# Computational Mechanics of the Two Dimensional BTW Model

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**Abstract**

Some aspects of computational mechanics in two dimensions are investigated in this project. The specific model studied is the two dimensional Bak-Tang-Wiesenfeld model, a model that exhibits self-organized critical behavior. This particular model was chosen for its simple definition, allowing us to focus on the issues involved with the computational mechanics in two dimensions. Some of the problems encountered were issues with a large/non-binary alphabet, ambiguities in defining block entropy in two dimensions, etc.

# 1 Introduction

Causal Dynamical Triangulations (CDT) [1],[2] is a fairly recent approach to quantizing gravity. CDT is a non-perturbative approach based on the gravitational path integral. The basic idea is to approximate smooth manifolds using simplicial complexes, an idea originally proposed by Regge as technique for discretizing general relativity [7]. This simplicial approximation is then used to discretize the Einstein-Hilbert action, transforming the gravitational path integral into a state sum over all possible random triangulations. The space of all possible triangulations is sampled using the importance sampling technique, each triangulation being weighted according to its effect on the discretized Einstein-Hilbert action.

CDT has many novel and interesting properties and results. From the computational mechanics point of view, the most important aspect of the CDT model is the existence of a phase structure and the emergence of the extended physical phase when the values of the (bare) Newton's constant and (bare) cosmological constant are suitably chosen. A snapshot of the 2+1 dimensional spacetime in the extended phase is shown in figure 1. Starting from some arbitrary initial conditions, the

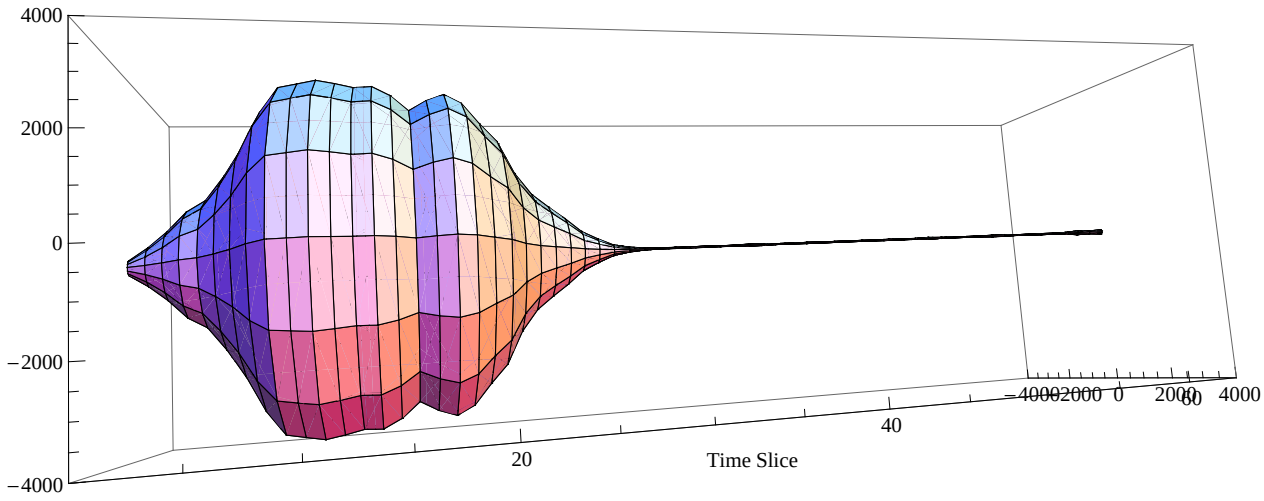


Figure 1: A snapshot of the spacetime in the extended phase.

system gradually settles into the extended geometric phase, and once in this phase, it stays there. This process is reminiscent of a machine going through a series of transient states and then settling into a sequence of one or more recurrent states. Hence the primary motivitation for this project was to investigate if there is some addtional structure to the CDT model, structure beyond what was put in to formulate the model.

While CDT is an interesting model that is worth exploring from a computational mechanics point of view, it is also an extremely challenging model for a couple of reasons. First, the model, even in its simplest 1+1 dimensional formulation, is a spatially extended system. The second problem, which is a more challenging one, is the lack of a fixed lattice in the CDT model. This is a good thing as far as quantizing gravity is concerned, but poses a challenge for computational mechanics.

With these issues in mind, I decide to analyze a simpler model, as a way to familiarize myself with computational mechanics in spatially extended and other higher dimensional systems. The model I decided to study was the two dimensional Bak-Tang-Wiesenfeld (BTW) model, a sandpile model that exhibits self-organized behavior. Like CDT, the BTW model is also a spatially extended model. But unlike CDT, the BTW model is formulated on a fixed lattice. Unfortunately, the BTW

model has a non-binary alphabet, which makes the analysis more difficult.

## 2 The Dynamical System

The BTW model, originally proposed in [3], is a sandpile model that exhibits self-organized criticality. In this model, we start with an initial random distribution of grains of sand on a two dimensional grid. When the number of grains in a cell exceeds a certain threshold value, the cell relaxes by "toppling", i.e. by losing grains of sand to its nearest-neighbor cells. As the system is driven by adding grains of sand, the number of sites that topple upon the addition of a single grain of sand diverges with the system size [4].

### 2.1 Algorithm

In two dimensions, we use a square lattice, $(i, j), 1 \leq i, j \leq L$ and a grain of sand is added to a random site $(i, j)$. If the "height" of the sand at this site exceeds a threshold value $h_{th}$, the site relaxes by distributing 4 grains to its 4 nearest neighbors.

1. Initialize each site in the lattice to a random value $s_{i,j} \in \{0, 1, \cdots, h_{th}\}$.
2. Record the state of the system by recording the number of grains in each site of the lattice.
3. Select a random site $(i, j)$ where $1 \leq i, j \leq L$
4. Add a grain of sand, $h(i, j) \rightarrow h(i, j) + 1$
5. If $h(i, j) > h_{th}$ relax site $(i, j)$:

$$\begin{aligned} h(i, j) &\rightarrow h(i, j) - 4 \\ h(i \pm 1, j) &\rightarrow h(i \pm 1, j) + 1 \\ h(i, j \pm 1) &\rightarrow h(i, j \pm 1) + 1 \end{aligned} \tag{1}$$

6. Scan the grid for any further sites that need to be relaxed. Continue relaxation till there are no further sites to be relaxed. The total number of sites that have been relaxed is called the "avalanche size".
7. Record the state of the system by recording the number of grains in each site of the lattice.
8. Return to step 3.

When sites on the edge of the grid relax, grains of sand leave the system. We choose $h_{th} = 3$, and record the state of the system only when all sites have relaxed. Thus the model can be described by an alphabet $\mathcal{A} = \{0, 1, 2, 3\}$. Figure 2 is a snapshot of a BTW sandpile during relaxation.

## 3 Methods

A one dimensional spatially non-extended system can be studied from an information theoretic point of view by analyzing the Shanon entropy of a sequence of $L$ random variables $S^L \equiv S_1 S_2 \cdots S_L$. This "block entropy" is defined as

$$H(L) = -\sum_{s^L \in \mathcal{A}^L} Pr(s^L) \log_2 Pr(s^L) \tag{2}$$

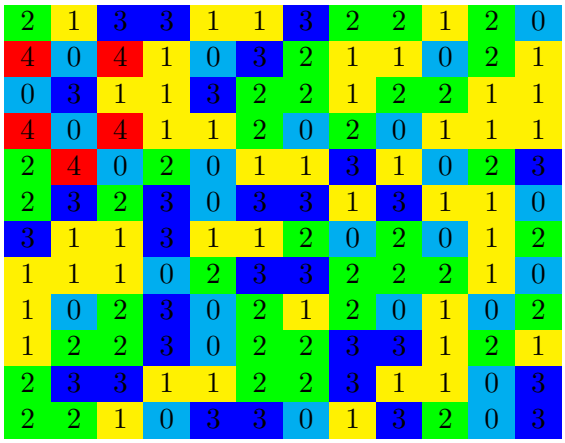


Figure 2: A snapshot of the BTW model with five sites (red) about to relax.

The entropy rate is then defined as

$$h_\mu = \lim_{L\to\infty} \frac{H(L)}{L} \tag{3}$$

These ideas can be extended to two dimensions in a straightforward way. The block entropy can be defined for a $M \times N$ block as ([6],[5])

$$H(M,N) = -\sum_{\mathcal{B}} Pr(\mathcal{B}^{\mathcal{M},\mathcal{N}}) \log_2 Pr(\mathcal{B}^{\mathcal{M},\mathcal{N}}) \tag{4}$$

where

$$\mathcal{B}_{i,j}^{M,N} \equiv \begin{matrix} S_{i,j} & S_{i,j+1} & S_{i,j+2} & \cdots & S_{i,j+M-1} \\ S_{i+1,j} & S_{i+1,j+1} & S_{i+1,j+2} & \cdots & S_{i+1,j+M-1} \\ S_{i+2,j} & S_{i+2,j+1} & S_{i+2,j+2} & \cdots & S_{i+2,j+M-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{i+N-1,j} & S_{i+N-1,j+1} & S_{i+N-1,j+2} & \cdots & S_{i+N-1,j+M-1} \end{matrix} \tag{5}$$

The entropy rate is defined as

$$h_\mu = \lim_{M,N\to\infty} \frac{H(M,N)}{MN} \tag{6}$$

The above extensions are the simplest way of doing two dimensional computational mechanics. The two dimensional blocks could be two spatial dimensions at an instant in time, as in the case of the BTW model, or they could be one temporal and one spatial dimension; i.e. the time evolution of a one dimensional spatial system such as one dimensional spin chain.

Equations (4), (5), (6) already bring to light a problem with two dimensional computational mechanics — the exponential growth in the number of possible values for $\mathcal{B}_{i,j}^{M,N}$. Table 1 illustrates this problem for a system with a four symbol alphabet. So a four symbol alphabet, which would not be considered a large alphabet, is already causing serious data collection problems.

Another problem is the sequence in which the $M \times M$ block is assembled. Different choices might yield different results, and any arbitray choice might impose addtional structure onto the system.

In one dimension, entropy rate $h_\mu$ can also be defined as the limit of a conditional entropy. First we define the entropy of a single site conditioned on a block of $L-1$ adjacent sites ([6])

$$h_\mu(L) \equiv H[S_L | S_{L-1}, S_{L-2} \cdots S_1] \tag{7}$$

| $M$ | Number of unique possible $M \times M$ blocks |
|---|---|
| 1 | 4 |
| 2 | 256 |
| 3 | 262144 |
| 4 | 4294967296 |

Table 1: Number of possible $M \times M$ blocks for 4 symbol alphabet

The entropy rate is then defined as

$$h_\mu = \lim_{L \to \infty} h_\mu(L) \tag{8}$$

In two dimensions, conditioned block entropy can be defined in multiple ways. In this project, I looked at two of the techniques suggested in [6]. In the first technique, $h_\mu(M)$ is the Shannon entropy of a lattice site conditioned on a $(M+1)\times(2M+1)$ two dimensional neighborhood template. This is illustrated in figure 3. The two dimensional entropy rate is then defined as before

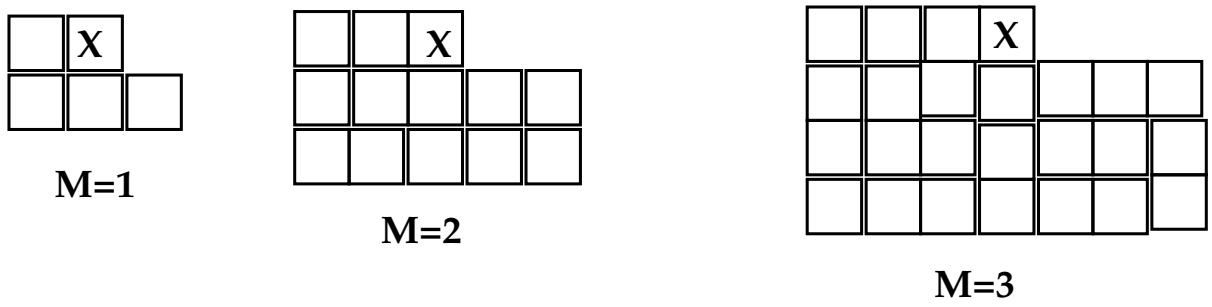


Figure 3: First version of a 2D conditional entropy template.

$$h_\mu = \lim_{M \to \infty} h_\mu(M) \tag{9}$$

Table 2 lists the number of possible unique blocks as a function of $M$ for a system with a four symbol alphabet.

| $M$ | Number of unique possible conditioned blocks |
|---|---|
| 1 | 256 |
| 2 | 16777216 |
| 3 | 281474976710656 |

Table 2: Number of possible conditioned blocks for 4 symbol alphabet

For systems with only nearest neighbor interactions, such as the BTW model, the following conditioned block entropy template can be used instead. The numbers in figure (4) indicate the sequence in which the elements of the conditioned block are assembled into a one dimensional string that is stored in a data structure. The rule followed is that sites closer (as defined by their Euclidean distance) to the target site are added first, and in case of a tie, the leftmost site is added first. Table 3 lists the number of possible unique blocks as a function of $M$ for a system with a four symbol alphabet. This version of the conditioned block has fewer unique possible values compared to the first version.

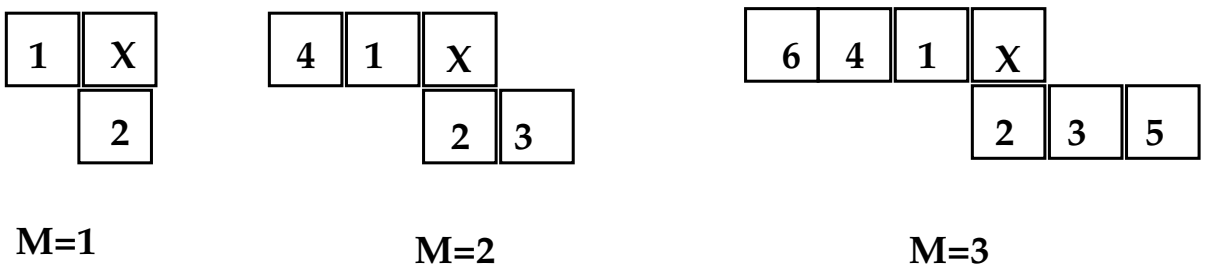


Figure 4: Second version of a 2D conditional entropy template.

| $M$ | Number of unique possible conditioned blocks |
|---|---|
| 1 | 16 |
| 2 | 256 |
| 3 | 4096 |

Table 3: Number of possible conditioned blocks for 4 symbol alphabet

## 4 Results

I first studied the block entropy of 1 dimensional blocks in the grid. To do this, I scanned the grid, horizontally and vertically, and calculated $Pr(s^L)$ for $L = 1, 2, \cdots, 32$. The results are plotted in figure 5. We see that block entropy increases linearly upto $L = 12$ and then stays constant. This

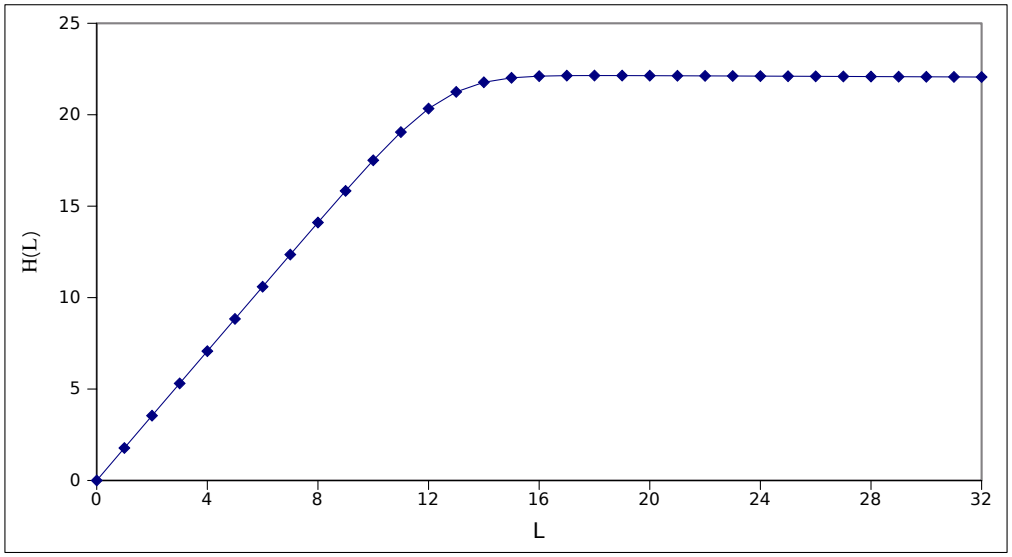


Figure 5: Block entropy of 1D blocks measured on a $256 \times 256$ grid.

could either mean that $L = 13$ onwards the system is completely predictable, or, more likely, we don't have sufficient data to carry out this analysis. $4^{13} = 67108864$ requires a large sample space for reliable analysis.

Next, the block entropy of two dimensional square blocks is studied, by calculating $Pr(s_{i,j}^{MM})$ for $M = 1, 2, \cdots, 16$. This is shown in figure 6. Once again we see that the block entropy increases linearly, upto $M = 4$ and then stays constant. This too might be most likely due to insufficient

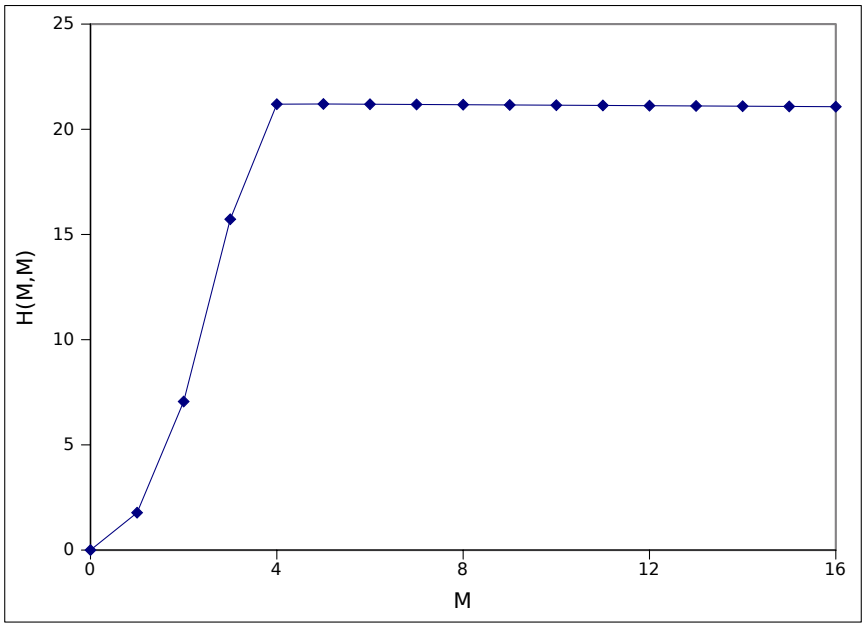


Figure 6: Block entropy of 2D blocks measured on a $256 \times 256$ grid.

data, since there are $4^{25} = 1125899906842624$ unique $5 \times 5$ blocks.

Figure 7 shows the conditioned block entropy $h_\mu(M)$as function of block size $M$. The conditioned blocks specialized for nearest neighbor only interactions (figure 4) are used for this calculation. Conditioning reduces entropy, so the decrease in $h_\mu(M)$ as $M$ increases is a good thing, but the abrupt drop in $h_\mu(M)$ at $M = 4$ is again probably due to insufficient data. If interpreted literally, this plot is suggesting that the entropy rate $h_\mu$ is 0, which would indicate a perfectly predictable system!.

## 5 Conclusions

The primary goal of this project was to get a basic feel for computational mechanics in two dimensions. While I feel this goal was achieved, the choice of the two dimensional BTW model was not a good one. The primary reason is that alphabet size for the BTW is too large — a system with a binary alphabet would have been a better choice. Another problem with the two dimensional BTW model is that it extends over two spatial dimensions. A system that extends over a single spatial dimension, such as a spin chain or 1D cellular automata, would have been a better choice for studying two dimensional computational mechanics.

What about computational mechanical analysis of CDT? Based on what I have learnt in this project, it would make sense to first analyze the $1+1$ dimensional CDT model. This model can be described by a binary alphabet, since there only two possible types of simplices in $1+1$ dimensions, an "up triangle" and a "down triangle". The system extends over a single spatial dimension, hence the dynamical system would be a two dimensional system. However, the lattice is itself dynamical, which should make the computational mechanical analysis interesting.

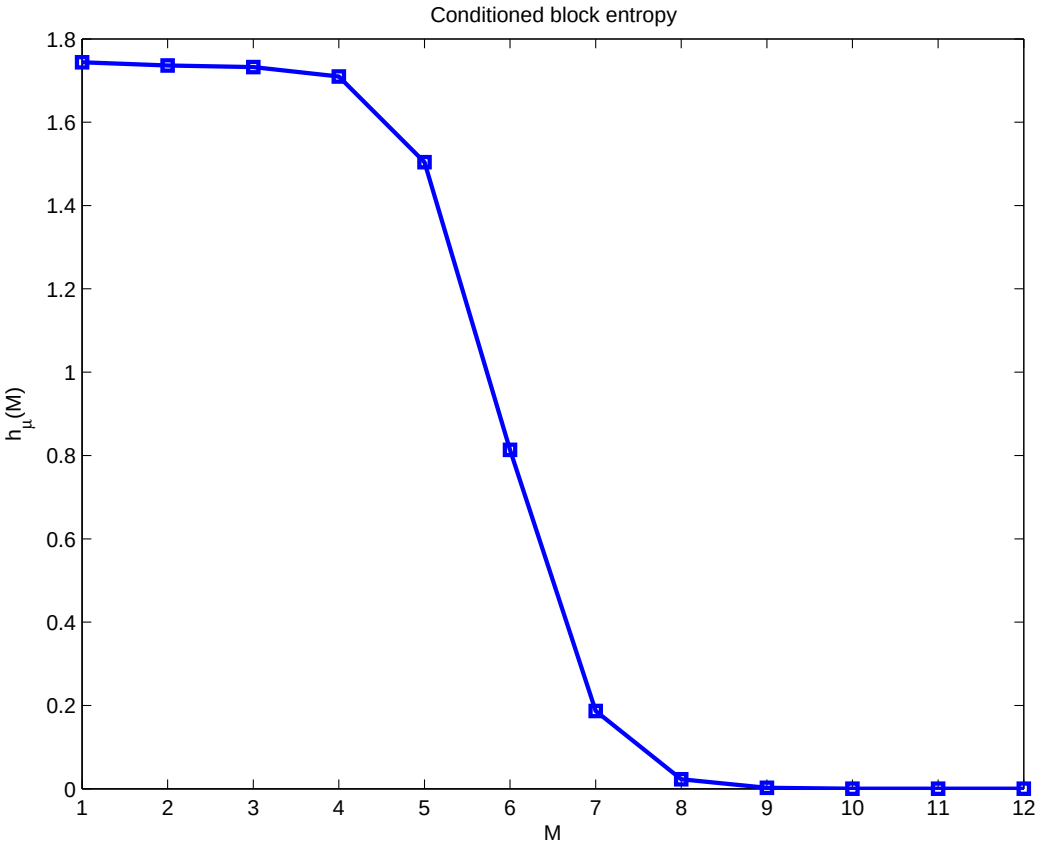


Figure 7: Conditioned block entropy using the specialized 2d conditioned blocks.

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