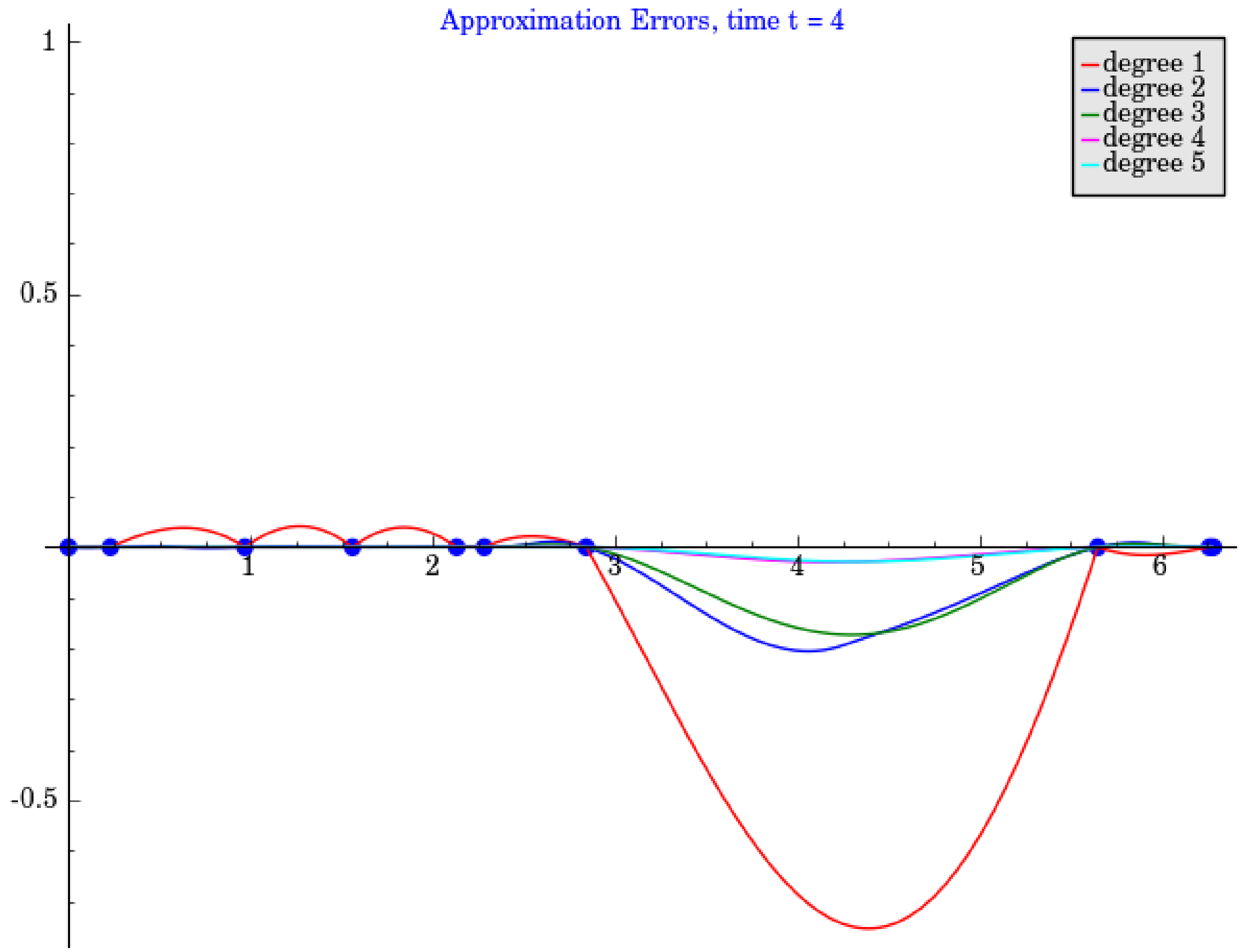


Approximations, time t = 3
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6

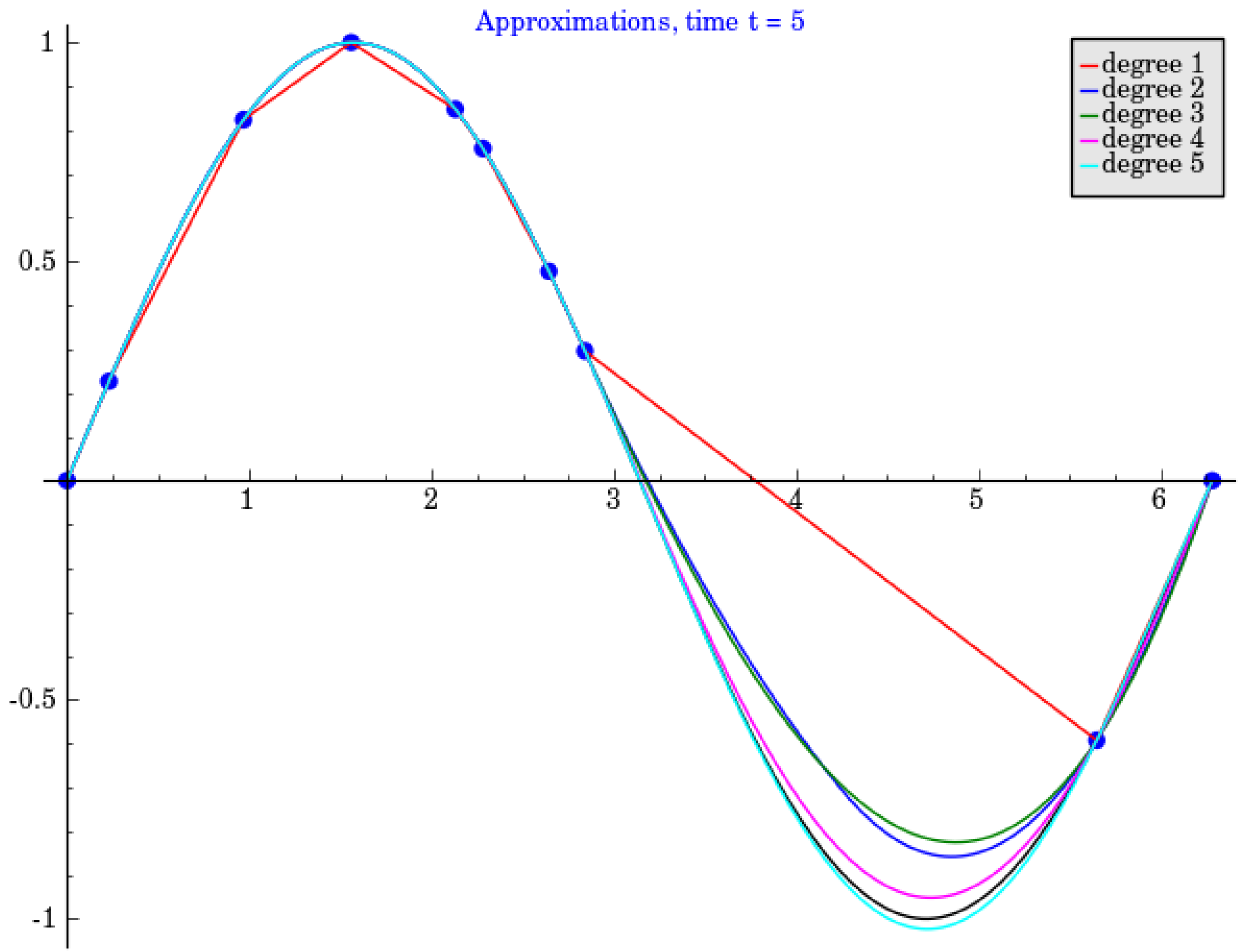

Approximation Errors, time t = 3
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
1
2
3
4
5
6



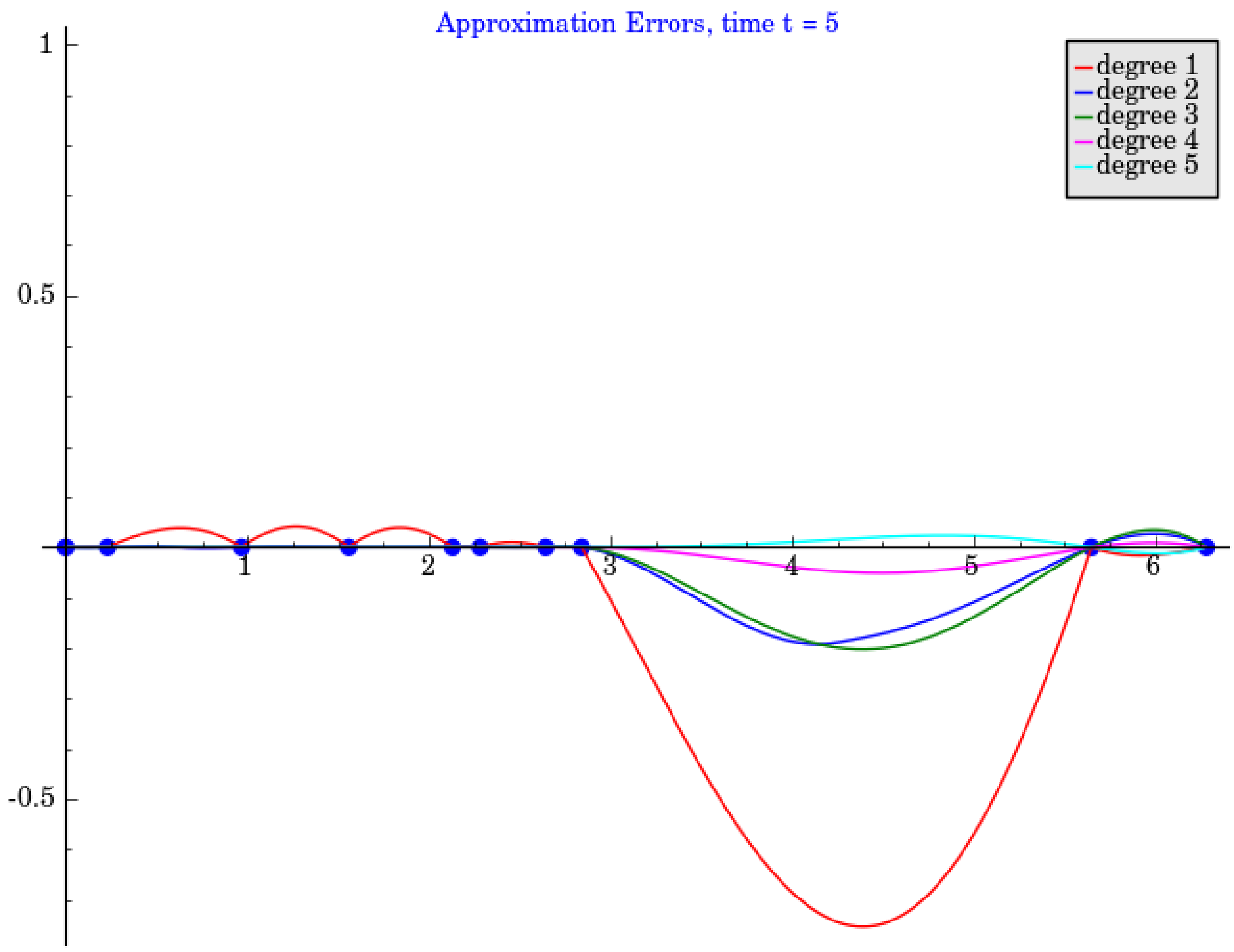
Approximations, time t = 4
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



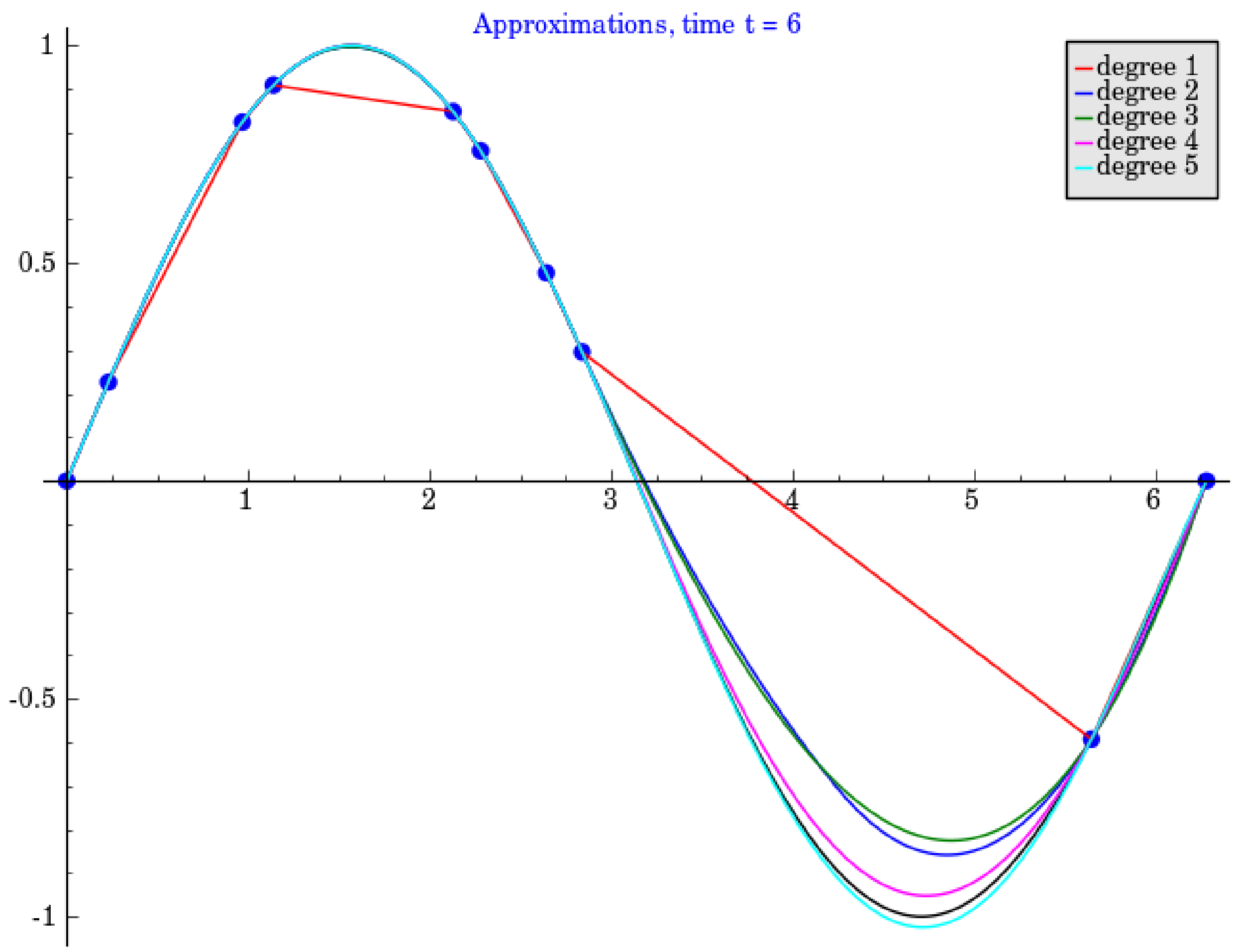
Approximation Errors, time t = 4
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
1
2
3
4
5
6



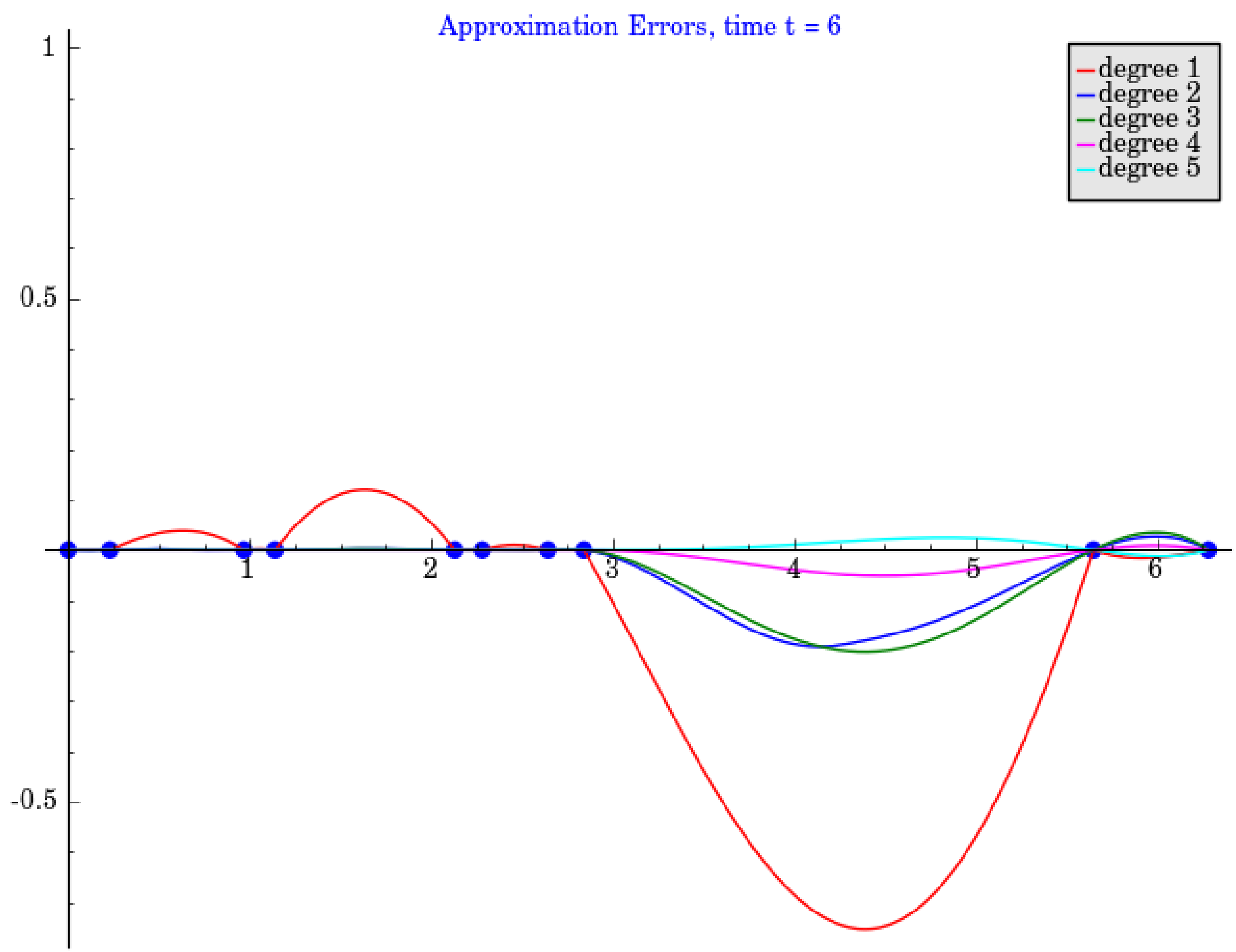
Approximations, time t = 5
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6

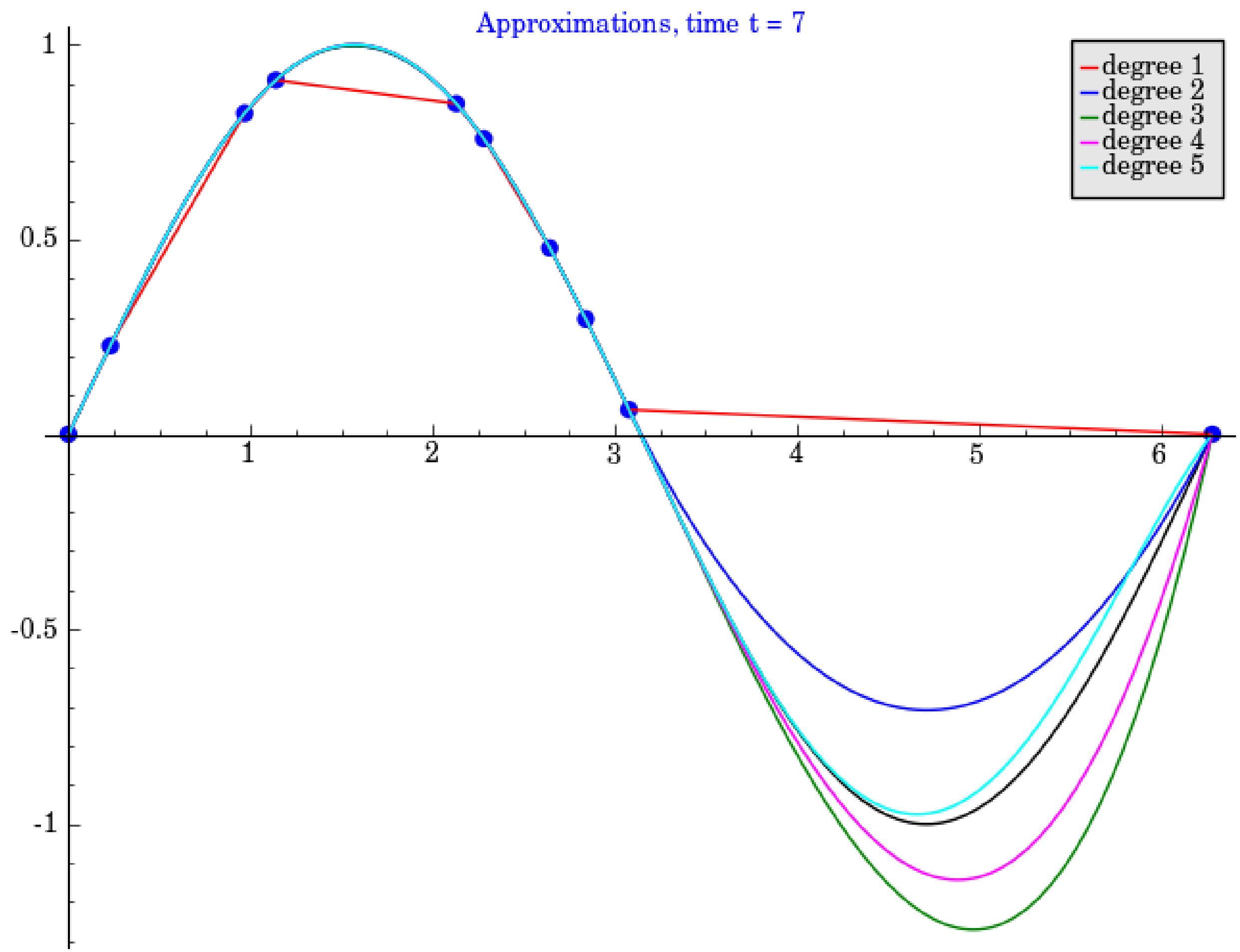


Approximation Errors, time t = 5
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
1
2
3
4
5
6

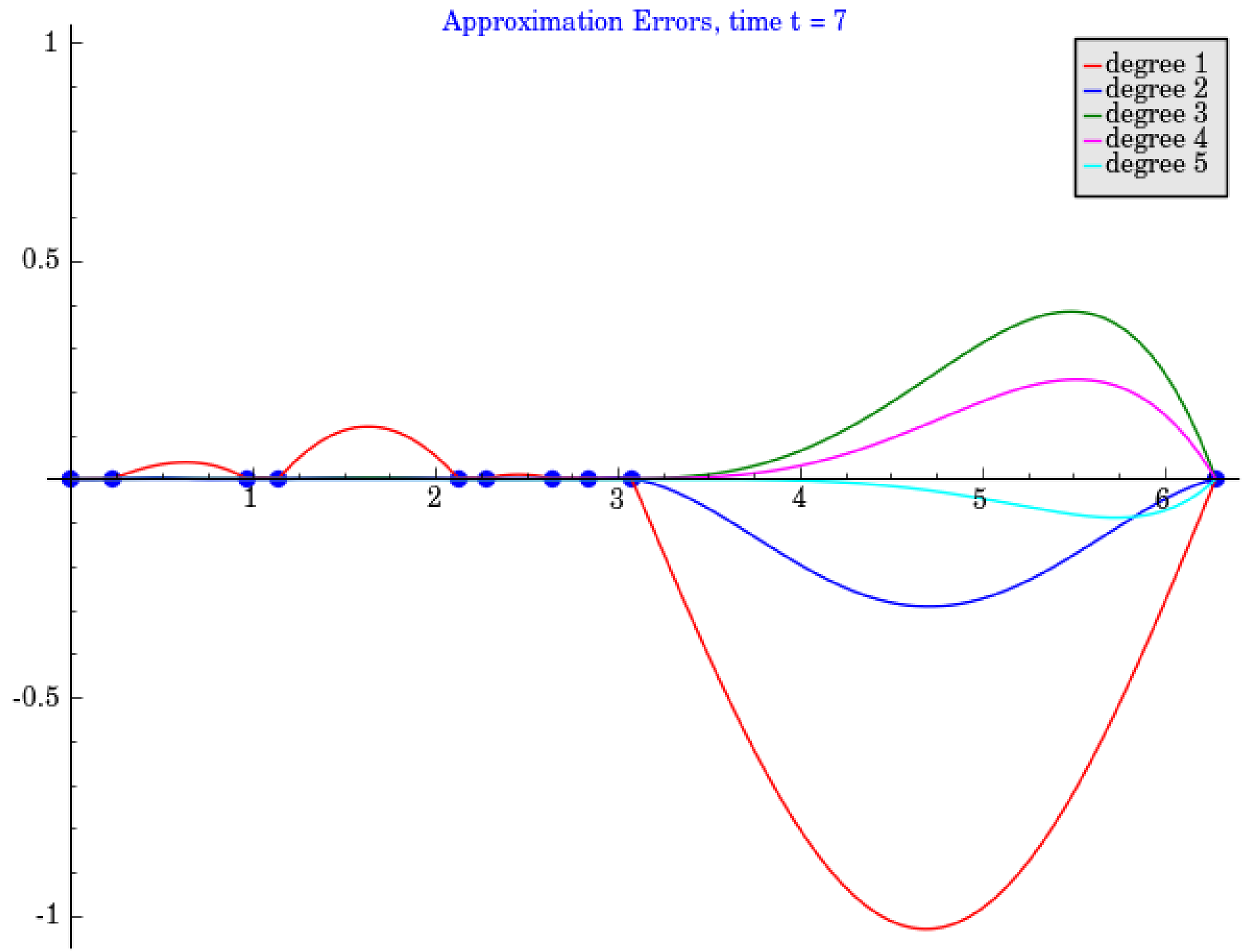


Approximations, time t = 6
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6

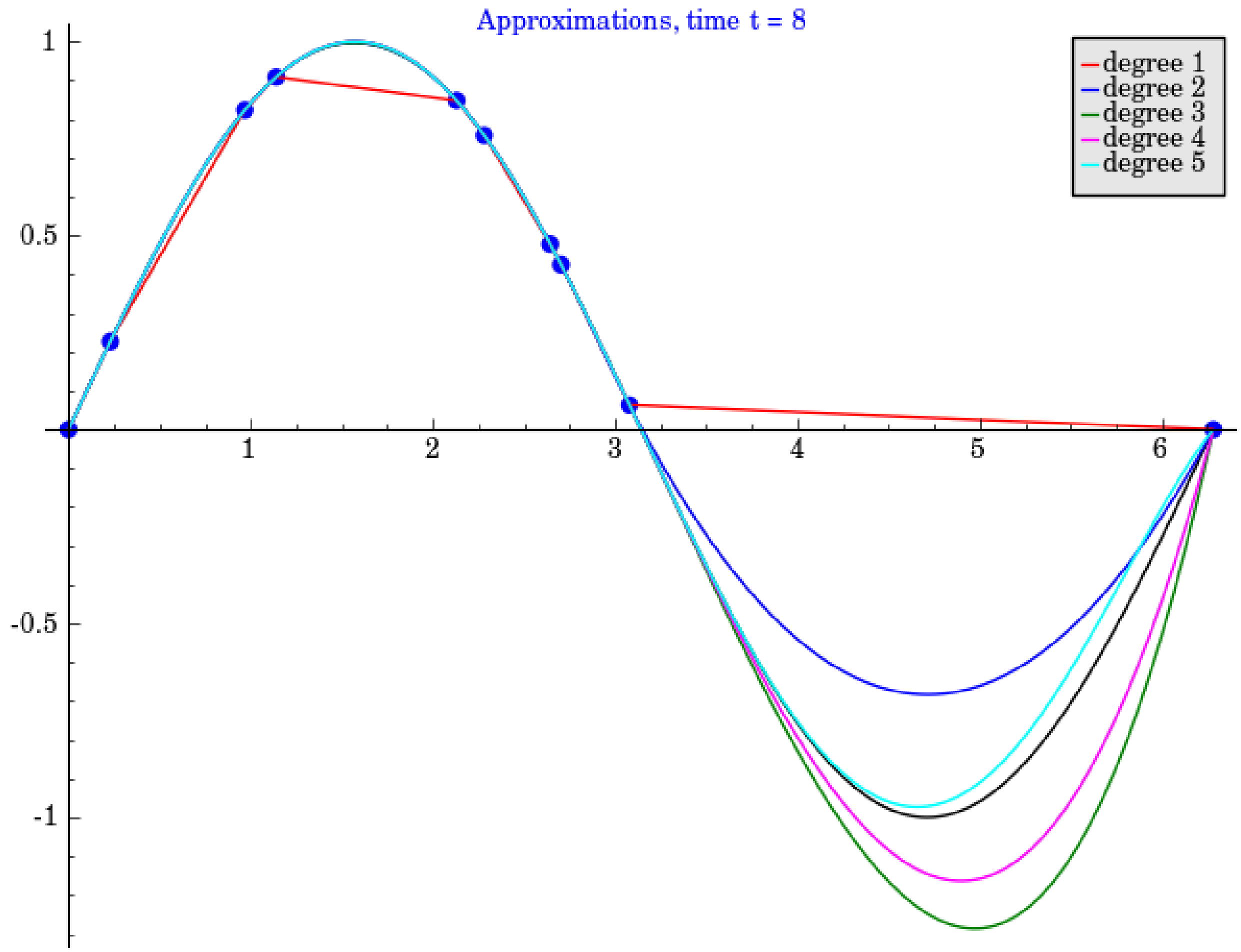

Approximation Errors, time t = 6
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
1
2
3
4
5
6



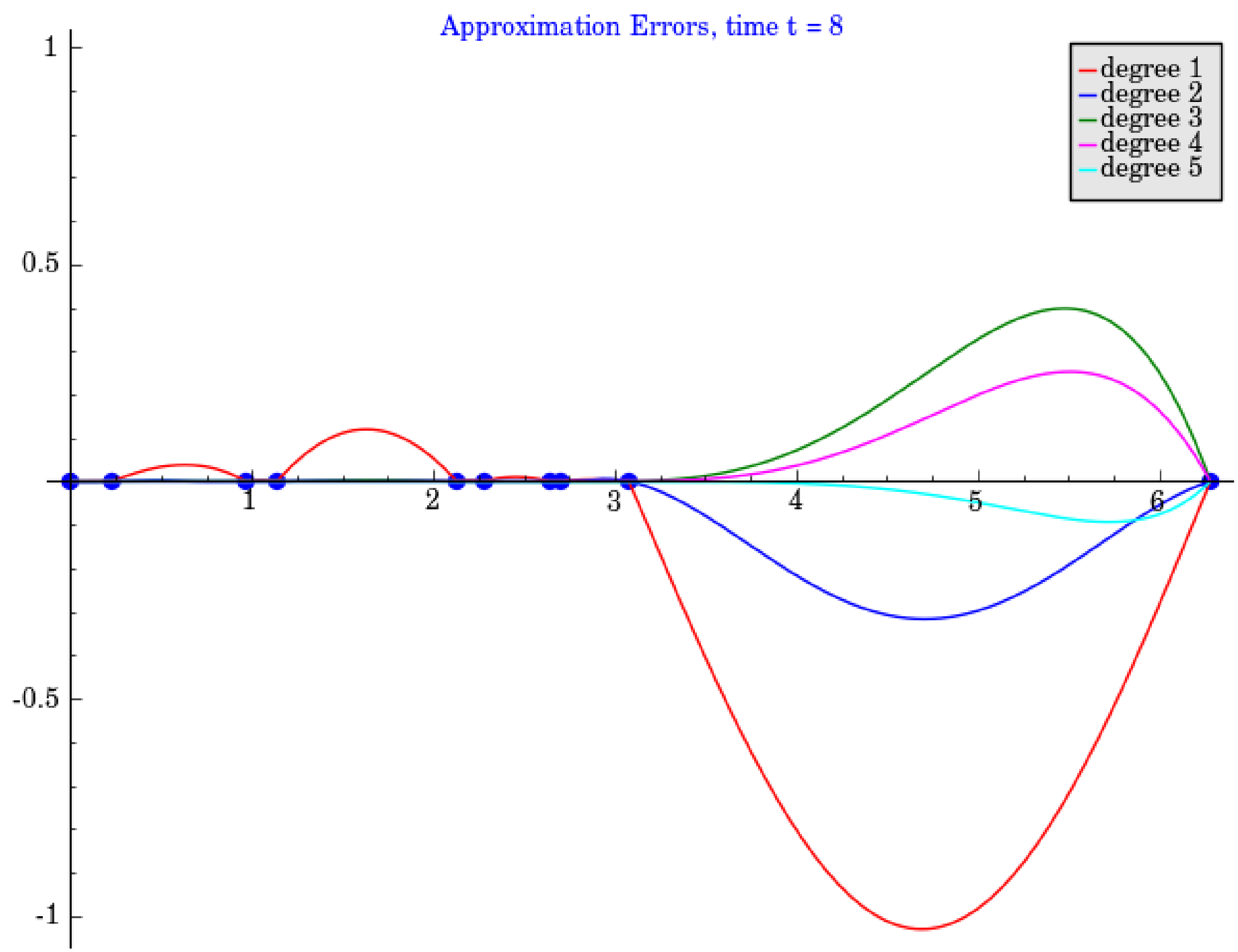
Approximations, time t = 7
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



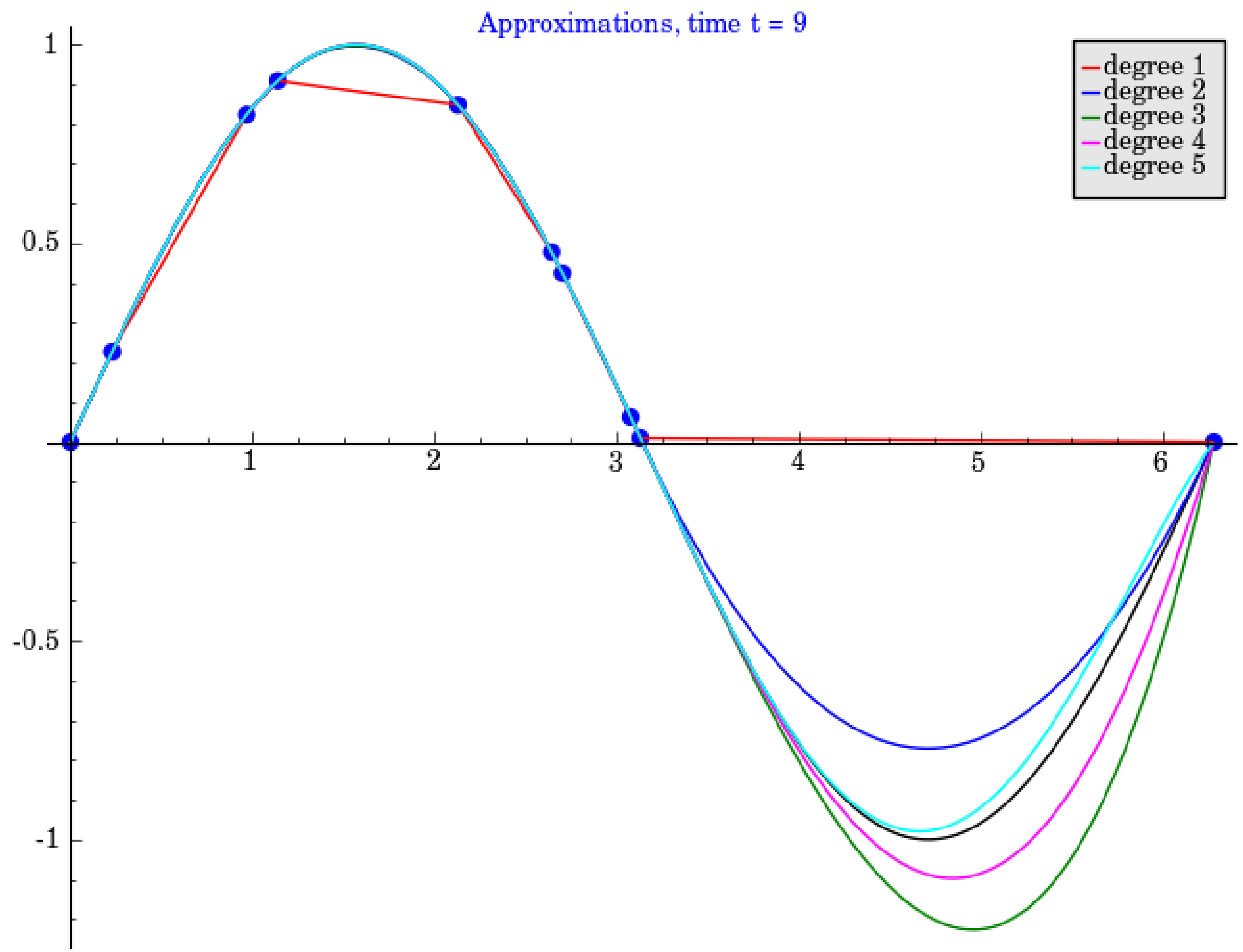
Approximation Errors, time t = 7
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



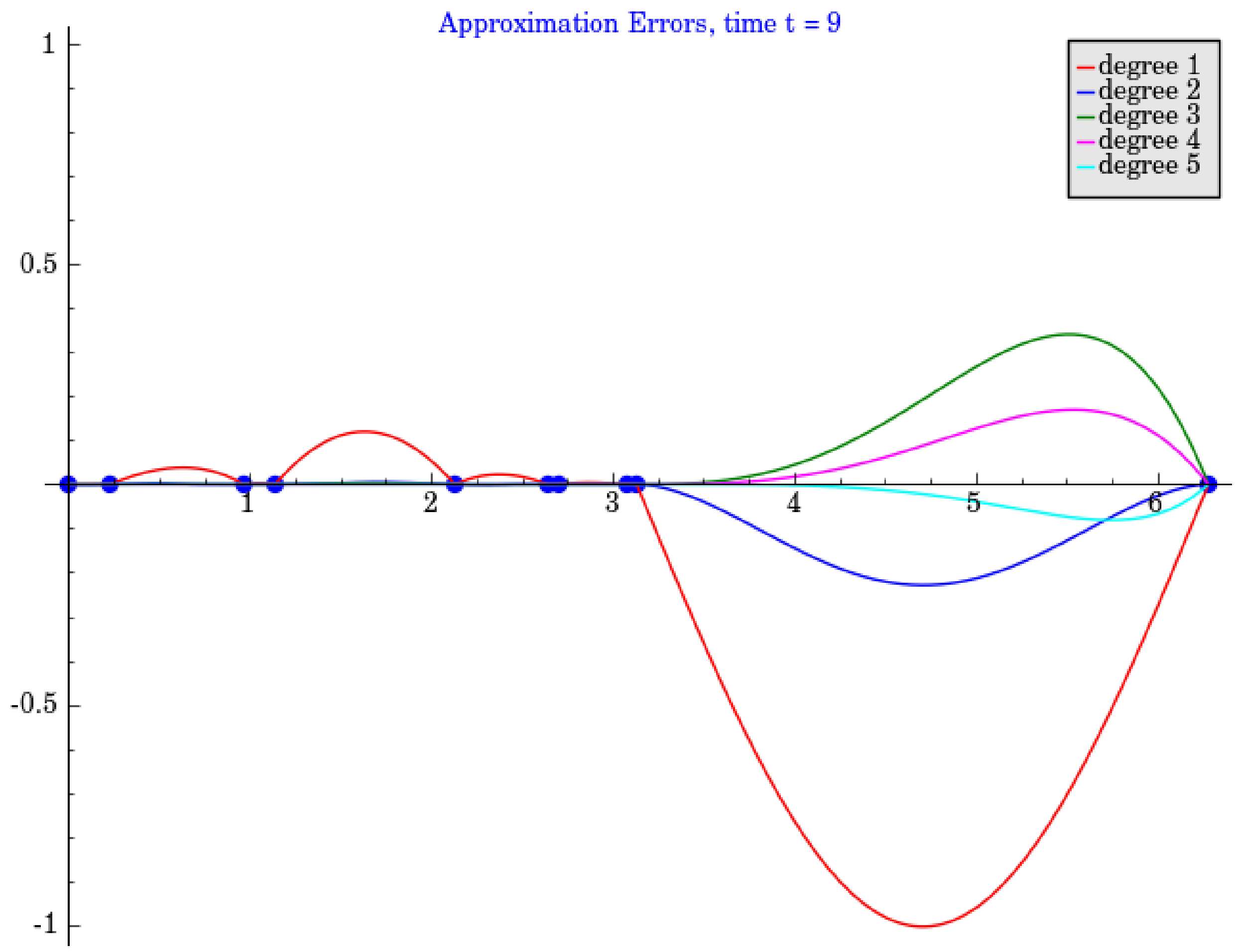
Approximations, time t = 8
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



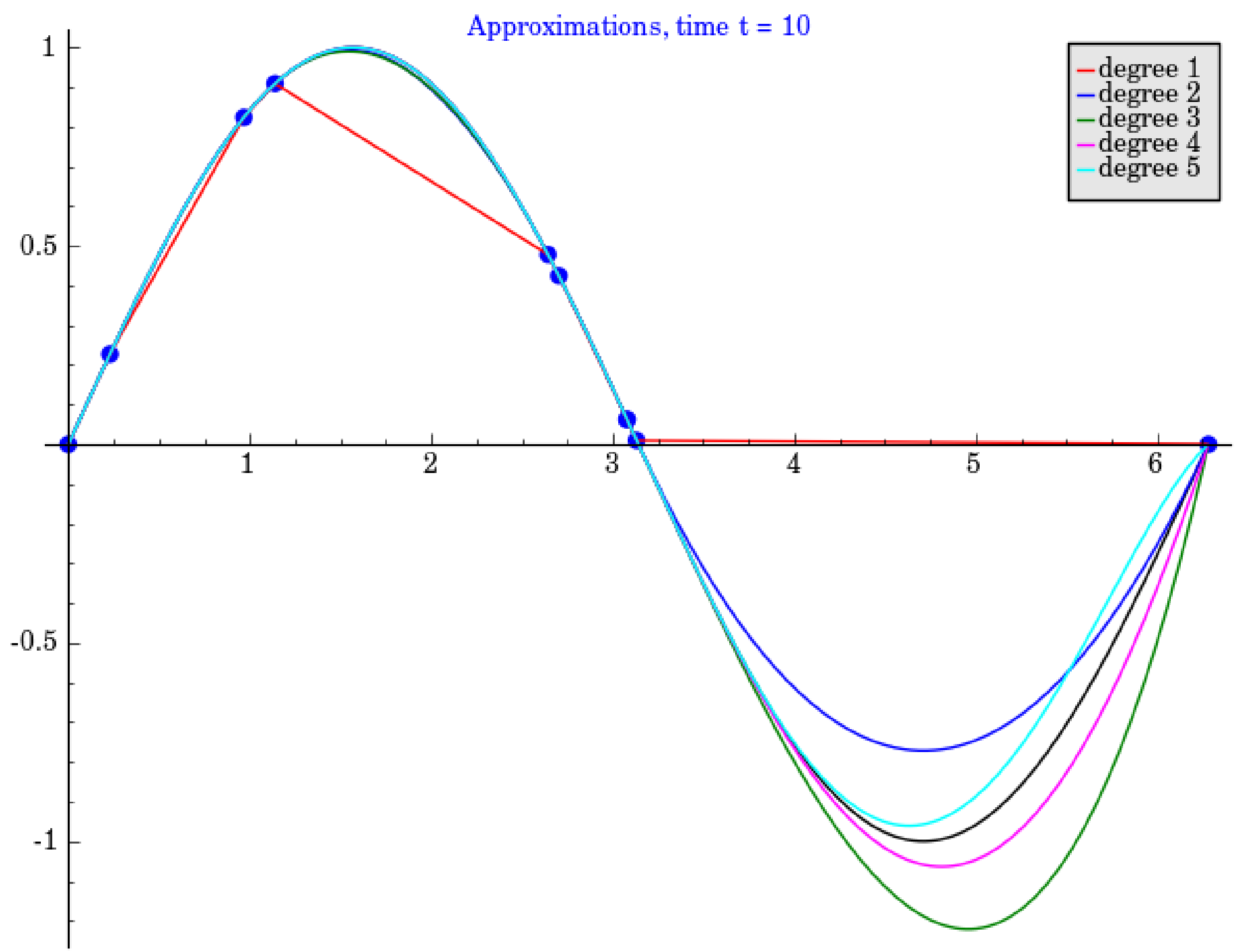
Approximation Errors, time t = 8
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



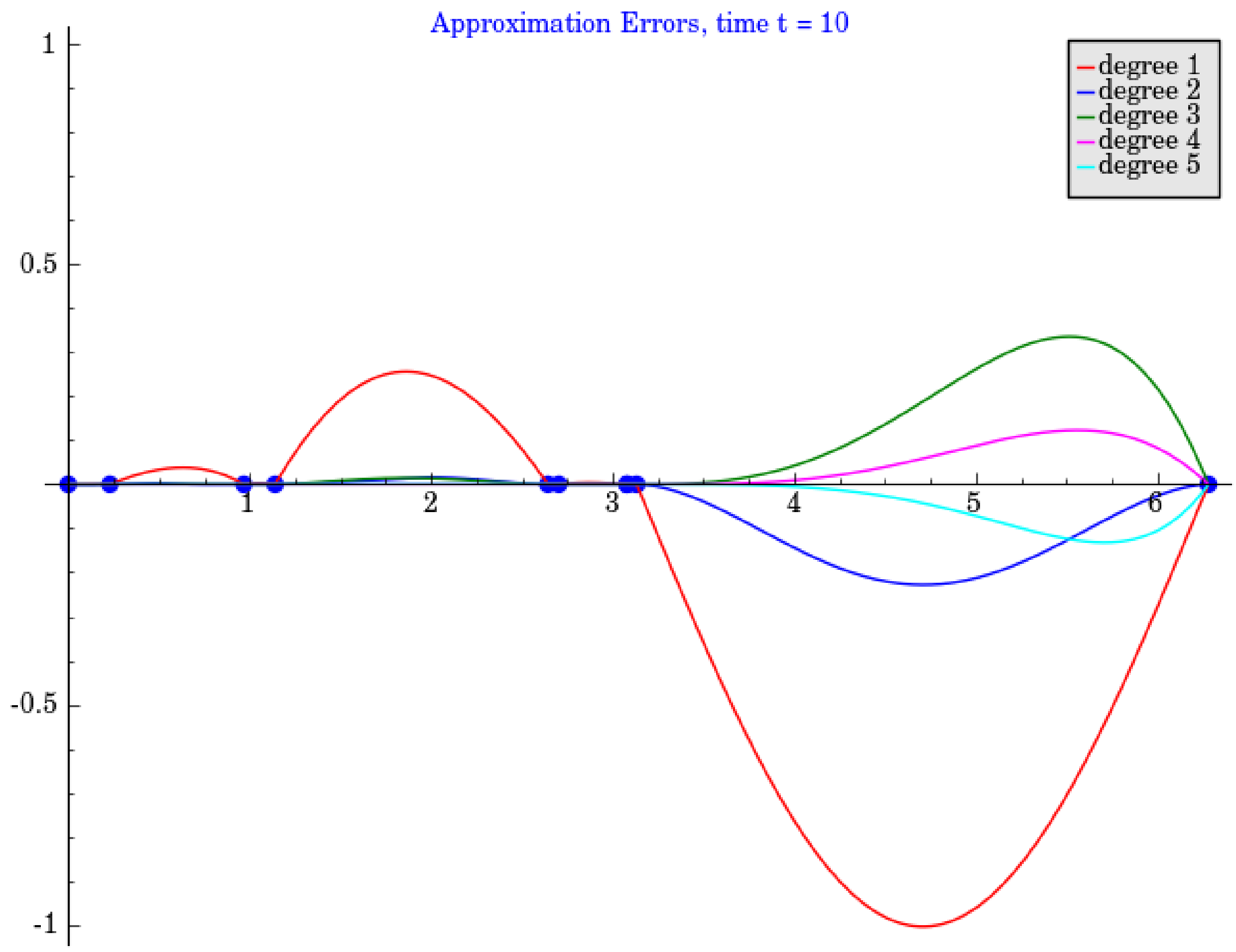
Approximations, time t = 9
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



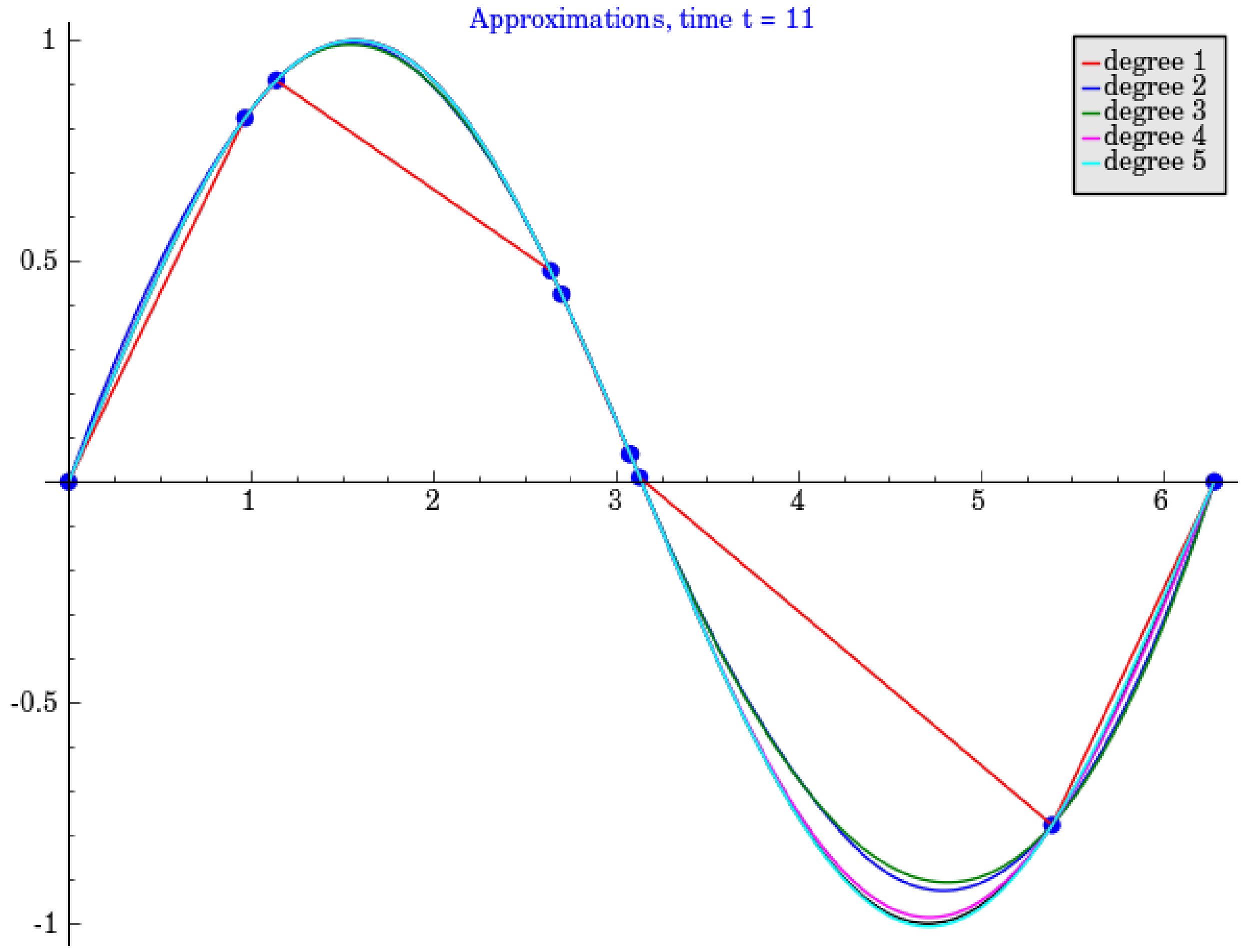
Approximation Errors, time t = 9
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6

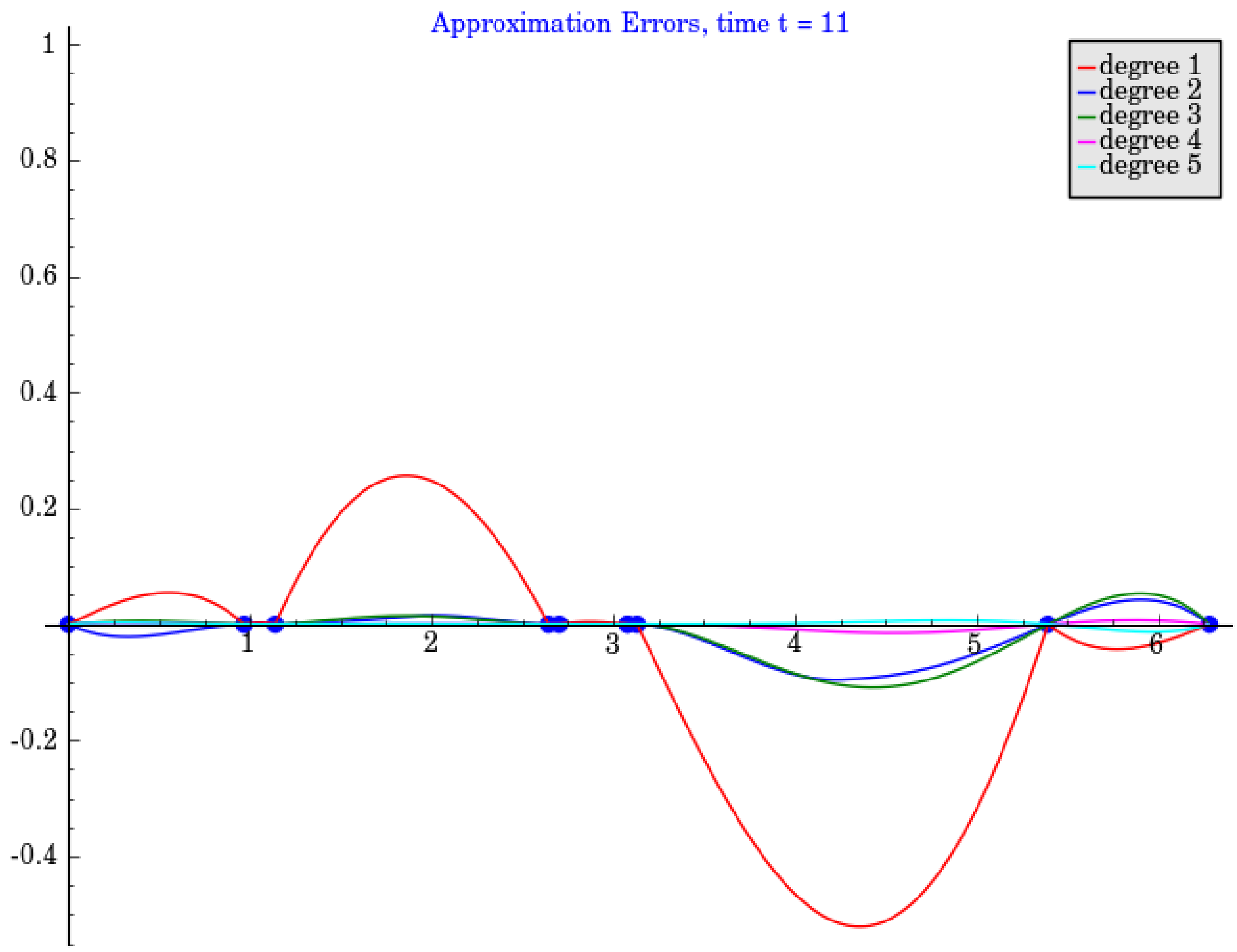


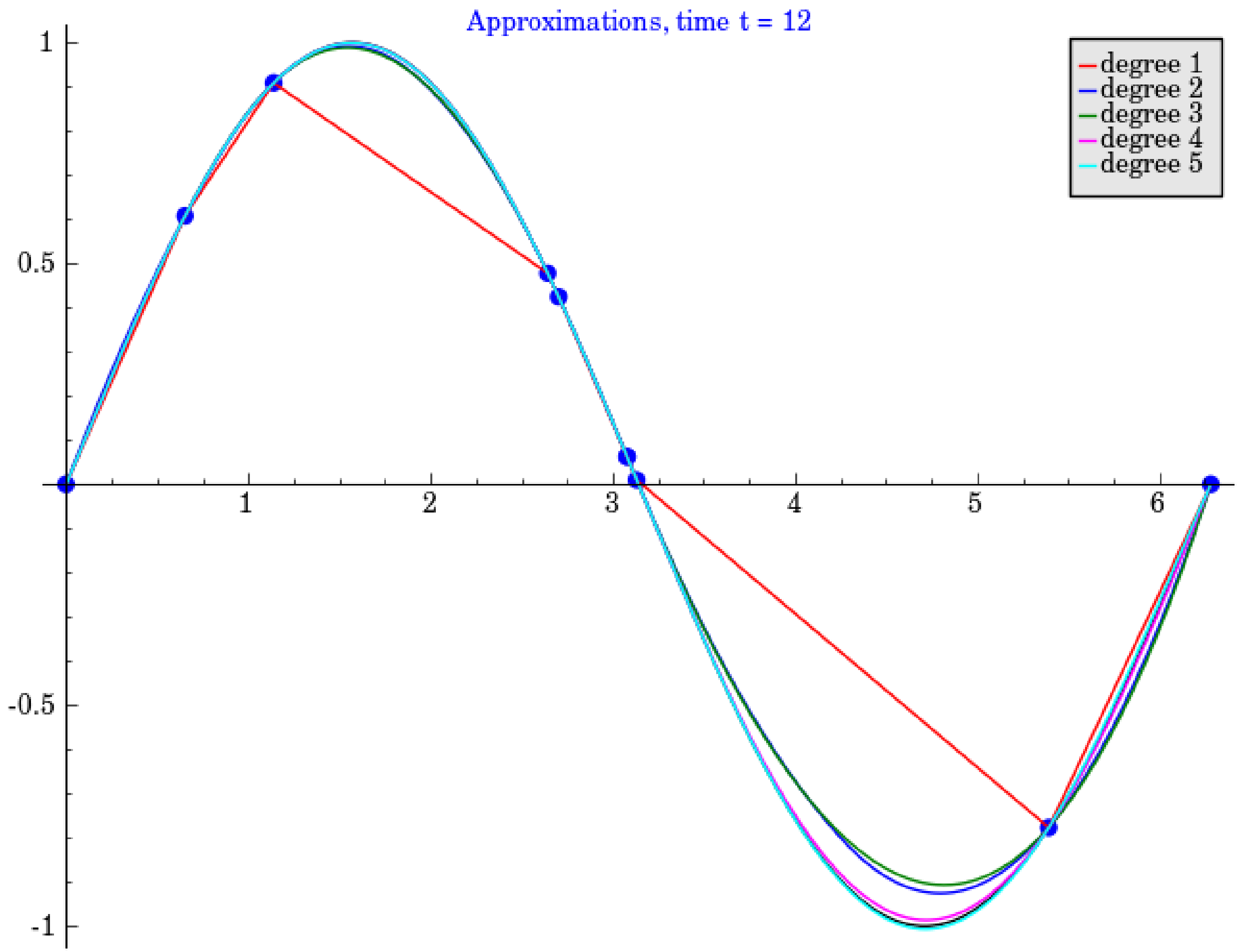
Approximations, time t = 10
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



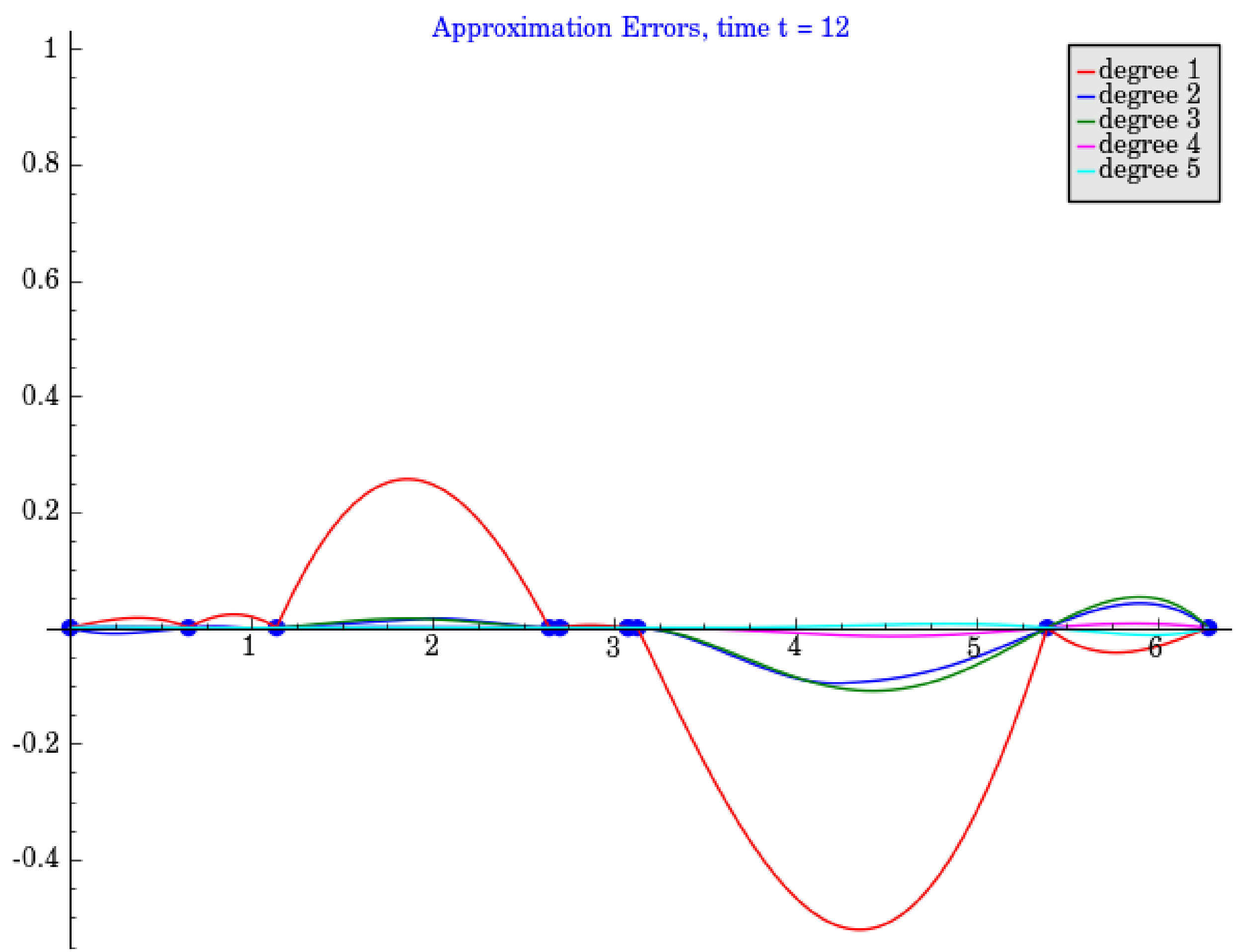
Approximation Errors, time t = 10
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6


Approximations, time t = 11
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6

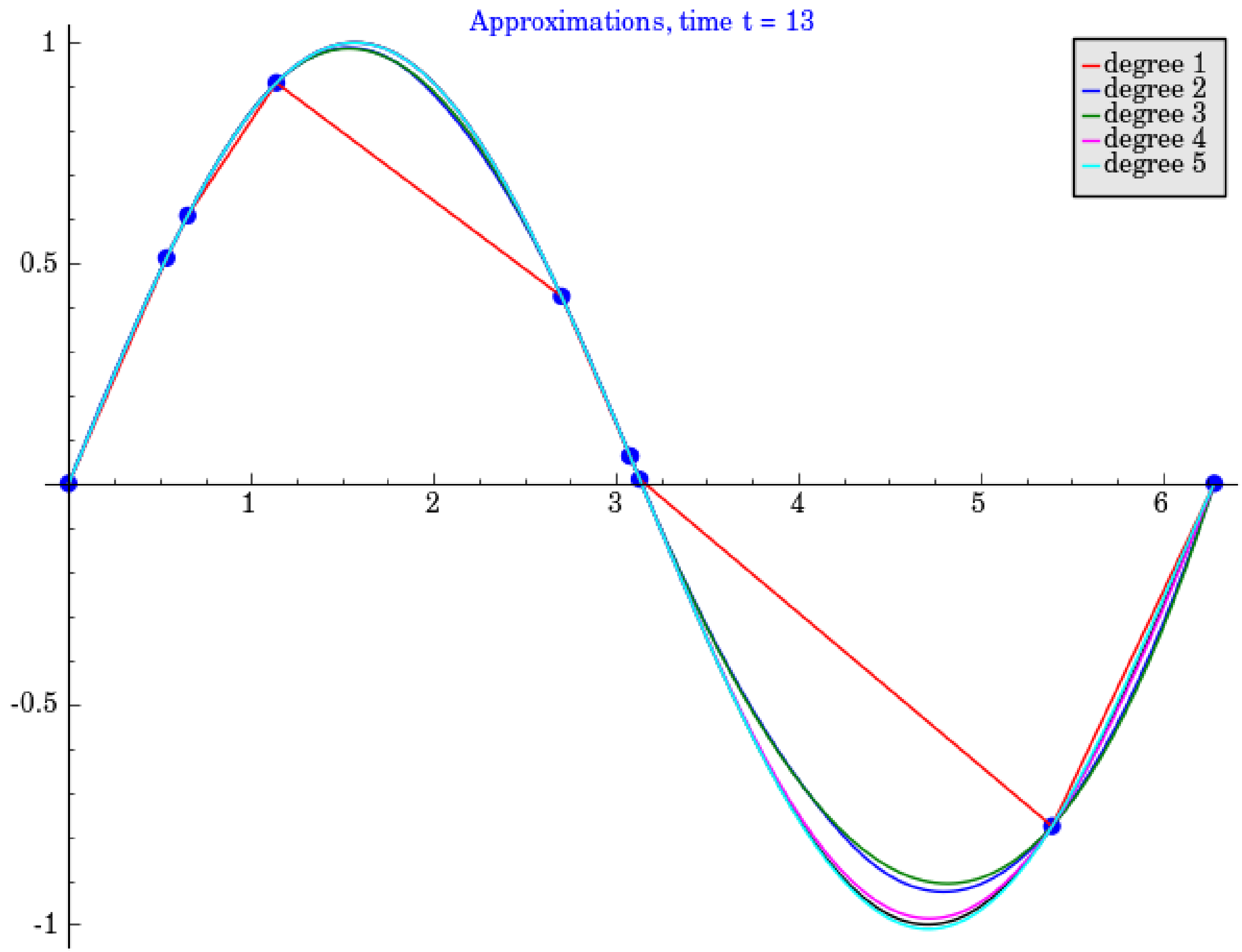

Approximation Errors, time t = 11
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
-0.4
1
2
3
4
5
6



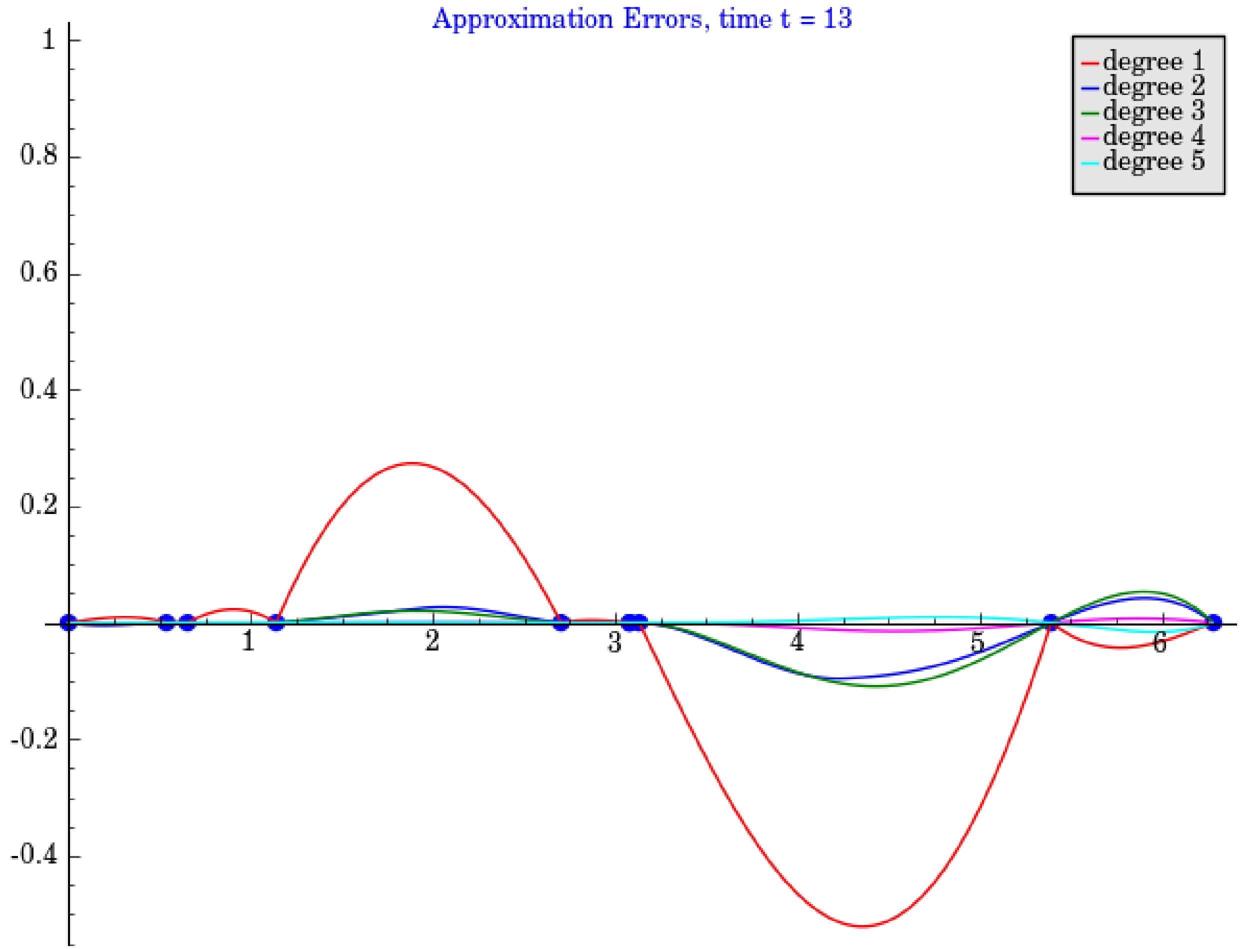
Approximations, time t = 12
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



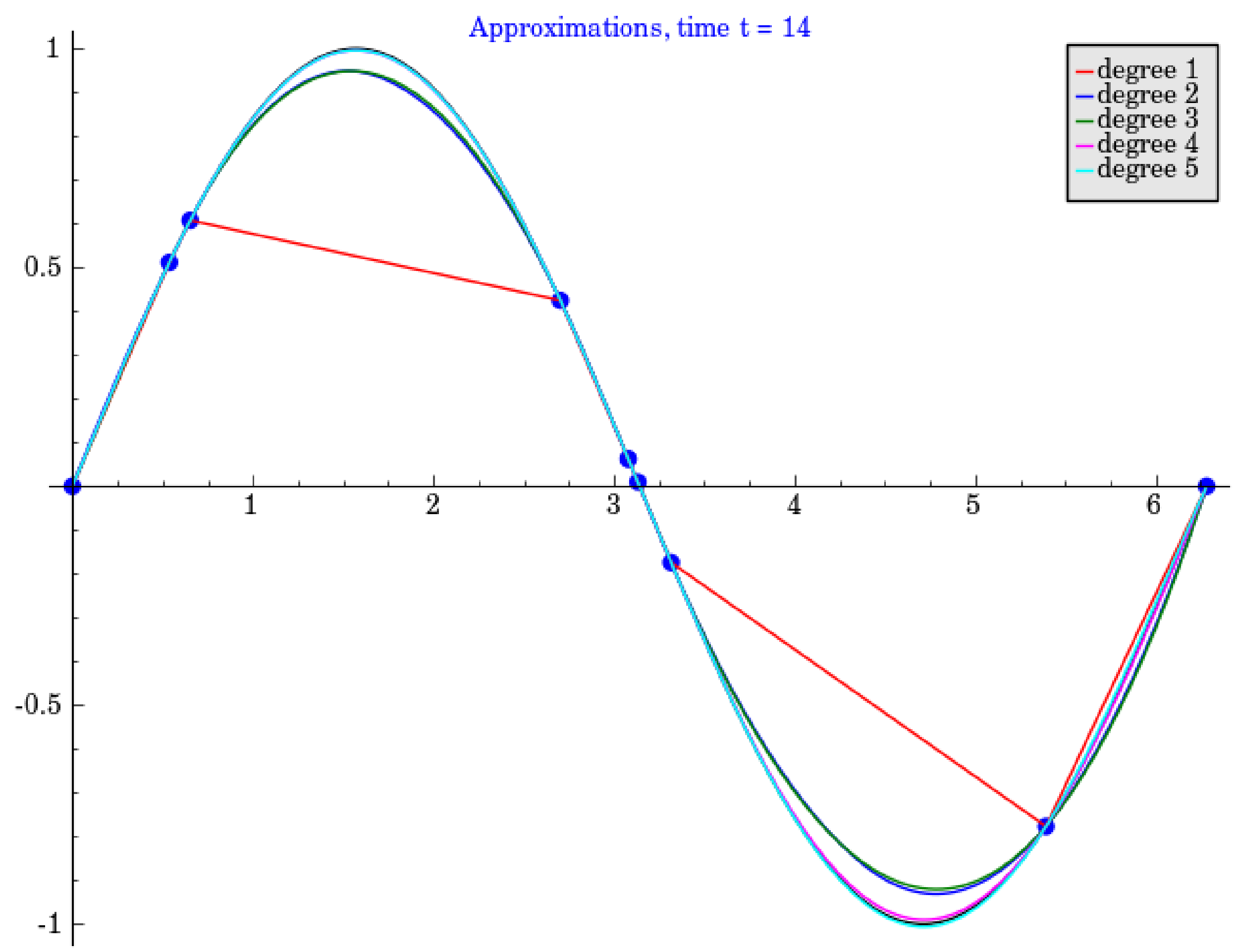
Approximation Errors, time t = 12
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
-0.4
1
2
3
4
5
6

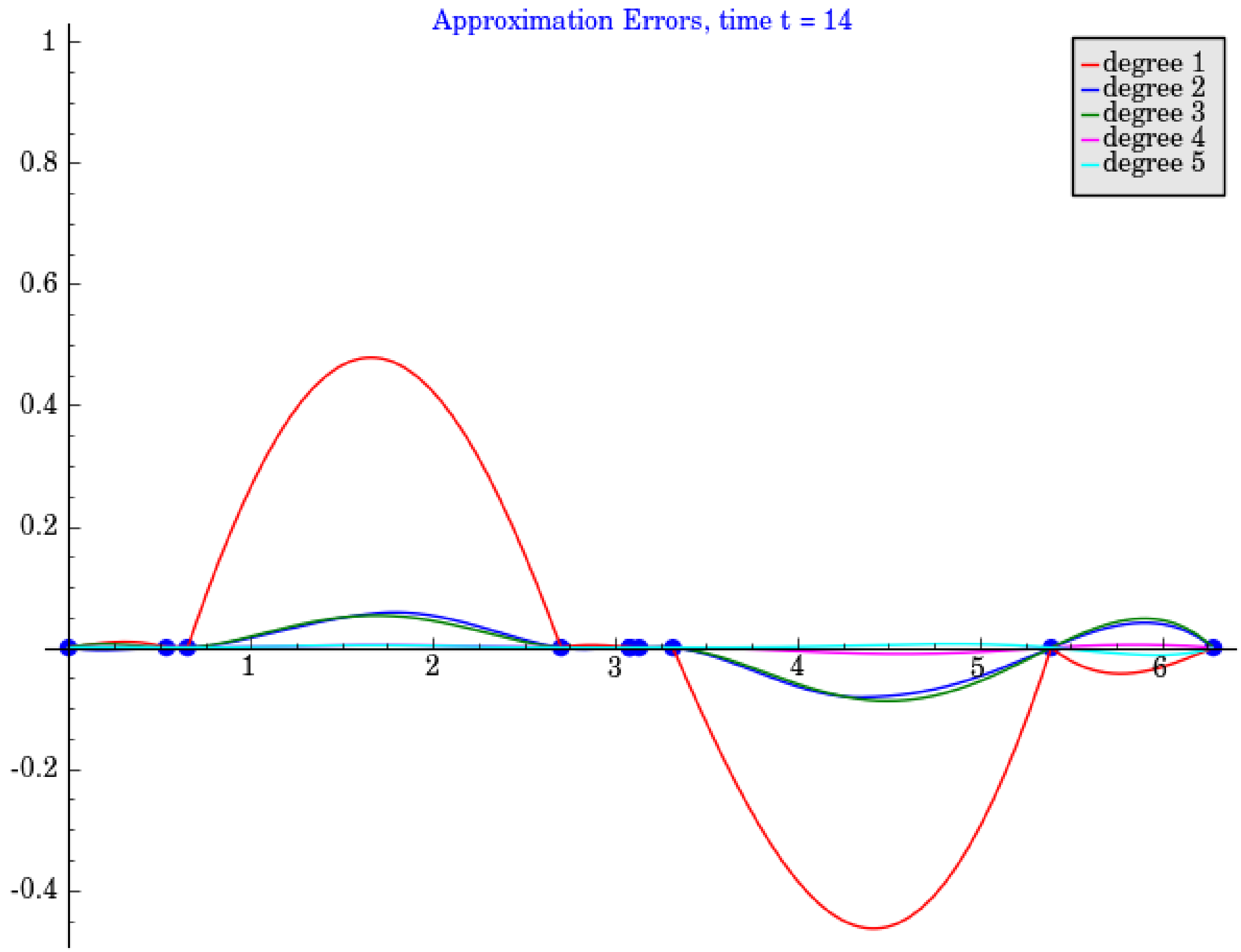


Approximations, time t = 13
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6

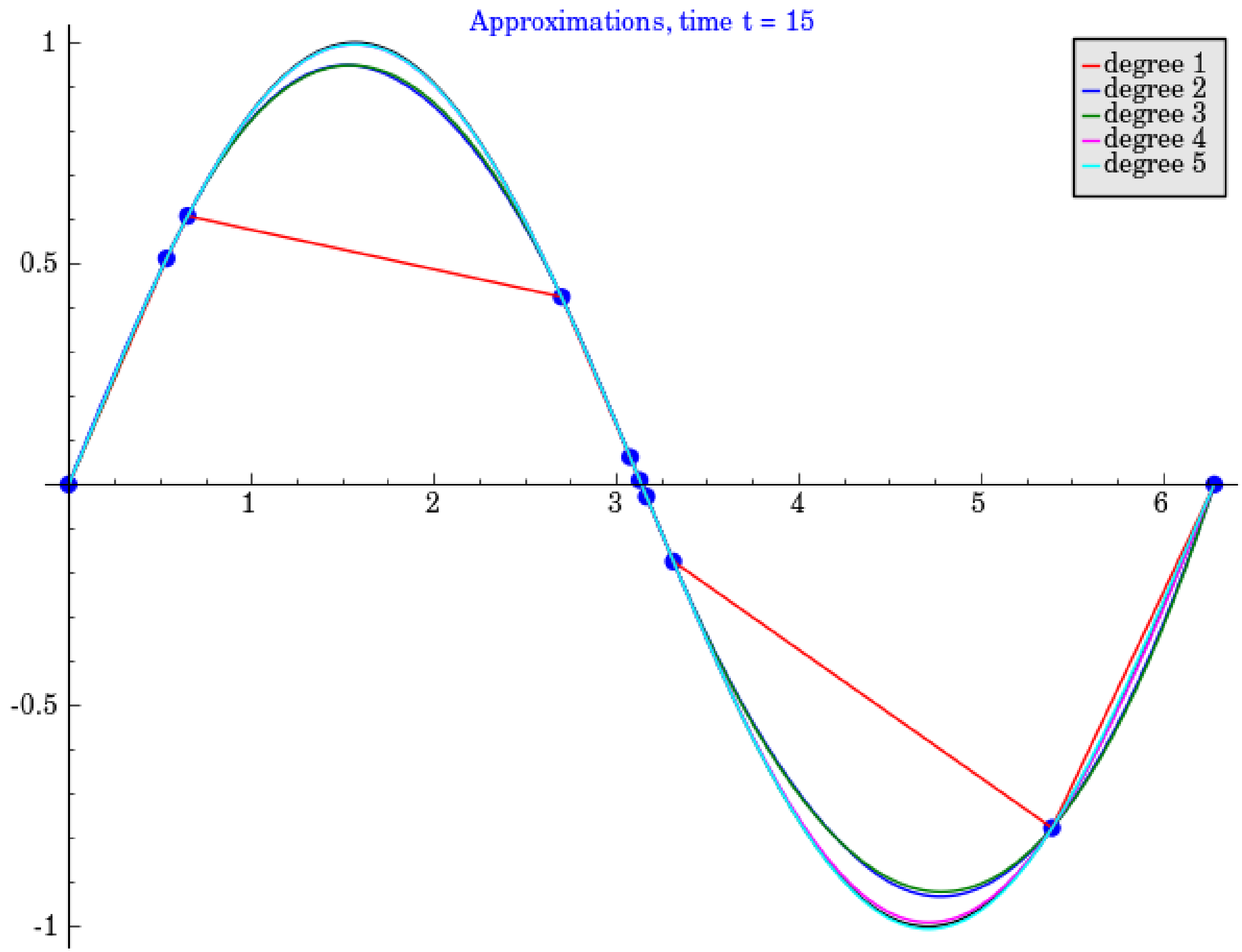


Approximation Errors, time t = 13
1
0.8
0.6
0.4
0.2
-0.2
-0.4
1
2
3
4
5
6
degree 1
degree 2
degree 3
degree 4
degree 5

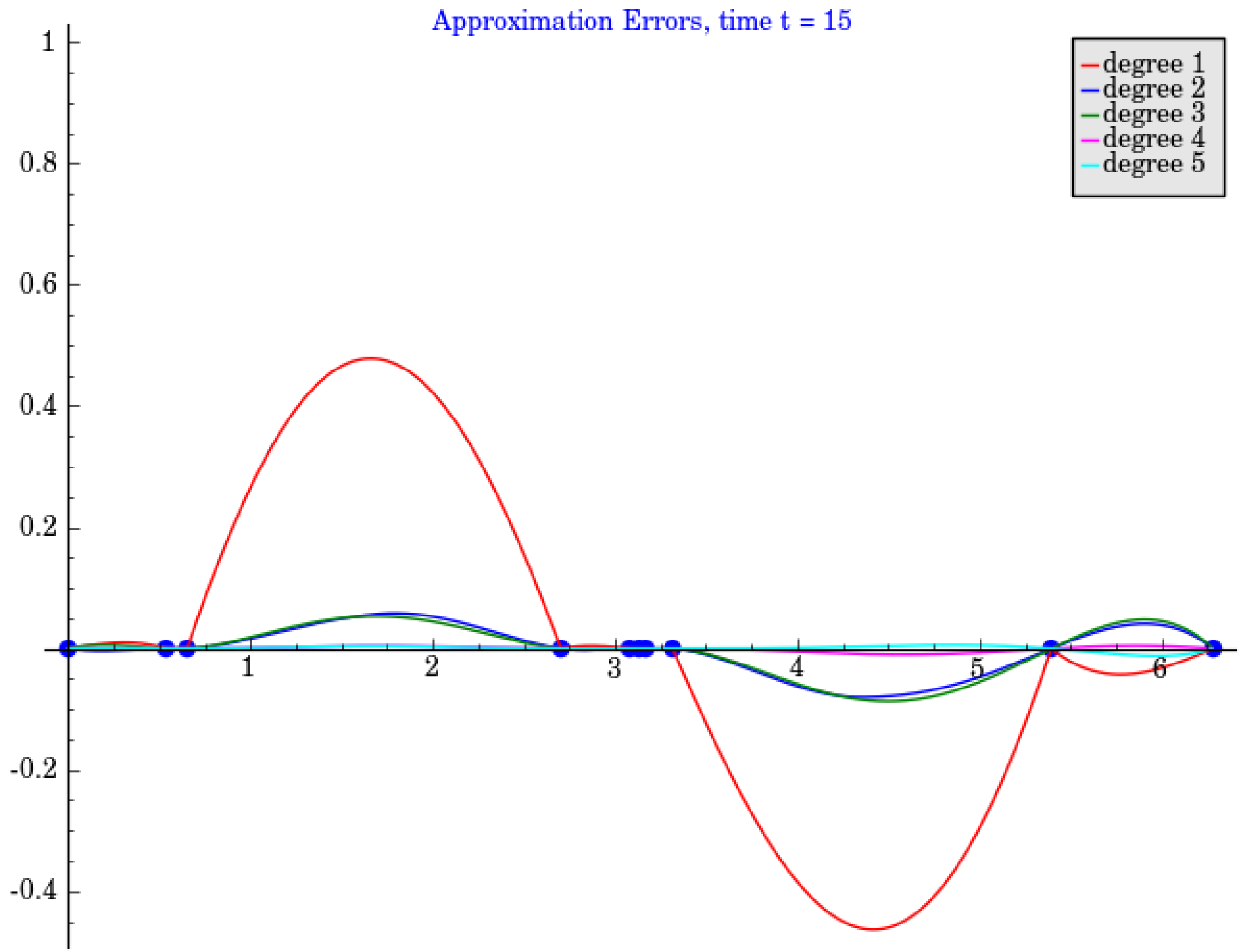

Approximations, time t = 14
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



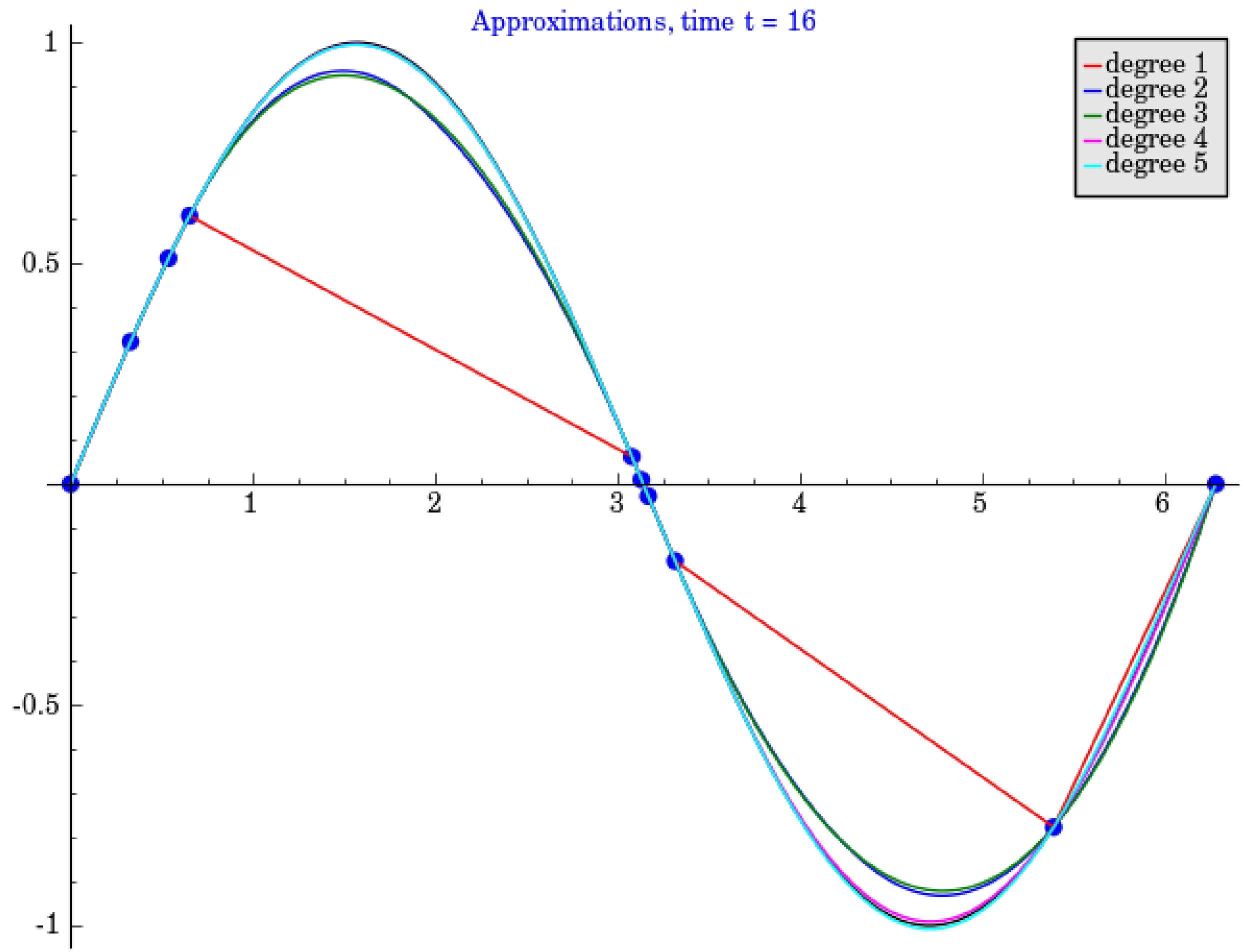
Approximation Errors, time t = 14
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
-0.4
1
2
3
4
5
6

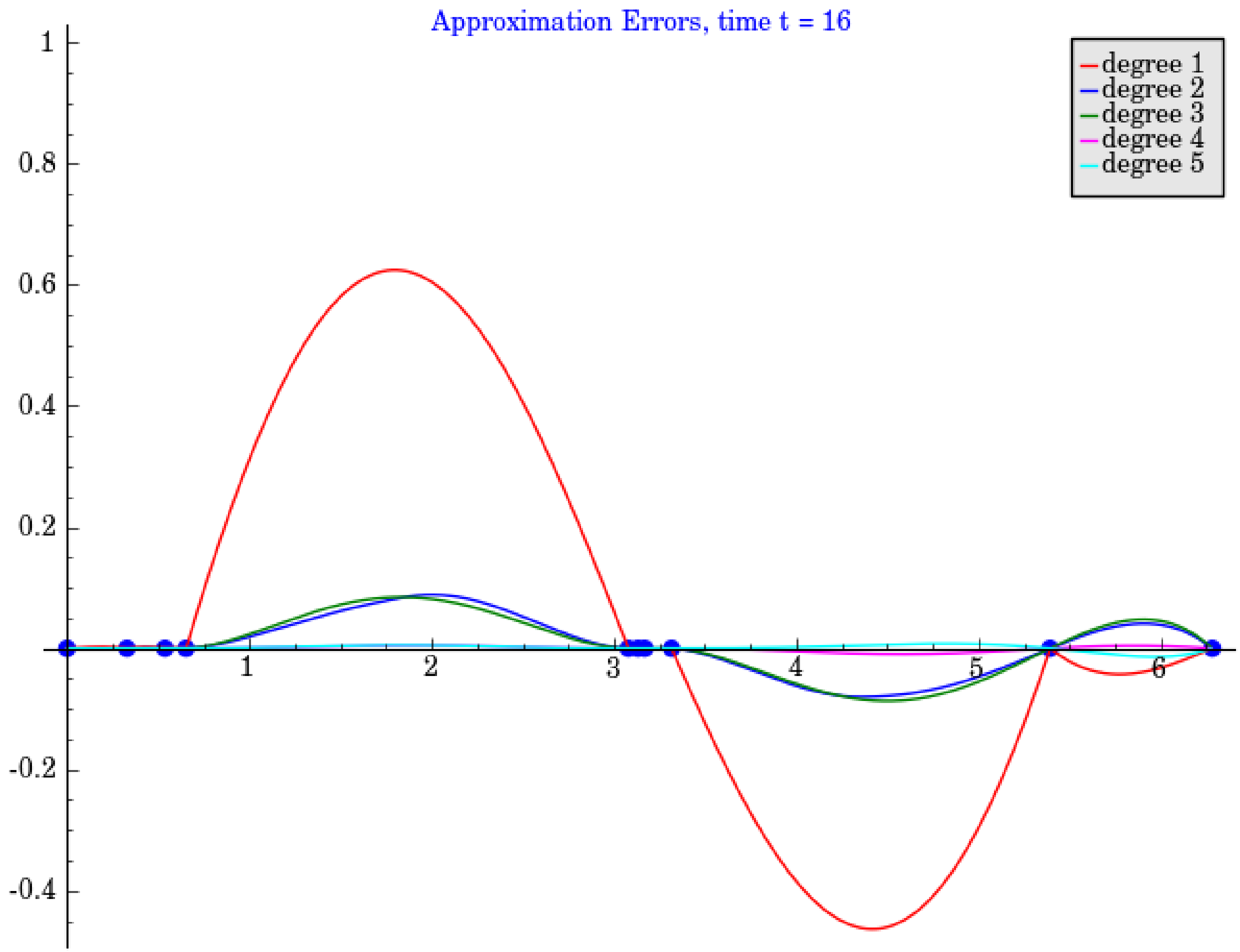


Approximations, time t = 15
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6

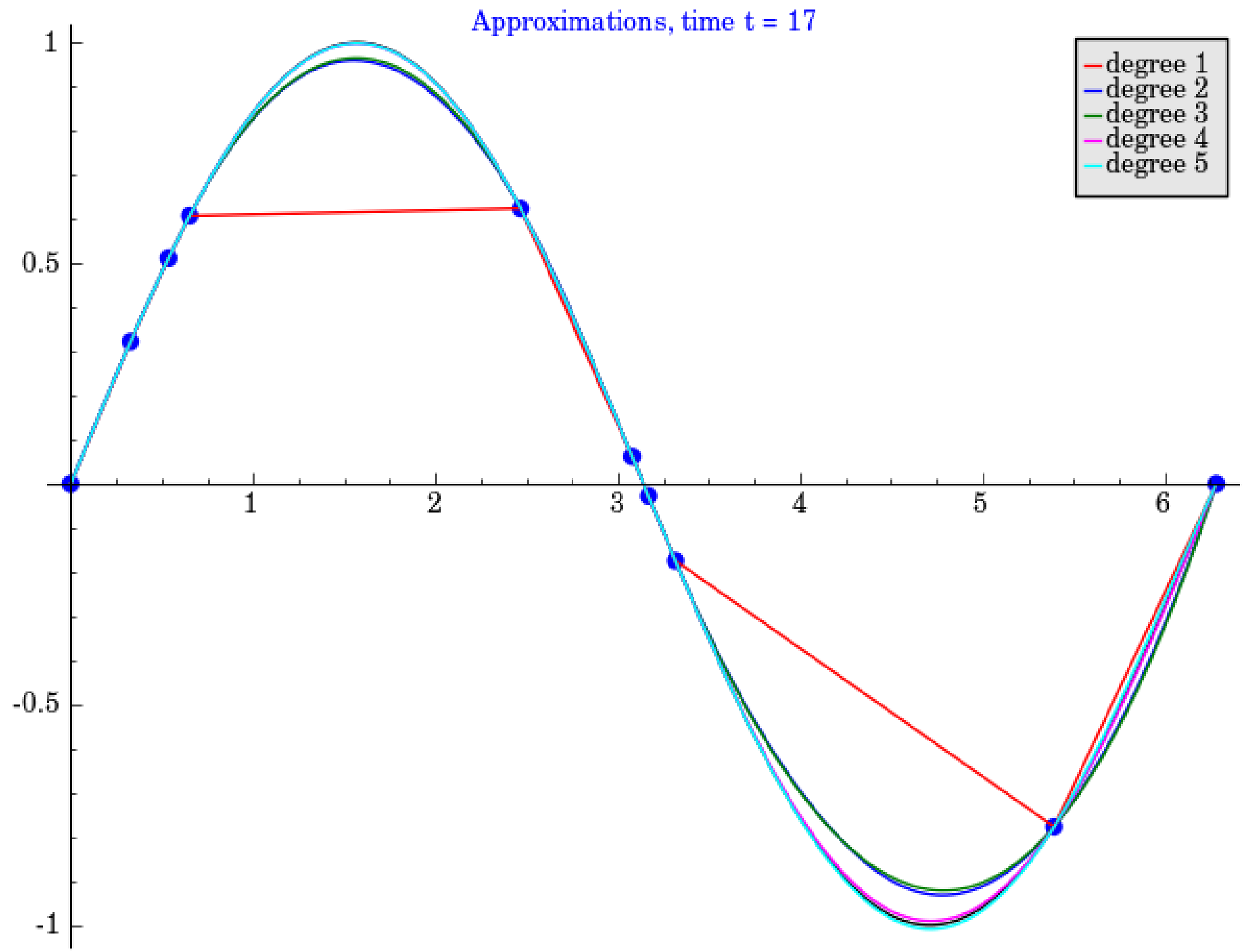


Approximation Errors, time t = 15
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
-0.4
1
2
3
4
5
6

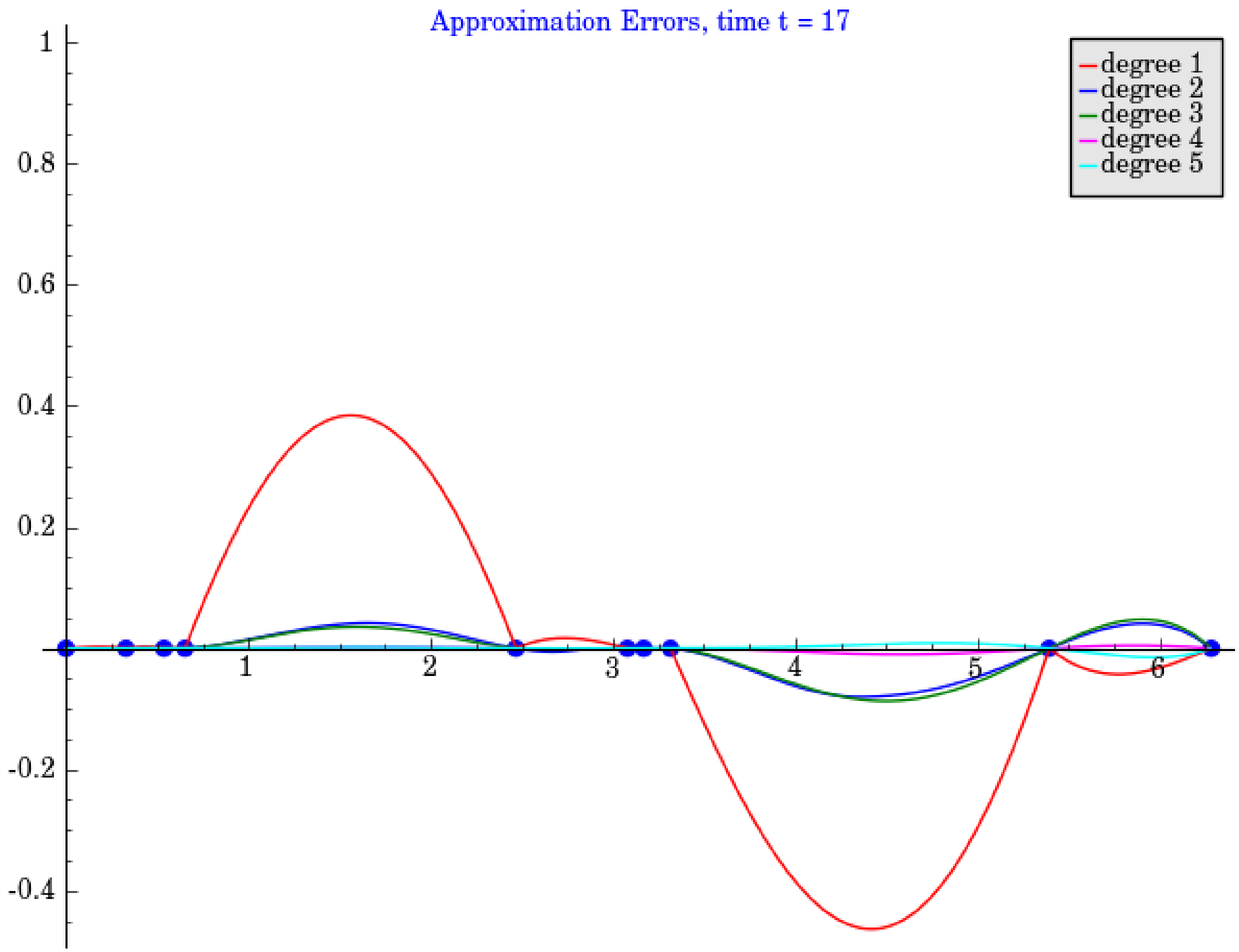

Approximations, time t = 16
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



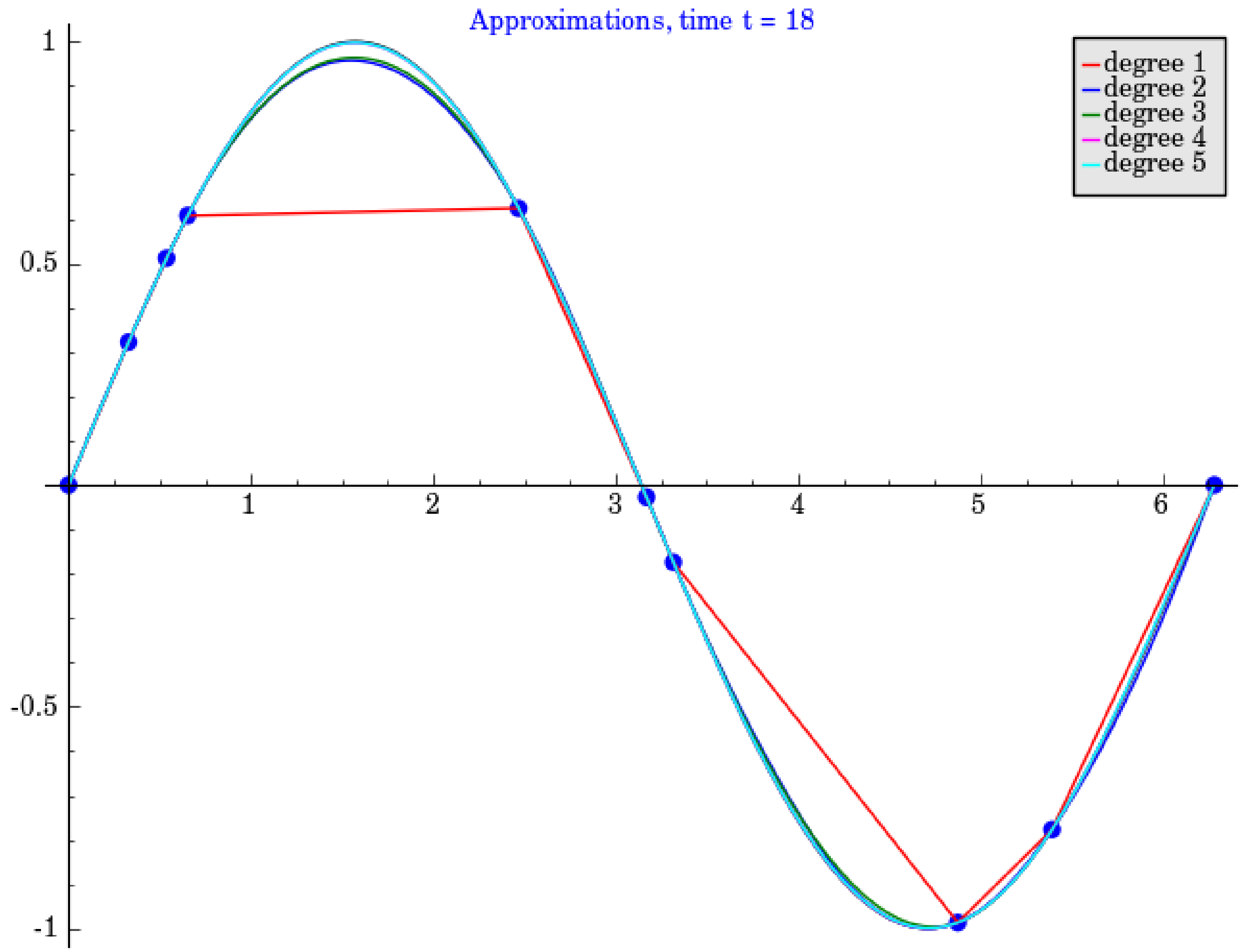
Approximation Errors, time t = 16
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
-0.4
1
2
3
4
5
6

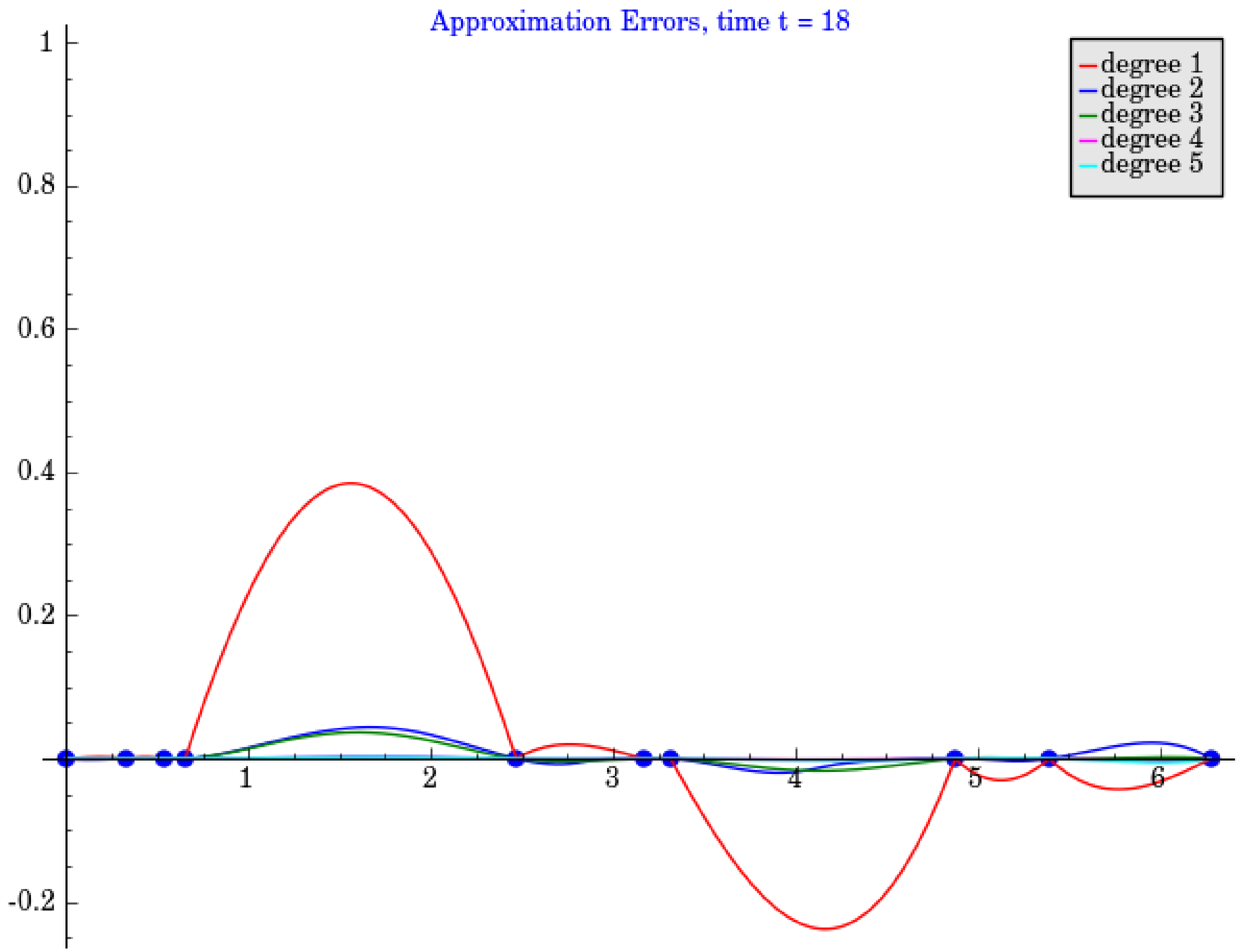


Approximations, time t = 17
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6

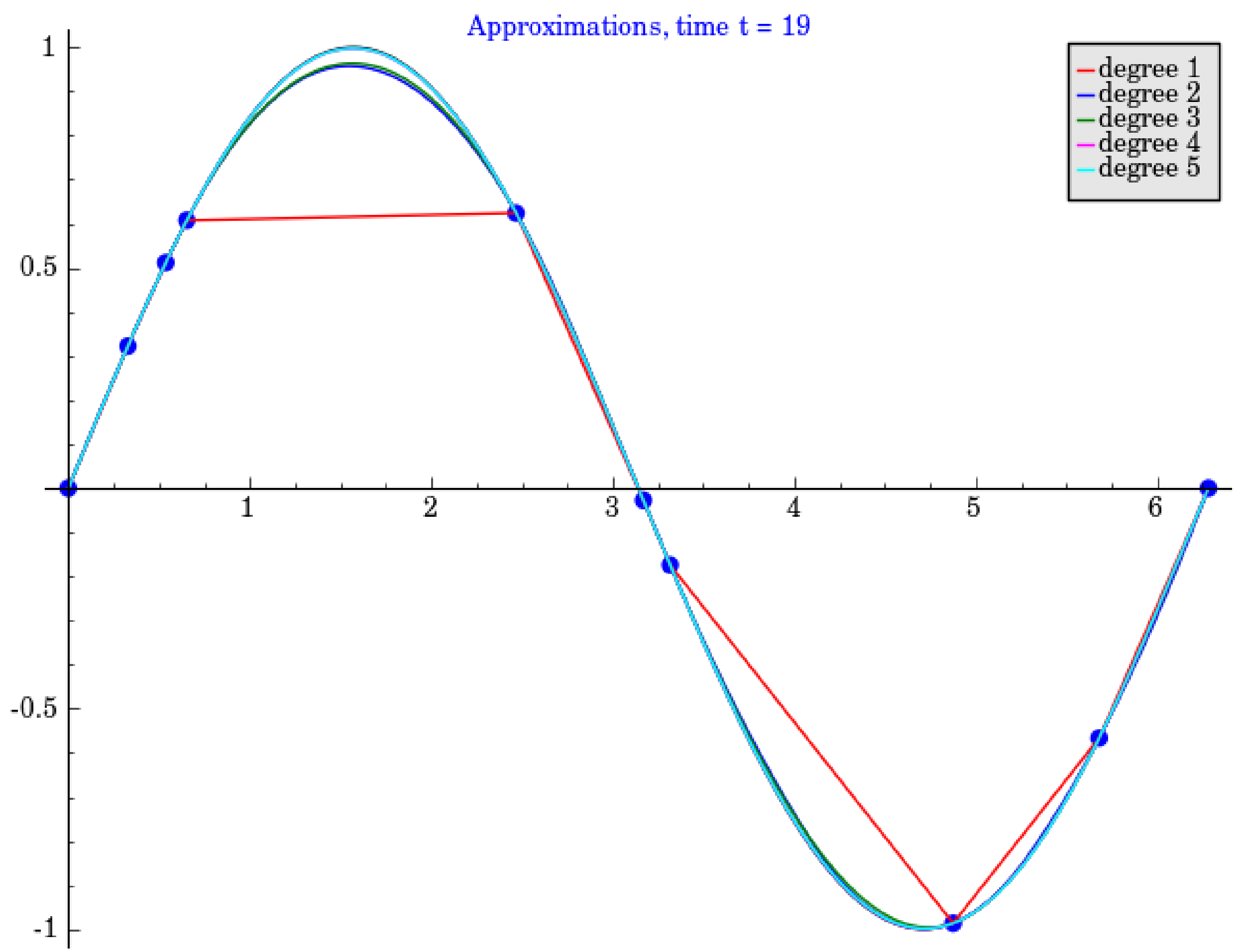


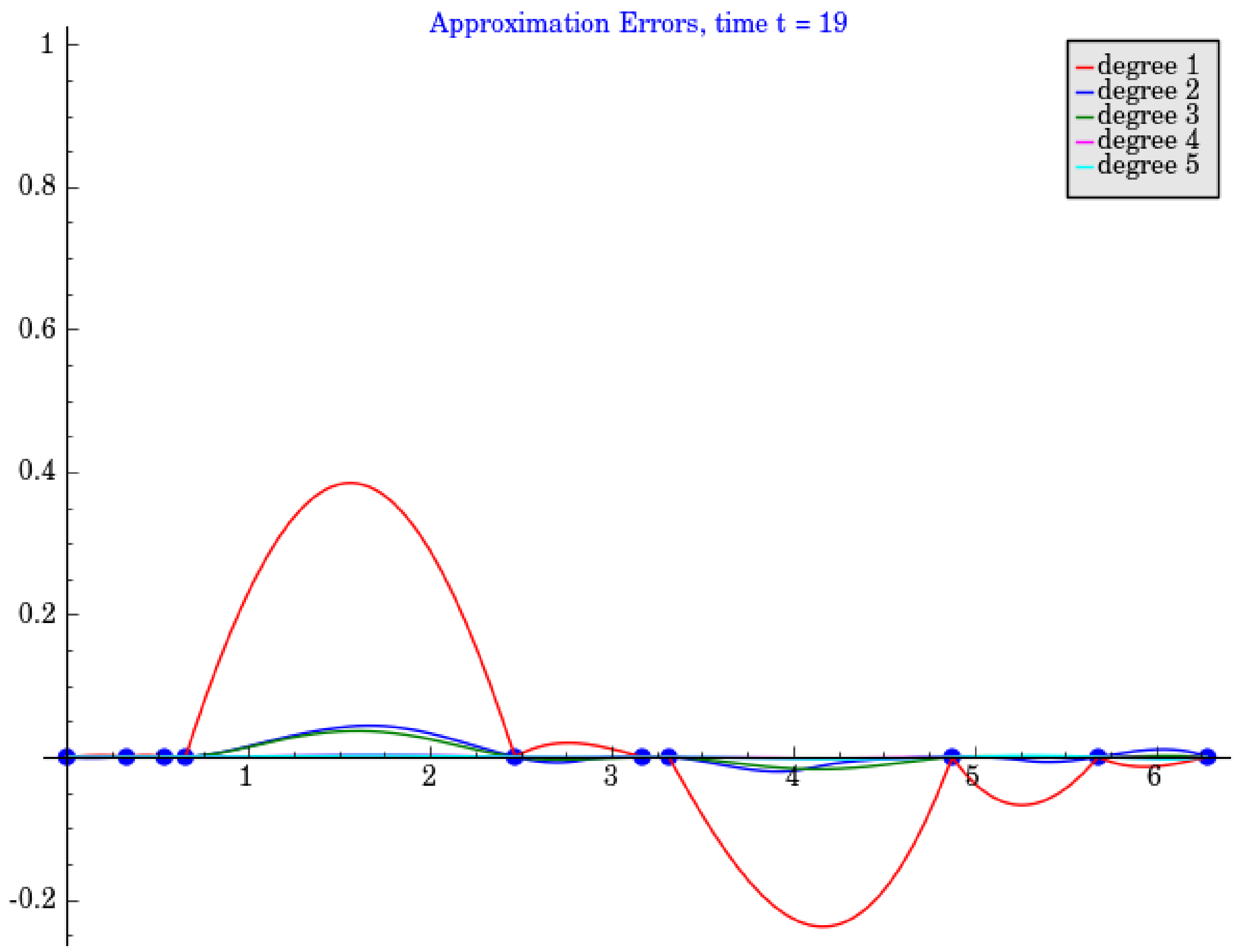
Approximation Errors, time t = 17
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
-0.4
1
2
3
4
5
6


Approximations, time t = 18
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6

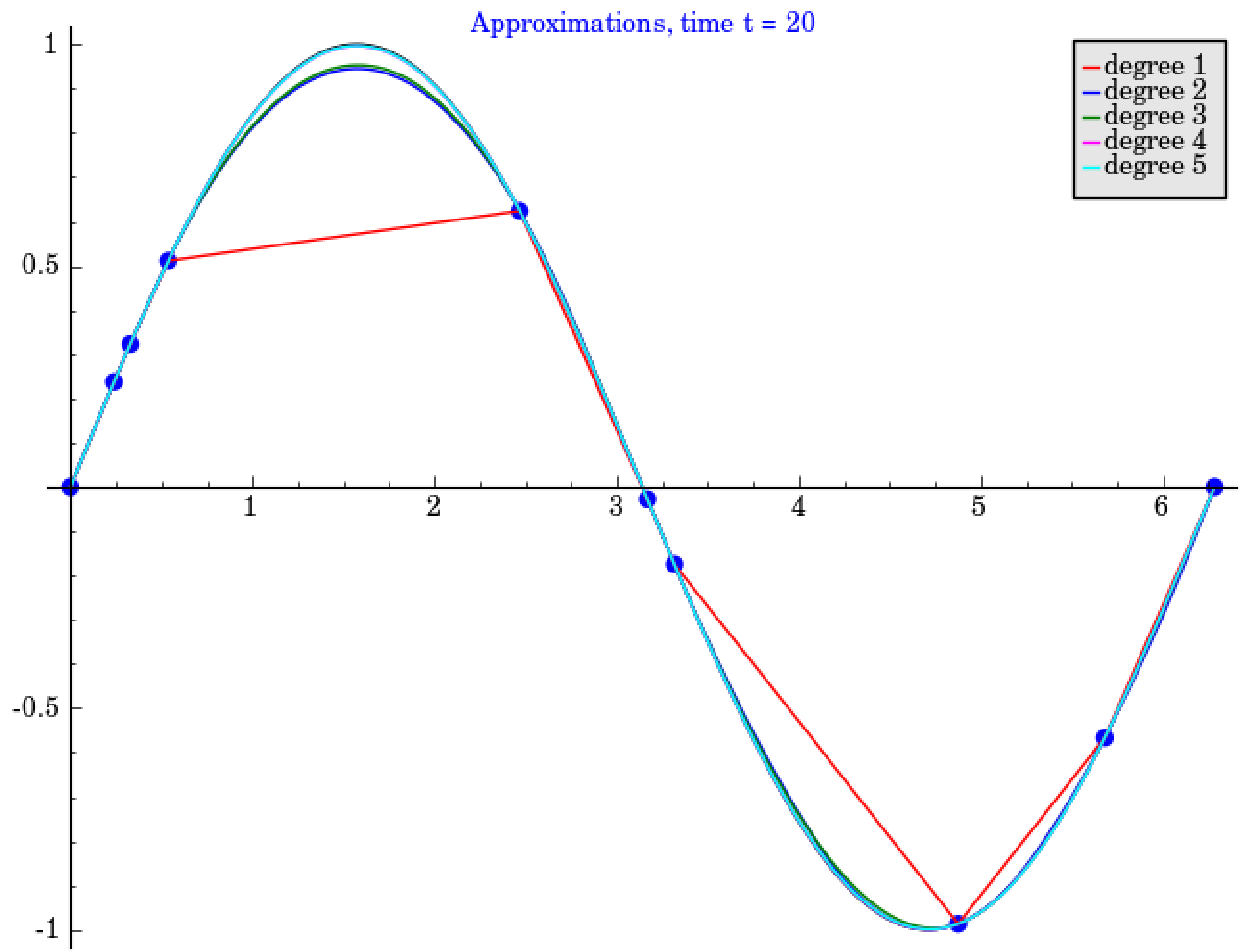


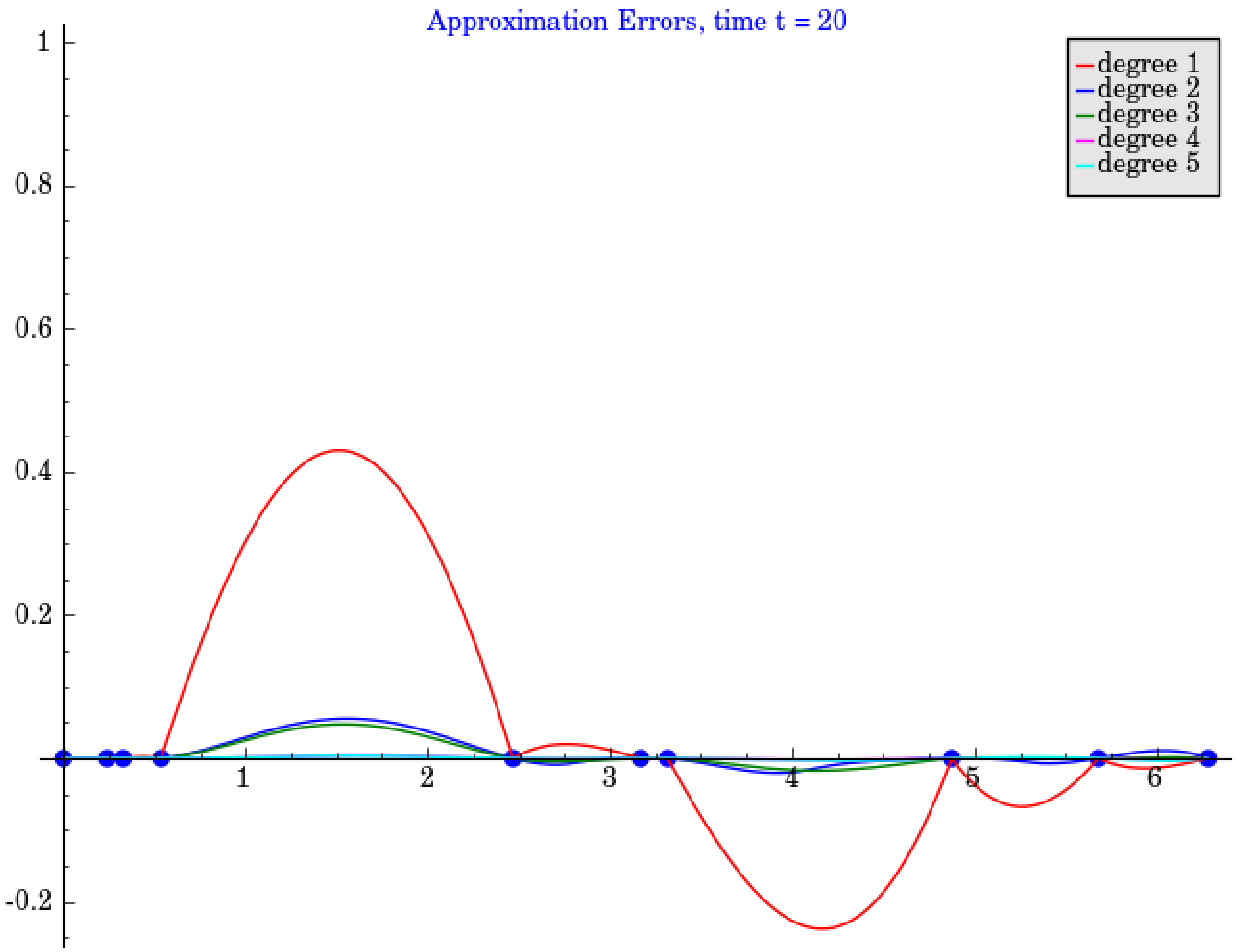
Approximation Errors, time t = 18
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
1
2
3
4
5
6


Approximations, time t = 19
1
0.5
-0.5
-1
1
2
3
4
5
6
degree 1
degree 2
degree 3
degree 4
degree 5

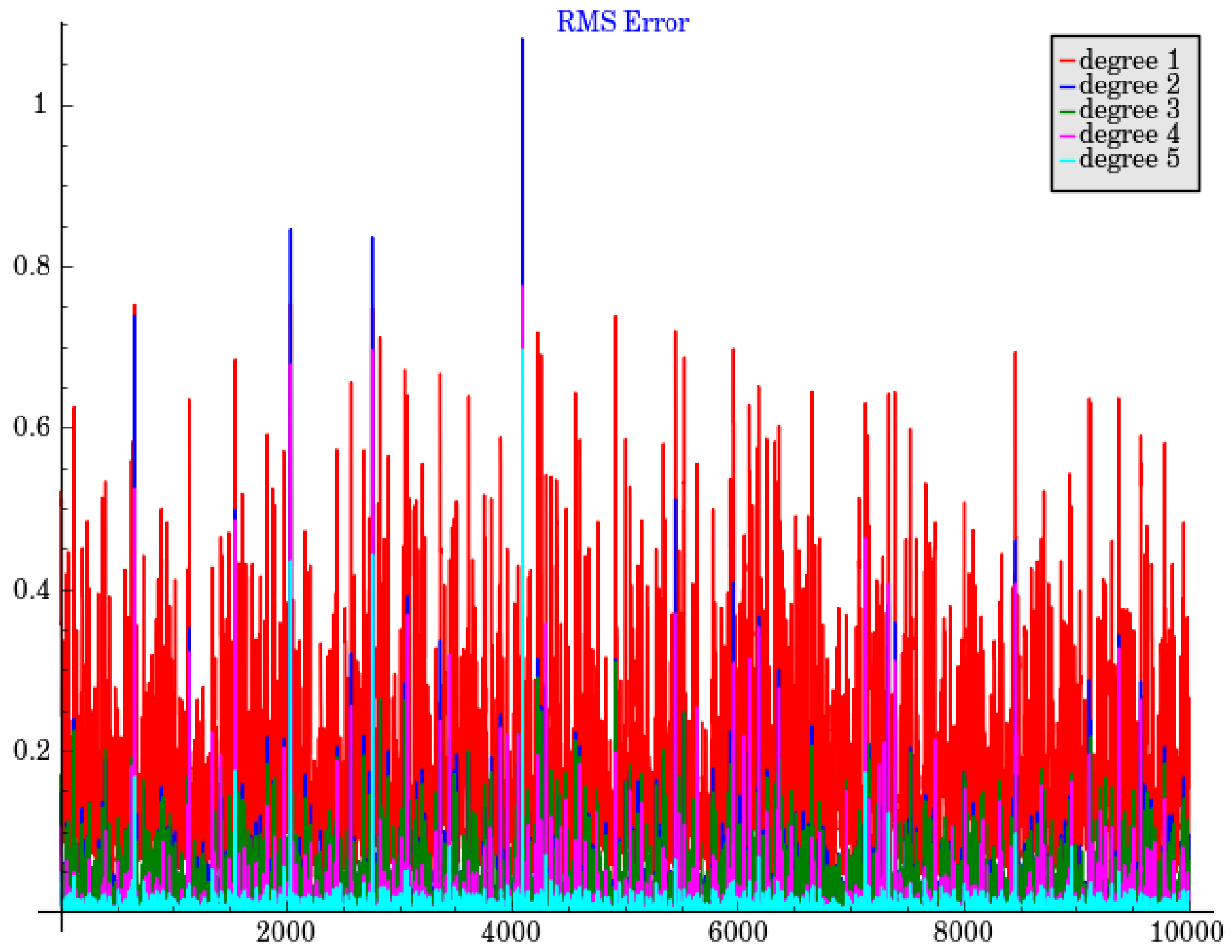


Approximation Errors, time t = 19
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
1
2
3
4
5
6

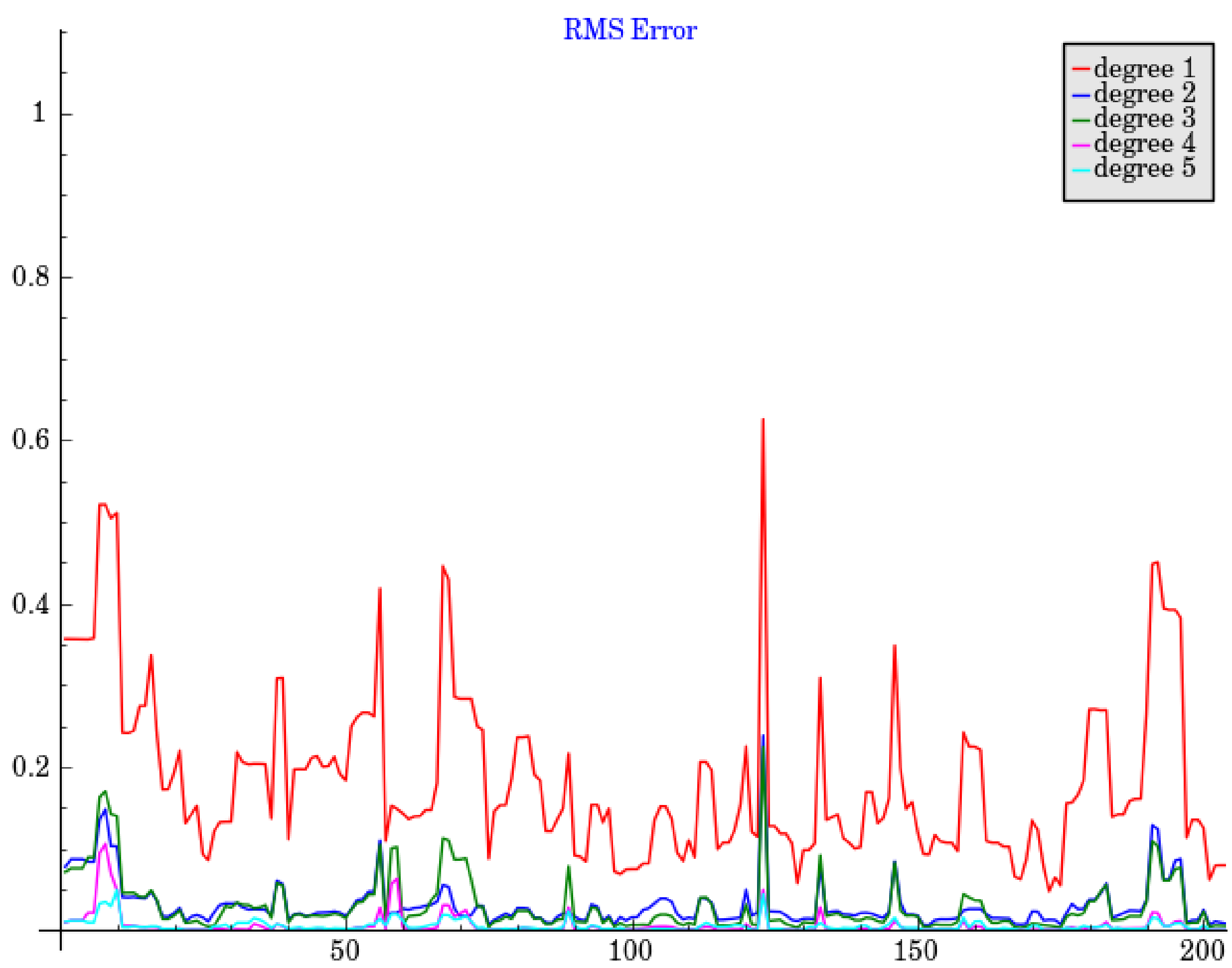

Approximations, time t = 20
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.5
-0.5
-1
1
2
3
4
5
6



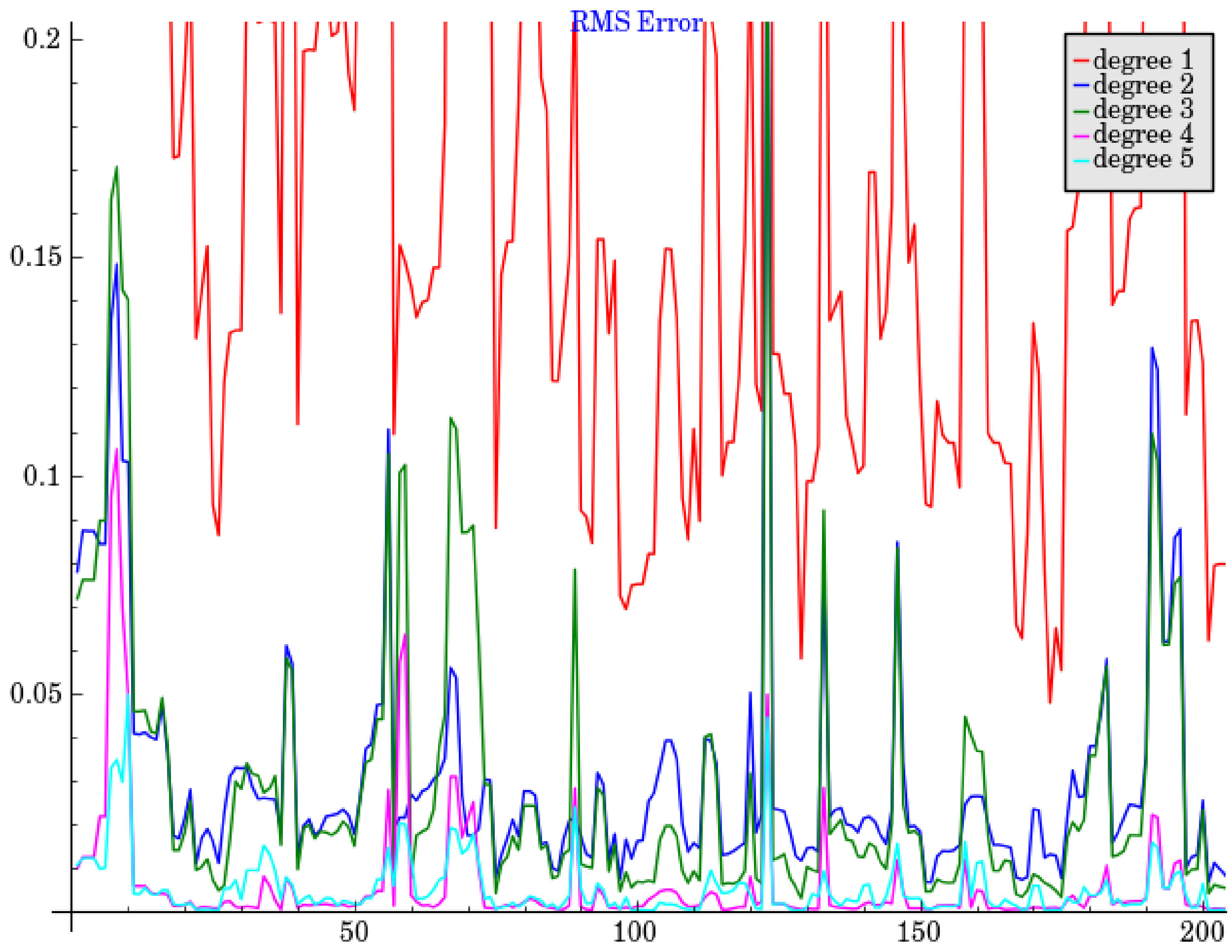
Approximation Errors, time t = 20
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
-0.2
1
2
3
4
5
6



RMS Error
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
2000
4000
6000
8000
10000



RMS Error
degree 1
degree 2
degree 3
degree 4
degree 5
1
0.8
0.6
0.4
0.2
50
100
150
200



RMS Error
degree 1
degree 2
degree 3
degree 4
degree 5
0.2
0.15
0.1
0.05
50
100
150
200

Discretization of RMS to Define Process Alphabets

- 11 symbol alphabet

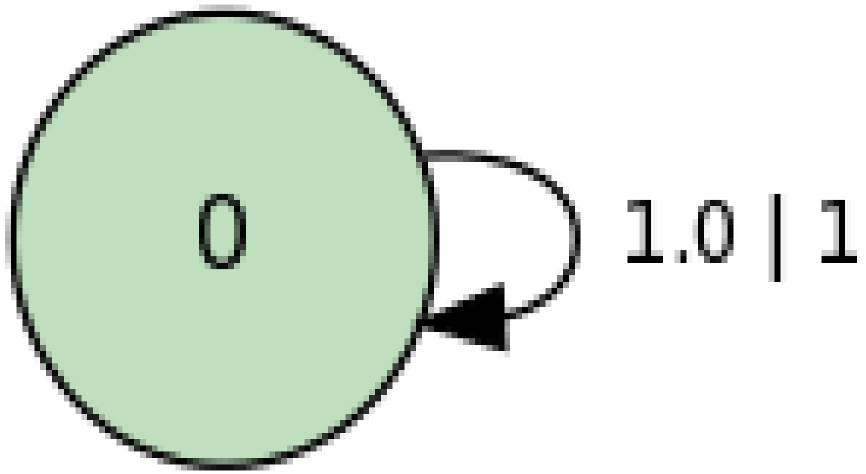
```
def get_symbol(RMS):
    if 0.000 <= RMS < 0.025 : return 0
    elif 0.025 <= RMS < 0.050 : return 1
    elif 0.050 <= RMS < 0.075 : return 2
    elif 0.075 <= RMS < 0.100 : return 3
    elif 0.100 <= RMS < 0.125 : return 4
    elif 0.125 <= RMS < 0.150 : return 5
    elif 0.150 <= RMS < 0.175 : return 6
    elif 0.175 <= RMS < 0.200 : return 7
    elif 0.200 <= RMS < 0.225 : return 8
    elif 0.225 <= RMS < 0.250 : return 9
    elif 0.250 <= RMS : return 10
    else : raise Exception('Range Error')
```

- Binary alphabet: Threshold of RMS = 0.01

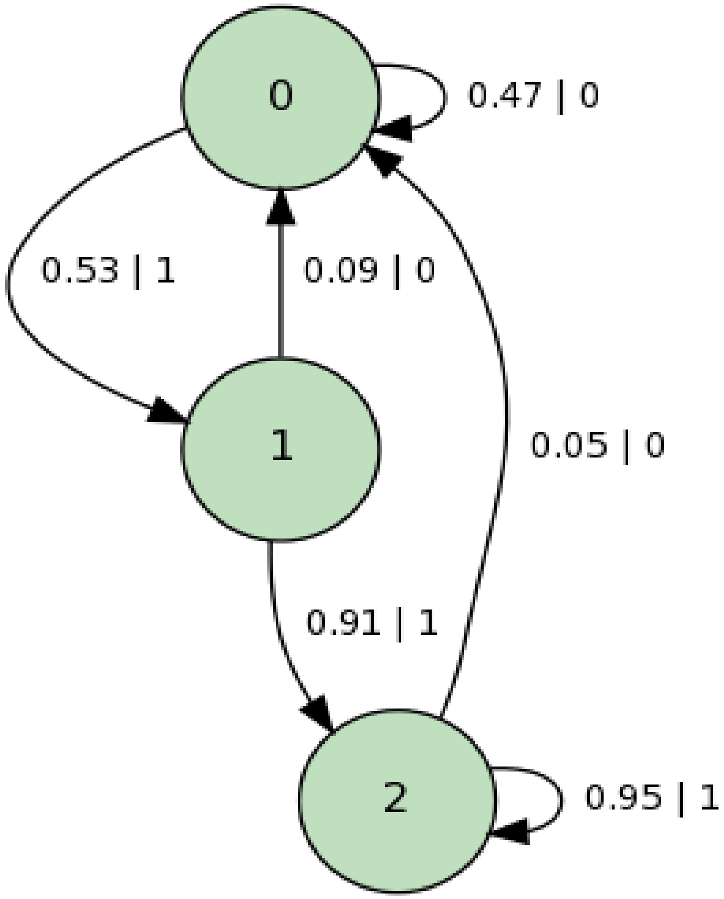
Preliminary Results

- Could not get epsilon machine inference to work for the 11 symbol alphabet
- For the discretized RMS bit string output, epsilon machines can be obtained

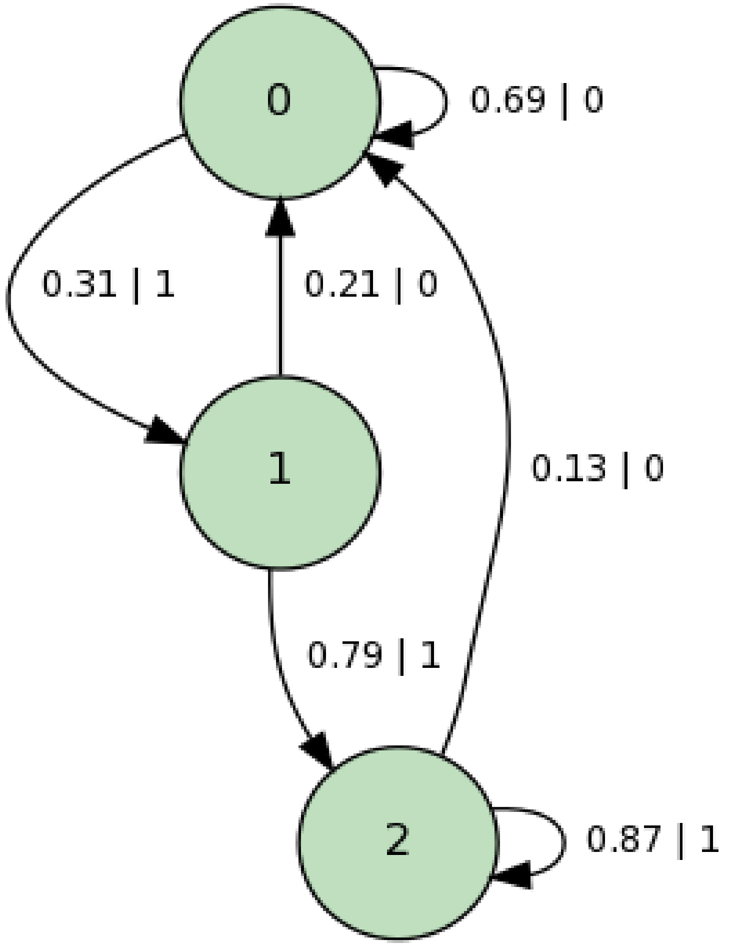
Epsilon Machine for degree=1



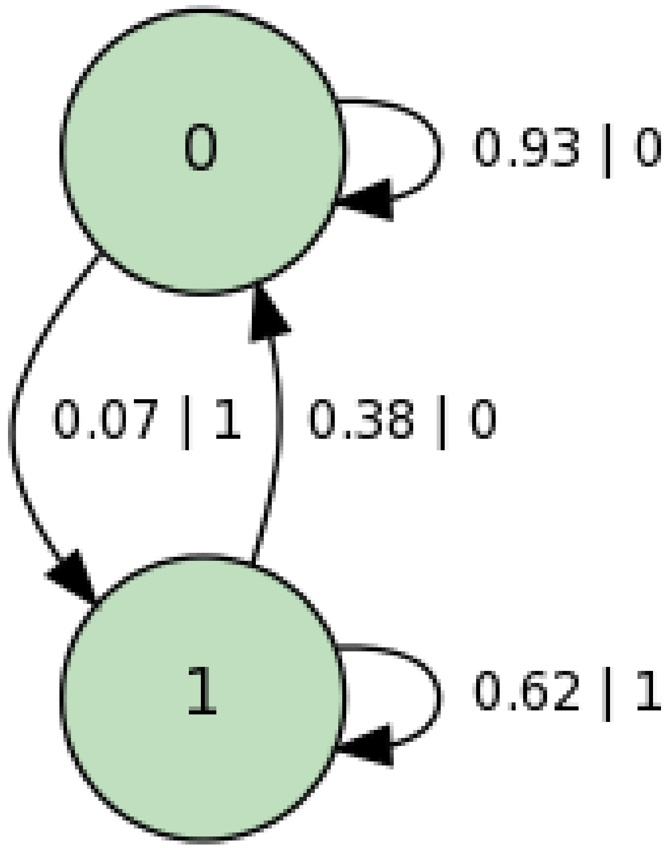
Epsilon Machine for degree=2



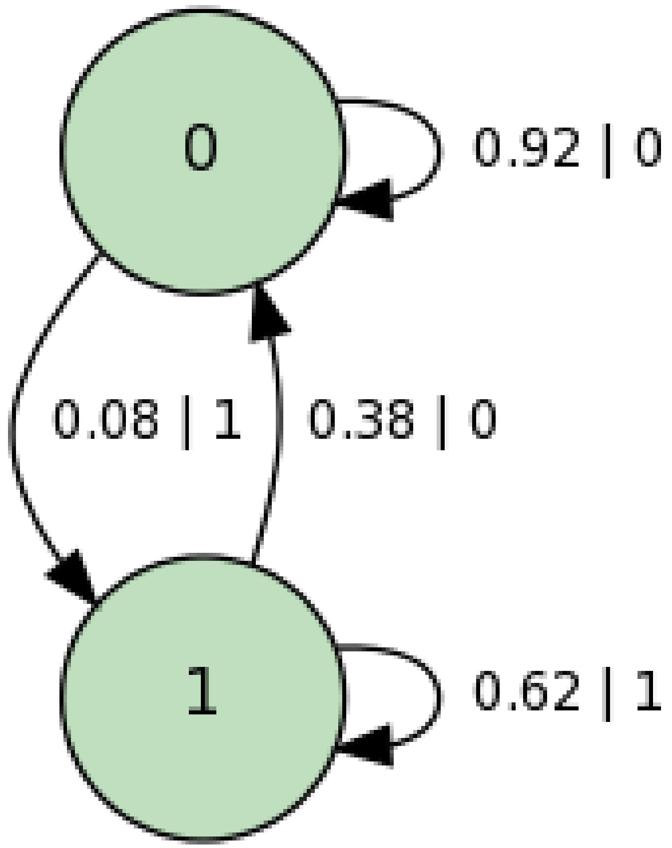
Epsilon Machine for degree=3



Epsilon Machine for degree=4



Epsilon Machine for degree=5



Entropy Rate, Statistical Complexity, and Excess Entropy

degree 1

	hμ	Cμ	E
Inferred Machine	0.00000	0.00000	0.00000

degree 2

	hμ	Cμ	E
Inferred Machine	0.36477	0.71544	0.07694

degree 3

	hμ	Cμ	E
Inferred Machine	0.67455	1.29305	0.21980

degree 4

	hμ	Cμ	E
Inferred Machine	0.46150	0.63045	0.16895

degree 5

	hμ	Cμ	E
Inferred Machine	0.49296	0.66112	0.16816

In progress / To do

- Try to get results with a non-binary alphabet
- Try different discretization thresholds
- Try different functions
- Make sense of the results?
- Investigate spatial correlations?
- Try other approximation methods
- Try higher dimensions