An algebraic perspective on topological ϵ-machines

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Motivation

- Algebraic decomposition of an ϵ-machine
- Algebraic picture of possibility reduction
- Algebraic characterization of ϵ-machine optimality

Semigroups

Definition

A semigroup is a pair $(\mathbf{S}, *)$ where

- $\mathbf{S}$ is a set
- $* : \mathbf{S} \times \mathbf{S} \to \mathbf{S}$ is a binary operation
- $*$ is associative: $a * (b * c) = (a * b) * c$

Definition

An element $e \in S$ is an identity element if $e * s = s * e = s$ for all $s \in S$.

Definition

A monoid is a semigroup with an identity element.

Example 1: Free monoid

Alphabet: $\mathcal{A}$

Set of words: $\mathcal{A}^*$

Free monoid on $\mathcal{A}$: $\mathbf{M} = (\mathcal{A}^*, *)$

- $*$ is word concatenation

$$x_0 \cdots x_m * y_0 \cdots y_n = x_0 \cdots x_m y_0 \cdots y_n$$

- empty word λ is the identity element

$$\lambda * w = w * \lambda = w$$

Example 2: $\mathbf{PF}(X)$

Monoid of partial functions on X: $(\mathbf{PF}(X), \circ)$

- $\circ$ is restricted function composition

$$(f \circ g)(x) = y \iff f(g(x)) = y$$

- Identity function $1_X(x) = x$ is the identity element

$$1_X \circ f = f \circ 1_X = f$$

Topological ϵ-machines

Definition

A topological ϵ-machine $\mathcal{M}$ is a 3-tuple $(\boldsymbol{\mathcal{S}}, \mathcal{A}, T)$ where

- $\boldsymbol{\mathcal{S}}$ is a set of states
- $\mathcal{A}$ is an output alphabet
- $T : \boldsymbol{\mathcal{S}} \times \mathcal{A}^* \to \boldsymbol{\mathcal{S}}$ is a transition (partial) function
- $\mathcal{M}$ is connected and non-halting.

Notation: $T(\mathcal{S}, w) = \mathcal{S}T^w$

Monoid of a machine (via word equivalence)

Start with the free monoid $\mathcal{A}^*$. Define an equivalence relation on words:

$$u \sim w \iff T^u = T^w$$

Definition

The monoid $\underline{\mathbf{M}(\mathcal{M})}$ of a machine $\mathcal{M}$ is $(M, *)$ where

- $M = \mathcal{A}^*/\sim = \{[w] \mid w \in \mathcal{A}^*\}$
- $[u] * [w] = [uw]$
- $[\lambda] * [w] = [w] * [\lambda] = [w]$

Monoid of a machine (via transformations)

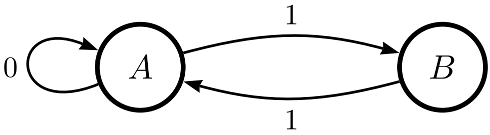
Definition

The monoid $\underline{\mathbf{M}(\mathcal{M})}$ of a machine $\mathcal{M}$ is $(M, *)$ where

- $M = \{T^w \mid w \in \mathcal{A}^*\}$
- $T^u * T^w = T^{uw} = T^u \circ T^w$
- $T^\lambda * T^w = T^w * T^\lambda = T^w$

The two monoids are isomorphic through $[w] \mapsto T^w$

Example



$*$	λ	1	0	01	10	010	101
λ	λ	1	0	01	10	010	101
1	1	λ	10	101	0	010	01
0	0	01	0	01	010	010	010
01	01	0	010	010	0	010	01
10	10	101	10	101	010	010	010
010	010	010	010	010	010	010	010
101	101	10	010	010	10	010	101

Table: Multiplication table of $\mathbf{M}(\mathcal{M})$

Forbidden words and zeros

Definition

An element z in a semigroup $\mathbf{S}$ is a zero if for all $s \in \mathbf{S}, z * s = s * z = z$.

Definition

A word w is a forbidden word if $T^w = 0_{\mathcal{S}}$.

Proposition

[1]If the set of forbidden words is nonempty it is equal to the unique zero of $\mathbf{M}(\mathcal{M})$. If there are no forbidden words then $\mathbf{M}(\mathcal{M})$ does not contain a zero.

[1]True with the addition of one condition

Idle words and identity elements

Definition

A word w is an idle word if $T^w = 1_{\mathcal{S}}$.

Proposition

The set of idle words is equal to the unique identity element of $\mathbf{M}(\mathcal{M})$.

Synchronizing words and ideals

Definition

A word w is a synchronizing word if $|\mathcal{S}T^w| = 1$.

Definition

A set I of a semigroup $\mathbf{S}$ is an ideal if for all $s \in \mathbf{S}$, $sI \subseteq I$ and $Is \subseteq I$.

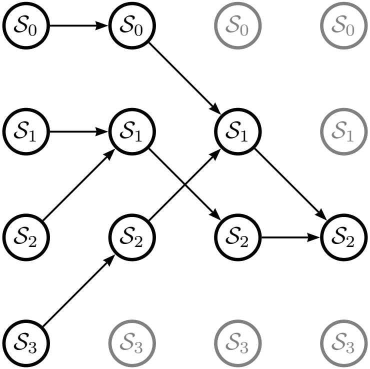
Proposition

Define

$$I_{\mathcal{M}} \equiv Sync(\mathcal{M}) \bigcup Forbid(\mathcal{M}) \qquad I_{\mathbf{M}(\mathcal{M})} \equiv \pi(I_{\mathcal{M}})$$

where π is the projection map $\pi(w) = [w]$. Then $I_{\mathcal{M}}$ is an ideal of $\mathcal{A}^*$ and $I_{\mathbf{M}(\mathcal{M})}$ is an ideal of $\mathbf{M}(\mathcal{M})$.

Synchronization



Inert words and filters

Definition

Let $|\mathcal{M}| = n$. A word w is an inert word if $|\boldsymbol{\mathcal{S}}T^w| = n$.

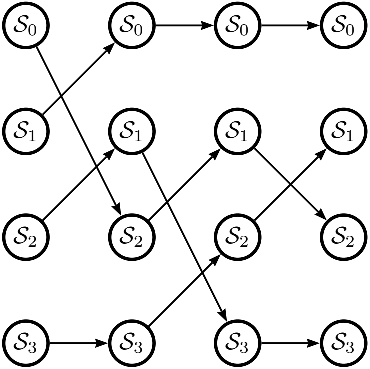
Definition

A subset F of a language $\mathcal{L}$ is called a filter if F is closed under taking subwords.

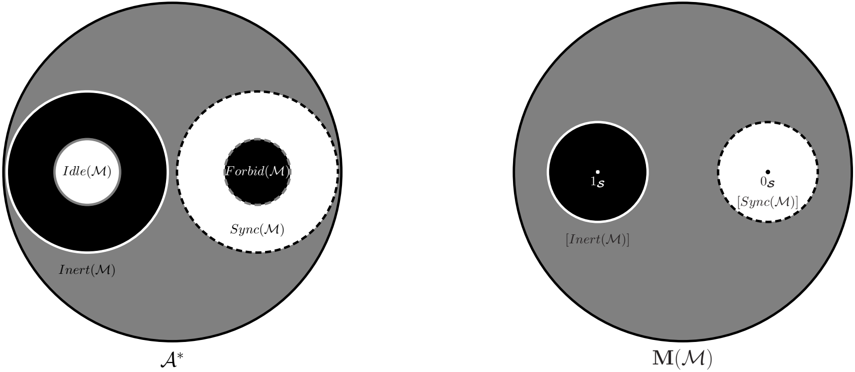
Proposition

The set of inert words is a filter of $\mathcal{A}^*$.

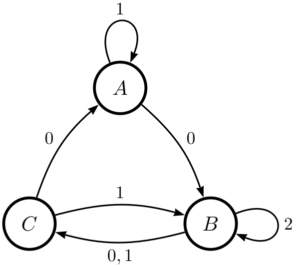
Inert transformation



An algebraic picture



Example



word set	contains
$Forbid(\mathcal{M})$	$\{*202*\}$
$Sync(\mathcal{M})$	$\{*2*\}$
$Idle(\mathcal{M})$	$\{(000)^n\}$
$Inert(\mathcal{M})$	$\{0,1\}^*$

Monoid hierarchy

- Q: What about the gray monoid stuff?
- A: Can be filled in with a hierarchy of words.
  - Organized by the degree to which they reduce uncertainty.
- Result: A new picture of synchronization.
- Check out written report.

Future directions

1. Consider the probabilistic monoid of the full ϵ-machine.
2. Algebraically characterize the optimality of ϵ-machines.
3. Approach to hierarchical ϵ-machine reconstruction.