Logistic Map: Ecologically Considered

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1 Introduction

The logistic map has been a heavily studied system for its fascinating dynamical behavior. However, despite its original development in ecology, the analysis, particularly in the context of information theory, has been seldom conducted through an ecological perspective with sufficient natural realism. Under modern quantitative advancements, it is, nonetheless, important to reassess some of the map's foundational assumption in order to properly navigate, if not reconstruct, its future research direction. This report briefly reviews the model in biological terms and attempts to expose some of the complications involved.

1.1 Logistic Map

$$x_{n+1} = f(x_n) = rx_n(1 - x_n)$$
$$x_n \in [0, 1]$$
$$r \in [0, 4]$$

The classic logistic map equation seen here is taken from the non-dimensionalized form of the theoretical ecology model that is used to analyze changes in population density over time. The *r* parameter relates to the intrinsic growth rate – or rate of offspring production – per individual. This is a value that can be measured easily either observationally or empirically. The time increment in the logistic map denotes non-overlapping generations instead of real time. With the intrinsic growth rate known, ecologists are often presented with statistical data of population density over a number of generations. The question for management strategy then becomes: what will the population size be in the next generation?

1.2 Ecological Meaning

In order to answer this practical concern, the usual mathematical approach to the problem of population density prediction using entropy or entropy rate as measurements fails to capture the environmental forcings and interactive complexity of ecological systems. Weather changes, nutrients cycling, and landscape transformations perpetually alter the local and regional dynamics upon which the studied species are governed by. More specifically, variations in moisture, temperature, and atmospheric chemical compositions directly impact the ecosystem's primary productivity (plants production), sometimes generating regime shift through its adaptive response to the external changes. Moreover, this effect trickles up the food web, strengthening or weakening the trophic interactions which potentially leads to extinction and reorganization of the linkages. This clearly has tremendous consequences for the inhabitants' density dynamics. Furthermore, spatial corridors can be created or destroyed within short time-scale due to both abiotic and anthropogenic influences. The resulting influx of invasive species and dispersal of native species can drastically transform the ecological compositions, thereby causing sudden population rises or crashes that are unforeseeable using the traditional logistic map analysis. Minor but frequent environmental perturbations, behavioral dynamics, and genetic drift are all factors neglected by the logistic map and their associated mathematical extrapolations. Therefore, acknowledging the model's questionable applications even under short, ecological time-scale, the study of its performance under extended, evolutionary time-scale – implicitly done through the entropy calculation of long block length – concludes patterns that do not reflect any aspect of the natural systems.

1.3 Transiency

Due to practicality and the demand of ecological realism, the entropy or entropy rate after lengthy iterations does not have great relevance to the management of natural systems. The state of species population is inherently difficult to predict due to the numerous external impacts it's continuously subjected to. Instead, control measures can be better directed by examining its transient dynamics since almost no ecosystem ever reaches the recurrent states. Because the logistic map relates present state and future state, short block length of symbolic sequence corresponds to few iterations of generations which then implies transiency. As a result, the case studies here focus on observed sequence of $L=1$.

1.4 Objective

Questions that are posed based on the above premise is: Given a local population that historically exhibits logistic trajectories, how does the accuracy of density measurement relate to the reliability of prediction for the following year? How is the outcome dependent upon the species life history which is characterized by its intrinsic growth rate, r? Since traditional data collecting technique such as catch-and-release and fur counts are monetarily costly and often yield misleading statistics, the minimum data sampling resolution must be decided upon, determined by the specific ecological problem that is being asked. The correct choice will reveal the species dynamics more realistically and lead to sound management strategies.

2 Method

An analytical instead of computational approach is taken here. In other words, the working data is not program-generated. Due to the adopted sequence length, the emitted symbolic data is limited and an analytical method provides greater precision in the results.

2.1 Entropy

In a sequence of $L=1$, there is no distinction between the value of entropy and entropy rate. To avoid confusion, this paper will evaluate the transient dynamics collectively using entropy rate. The entropy of a general length-L sequence, s^L, is calculated with the following equations

$$H(L) = -\sum_{s^L \in A} Pr(s^L) log_2(Pr(s^L))$$
$$H(0) = 0$$

where A is the alphabet.

2.2 Entropy Rate

The general equation for entropy rate is

$$h_\mu(L) = H(L) - H(L-1)$$

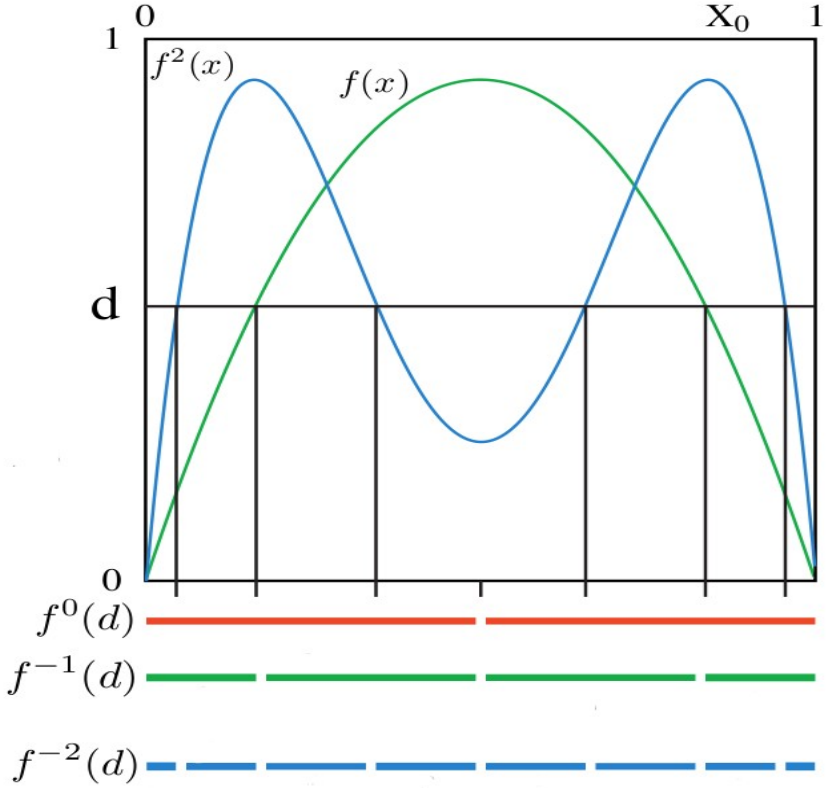
2.3 Partitioning

Partitioning is the process of translating a sequence of state data $\{x_n\}$ – i.e. numerical values of the population density for consecutive generations – to a new sequence of observed date $\{s_n\}$ from a finite alphabet of symbols.

$$x_n \in \mathbb{R}$$
$$s_n \in A \text{ where } |A| \text{ is finite}$$
$$s_n = o(x_n)$$

The data sampling resolution is analogous to the partition resolution. The finer the grain of partitioning upon the logistic map, the more numerous ranges population size can be “binned” within. As shall be soon illustrated, the life history of the studied species and the purpose of the research together dictate the optimal partition resolution to apply. There exist multiple types of partitions, including the highly revealing Markov partition and the specifically constructed generating partition. On account of the logistic map's geometric properties, Markov partition is not achievable for most of its parameter space. Hence, only the generating partition is considered here[1], employing the division points seen in the graph below.

1 In the follow-up paper (Celis 2009), equidistant partition will be investigated, qualitatively compared with the generating partition.



2.4 Partition Resolutions

Three separate generating partitions of size 2, 4, and 8 are presented here, along with their respective transient entropy rate. The division points are determined in accordance to the iterated trajectories of the map $f^m(x)$ where the slope is zero,

$$\frac{\partial f^m(x)}{\partial x} = 0$$

The relationship between the number of iterations m and the number of partitions n is $n = 2m$. For simplicity sake, the sampling data is assumed to be uniformly distributed along the state axis. That is to say, it is equally likely to find the population at any arbitrary density that is bounded by its carrying capacity. For example, a size-2 partition of alphabet {0, 1} consists of

$$P = \{0 \sim x \in [0, \frac{1}{2}], 1 \sim x \in (\frac{1}{2}, 1]\}$$

More generally, partition of size-n is represented by the following,

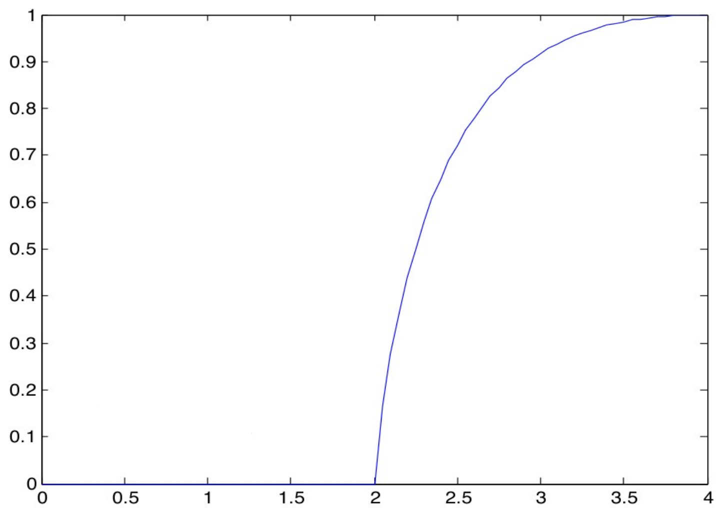
$$x_i < x_{i+1} \text{ for } i \in [0, n]$$

$$P(\mathbf{x}) = \left(\bigcup_{i=1}^{n-1} \{i \sim x \in (x_i, x_{i+1}]\} \right) \bigcup \{0 \sim x \in [x_0, x_1]\}$$

3 Results

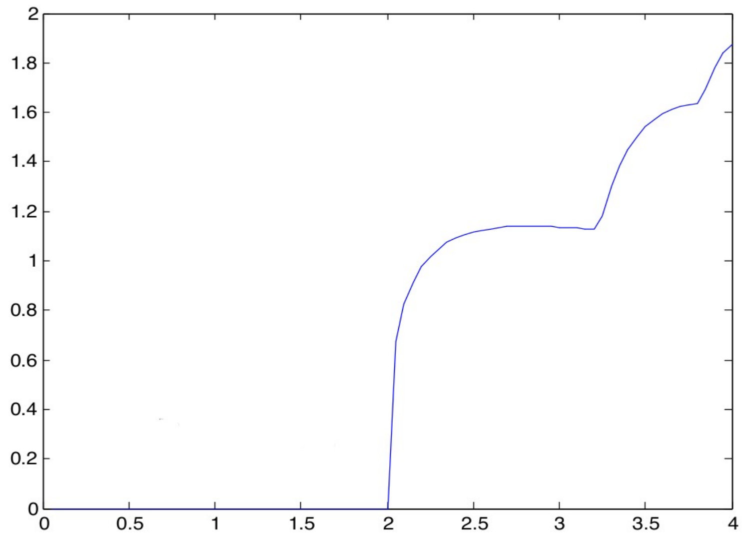
The figures below show the transient entropy rate plotted against the intrinsic growth rate for generating partitions of size 2, 4, and 8.

3.1 2 Partitions



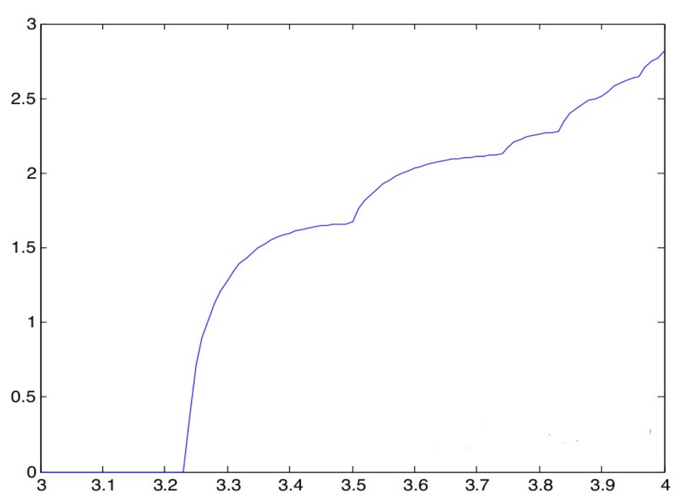
The most obvious condition to utilize a partition of size 2 is when managing the survival of an endangered species. In this scenario, the relevant density value is simply a critical point. Below this population size, the population will likely go extinct due to stochasticity, inbreeding, or lack of mating opportunity. A high resolution data set is unnecessary in this case when a coarse grain sampling may suffice. The graph above makes sense when interpreted with the logistic map. At $r = 2$, above which point the entropy rates become positive, the map intersects $x_n = x_{n+1}$ at the maximum x_{n+1}. This means that for all populations with intrinsic growth rate of 2 or lower, when their densities are crudely measured – partitioned twice through the midpoint – the observed states of the following year taken from the same apparatus or sampling technique are fully predictable as they are trapped in the left half of the map. Counter-intuitively, this result is neither interesting nor useful since the emitted sequence will merely be repetitions of a single symbol, therefore unable to reveal any dynamics of the actual system. The application of this partition resolution yields more in formation only for species of greater *r* value. The fastest rate of information gain occurs within the parameter space of 2 to 2.5. Biologically speaking, a crude sampling technique or apparatus is valuable only for species that reproductively doubles its population size at low density. At the maximum intrinsic growth rate, the transient entropy rate reaches 1, resembling the system's dynamics to that of a fair coin toss.

3.2 4 Partitions

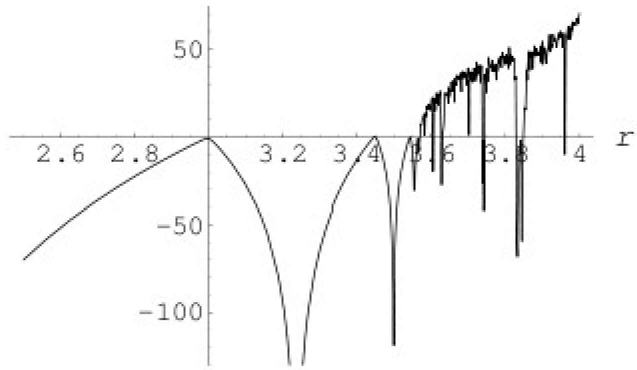


A finer resolution of density data becomes necessary when the species dynamics qualitatively changes as the population reaches certain threshold. This sudden state shift can be caused by the onset of coordinated group behavior as observed in swarms, increased dispersal rate due to heightened local intra-specific competition, mode switch in inter-specific interactions (i.e. from mutualism to predation), and other ecological phenomena that are not incorporated into the logistic model. When the partition granularity is raised to 4, a few curious traits emerged for the transient entropy rate as a function of r. Information gain once again occurs as $r = 2$. Moreover, the level of uncertainty as measured by the entropy rate no longer reaches its maximum monotonically. The trajectory is segmented by the local asymptotes into n' parts that is equivalent to the partition size. The construction of this generating partition ensures that the probability of reaching certain density range for the following generation diminishes for higher density level. This is observed by the decreasing "step" size in the above graph. Unlike the previous case, the transient entropy rate for a size 4 partition never reaches 2, the maximum entropy rate for a random variable of 4 probability distributions. In fact, the same is witnessed in the size 8 partition. This likely means that, as the sampling resolution sharpens, the *relative* information one can extract lowers. Hence, optimal sampling resolution depends on a balance between the entropy rate and the system's complexity.

3.3 8 Partitions



Here, positive entropy rate occurs at $r = 3.24$. At first glance, its behavior is erringly similar to the point of superstability for the lyaponov exponent.



The reason is actually rather simple: the construction of generating partitioning solves for the map when its first iterated slope equals zero. This is precisely when the lyaponov exponent shoots to negative infinity. More specifically, $r = 3.24$ is the intrinsic growth rate that generates a superstable period-2 cycle and the place where these orbits cross partitions are captured by the entropy rate. This topic will be further explored in Celis (2009).