Partition Matters and Entropy Measures with the Logistic Map

Shane Celis

March 20, 2009

Abstract

This paper uses the logistic map to relate some features of the lyapunov exponent and entropy, which are dependent on partitioning and outlines a specific property of generating partitions. It also examines the robustness of the entropy and entropy rate under partition variation.

1 Introduction

There are a number of measures of randomness that one can apply to a system. The lyapunov exponent is one such measure. The entropy and entropy rate are two others. The lyapunov exponent and entropy measures operate in different domains. The lyapunov exponent provides a measure of stability and instability over a continuous set of values. Entropy measures, however, are computed with a discrete set of symbols that theoretically extends infinitely into the future and the past. This paper will try to examine how these different measures of randomness relate and how partitioning can affect the entropy and entropy rate. The logistic map will be used as the system of study.

Questions The general question this paper would like to explore is, what does the entropy and entropy rate reveal about the dynamics of the logistic map? More specifically, how do the lyapunov exponent and the entropy and entropy rate relate? What kinds of randomness are they sensitive and insensitive to? How do the partitions affect the entropy and entropy rate? Are they robust to partition variation?

The Figure 1 shows the lyapunov exponent and the bifurcation plot. The lyapunov exponent nicely captures the points of instability where bifurcations can happen. Only for values of $\lambda \geq 0$ do bifurcations occur. Does the entropy or entropy rate have any such indicators either in relation to the bifurcation plot or to the lyapunov exponent? The answer will come in Section 3.

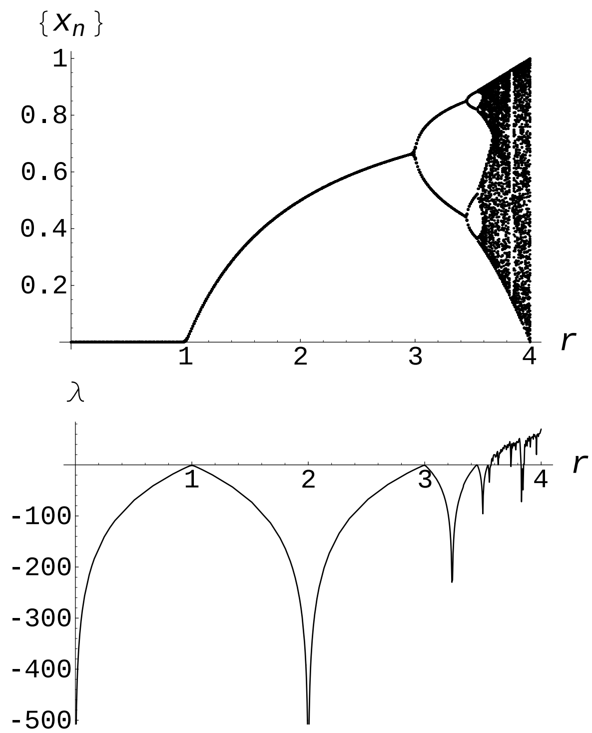

Figure 1: Bifurcation

2 Method

The equations of motion for the logistic map are given by Equation 1.

$$x_{n+1} = f(x_n) = rx_n(1 - x_n) \tag{1}$$
$$x_n \in [0, 1] \tag{2}$$
$$r \in [0, 4] \tag{3}$$

The value x_0 is the initial condition of the system, and r is the parameter of the system.

2.1 Lyapunov

The lyapunov exponent is calculated using the following formula. Let n_{trans} denote the number transients states that are typically not used for calculations. Calculations in this paper used $n_{trans} = 100$.

$$\lambda = \sum_{i=0}^{n} ln|\frac{\partial f}{\partial x}(x_{i+n_{trans}})| \tag{4}$$
$$\frac{\partial f}{\partial x} = r(1 - 2x) \tag{5}$$

2.2 Partitioning

One reason to focus on different partitions is to expose one of the distinctions between the lyapunov exponent and the measures of entropy and entropy rate. What is partitioning? Partitioning will take a sequence of real numbers $\{x_n\}$ and produce a new sequence $\{s_n\}$ from a finite alphabet of symbols.

$$x_n \in \mathbb{R} \tag{6}$$
$$s_n \in A \text{ where } |A| \text{ is finite} \tag{7}$$
$$s_n = o(x_n) \tag{8}$$

Two kinds of partitioning will be examined in this paper. Generating partitions are specially constructed so that hopefully one can study the symbol sequence to learn about the continuous system. However, generating partitions require that one know the dynamic of the system, which is not always available. Equidistant partitioning divides the state space into equal lengths and assigns a symbol to each piece. If one knew nothing about the system, equidistant partitioning would be a good first guess. One reason to explore these two partitioning schemes is they represent extremes of knowledge. Generating partitions require one know intimate details of the system. Equidistant partitions are a good guess if one knows nothing about the system. So it is worthwhile to see how the entropy and entropy rate change depending on what knowledge one has of the system. One would like for the entropy and entropy rate to still be useful even if a generating partition is not available.

An example of a partition that splits the state space of the logistic map in two is given by Equation 9. The finite set of symbols are $\{0, 1\}$. Incidentally, this is the partition used for many of calculations of entropy and entropy rate in this paper.

$$P = \{0 \sim x \in [0, \frac{1}{2}], 1 \sim x \in (\frac{1}{2}, 1]\} \tag{9}$$

A general way to form these partitions is based off of a vector $\mathbf{x}$ where

$$x_i < x_{i+1} \text{ for } i \in [0, n] \tag{10}$$

$$P(\mathbf{x}) = \left(\bigcup_{i=1}^{n-1} \{i \sim x \in (x_i, x_{i+1}]\} \right) \bigcup \{0 \sim x \in [x_0, x_1]\} \tag{11}$$

For example, one could rewrite (9) in terms of (11) as $P([0, \frac{1}{2}, 1])$.

Equidistant Partitions Generally for n equidistant partitions,

$$x_i = \frac{i}{n} \tag{12}$$

for $i \in [0, n]$. Let $\mathbf{x}_n^e$ denote an "equidistant" $\mathbb{R}^{n+1}$ vector where (12) holds, and let $P(\mathbf{x}_n^e)$ denote n equidistant partitions.

Generating Partitions Generating partitions are constructed in the following way. Let $f^2(x)$ denote $f(f(x))$.

$$\frac{\partial f^m(x)}{\partial x} = 0 \tag{13}$$

The function f is given by Equation 1. Solving (13) for x produces solutions $x_1, x_2, \dots, x_{n-1}$ where $x_i < x_{i+1}$ for $i \in [1, n-1]$. So that it fits within the formalism of Equation 11, let $x_0 = 0$ and $x_n = 1$. Let $\mathbf{x}_n^g$ denote this "generating" $\mathbb{R}^{n+1}$ vector, and let $P(\mathbf{x}_n^g)$ denote n generating partitions. The relationship between the number of iterations m and the number of partitions is $n = 2^m$. For example, the generating vector for $n = 4$ is $\mathbf{x}_2^g = (0, \frac{1}{4}(2-\sqrt{2}), \frac{1}{2}, \frac{1}{4}\ (2+\sqrt{2}), 1)$.

Interpolating Partitions Given the generating and equidistant partitions, one can compare the entropy and entropy rate calculated using each partition type, which this paper will do; however, rather than only comparing these distinct partitions, it might be interesting to see how the entropy and entropy rate change as the partitions are changed slightly. Toward that end, one can construct a means of interpolating between these two partitioning schemes.

$$\mathbf{x}_n^i(\alpha) = \mathbf{x}_n^e + (\mathbf{x}_n^g - \mathbf{x}_n^e)\alpha \tag{14}$$

$$\alpha \in [0, 1] \tag{15}$$

For the extreme values of α, it is equivalent to the generating or equidistant partitions.

$$\mathbf{x}_n^i(0) = \mathbf{x}_n^e \tag{16}$$

$$\mathbf{x}_n^i(1) = \mathbf{x}_n^g \tag{17}$$

So $P(\mathbf{x}_4^i(\frac{1}{2}))$ would denote four partitions that is midway between the generating partitions and the equidistant partitions.

2.3 Entropy

To calculate the length-L estimate for the entropy, the following formula is used

$$H(L) = -\sum_{s^L \in A} Pr(s^L) log_2(Pr(s^L)) \tag{18}$$

$$H(0) = 0 \tag{19}$$

where s^L is a string of L symbols, and A is the alphabet.

2.4 Entropy Rate

The length-L estimate for the entropy rate is given by (20).

$$h_\mu(L) = H(L) - H(L-1) \tag{20}$$

3 Results

The majority of the results for this paper are plots. Figure 3 shows $H(L)$ for $L \in [1, 9]$ for the generating partition $P(\mathbf{x}_2^g)$, which is equivalent to the partition described by Equation 9 . (Note that $P(\mathbf{x}_2^g) = P(\mathbf{x}_2^e)$ only for the case where $n = 2$.)

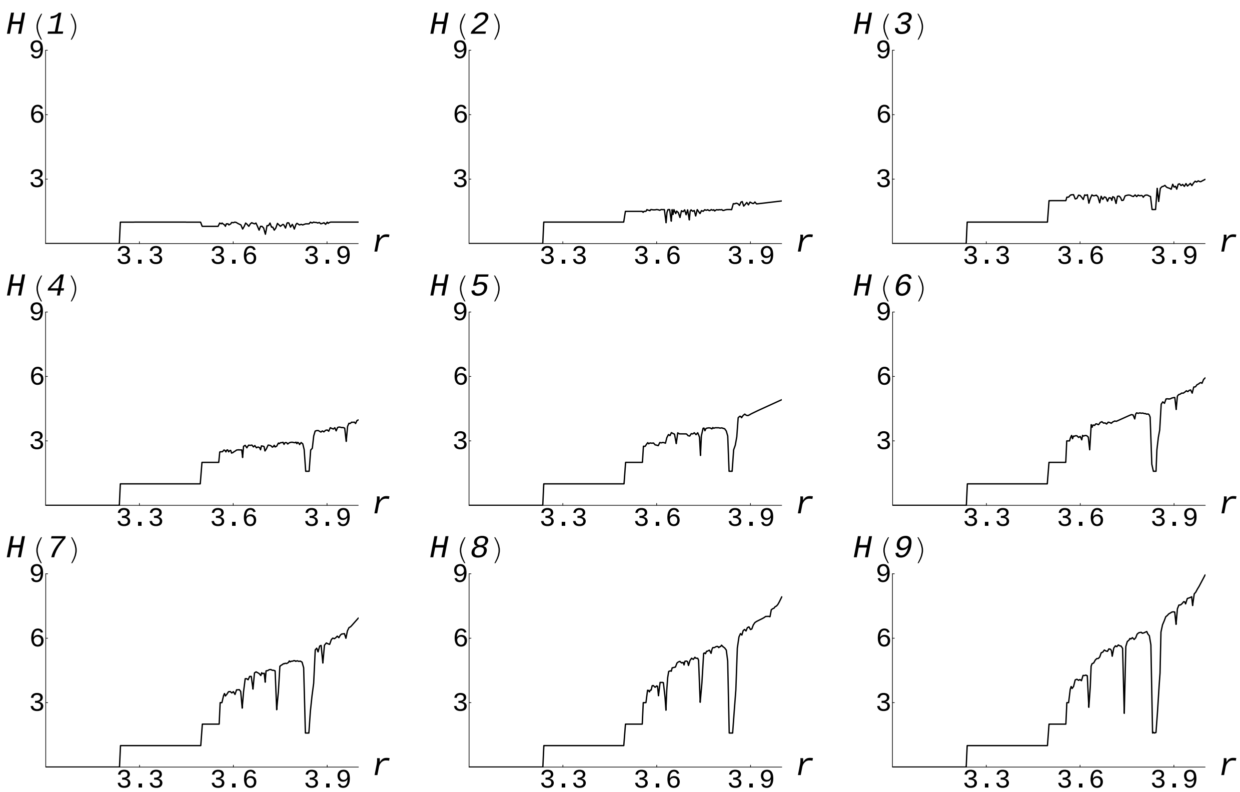

Figure 2: The entropy of the logistic map $H(L)$ partitioned with $P(\mathbf{x}_2^g)$

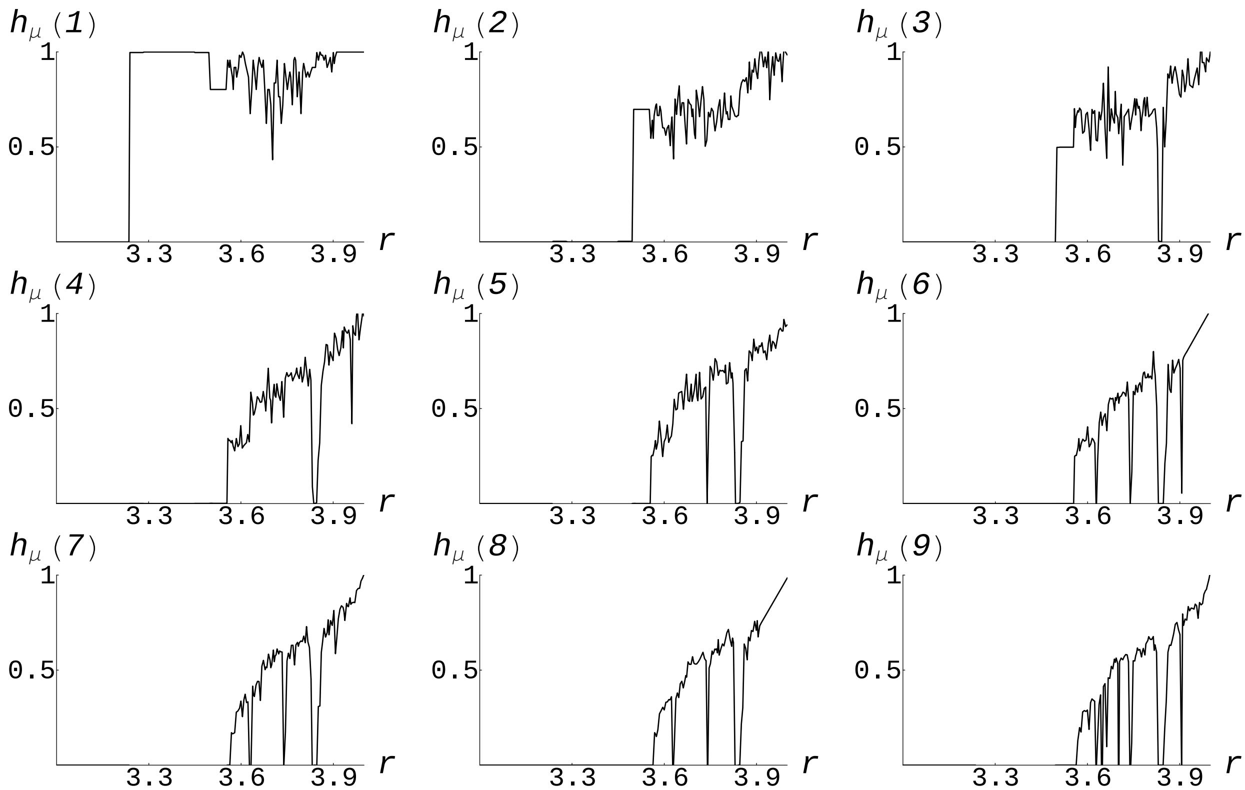

Figure 3: The entropy rate of the logistic map $h_\mu(L)$ partitioned with $P(\mathbf{x}_2^g)$

One of the important features of the lyapunov exponent becomes apparent when examining it in relation to the bifurcation plot. Does the plot of entropy have any readily identifiable features with respect to the bifurcation plot? See Figure 4. The superstable period 3 near $r = 3.8$ seems to correspond with the dramatic dip in entropy. What explains this? At the super stable period 3 orbit, the sequence $(x_1, x_2, x_3, x_1, \ldots)$ will produce sequences $(s_1, s_2, s_3, s_1, \ldots)$, so it makes sense that less variety of sequence symbols exists and therefore the entropy is less.

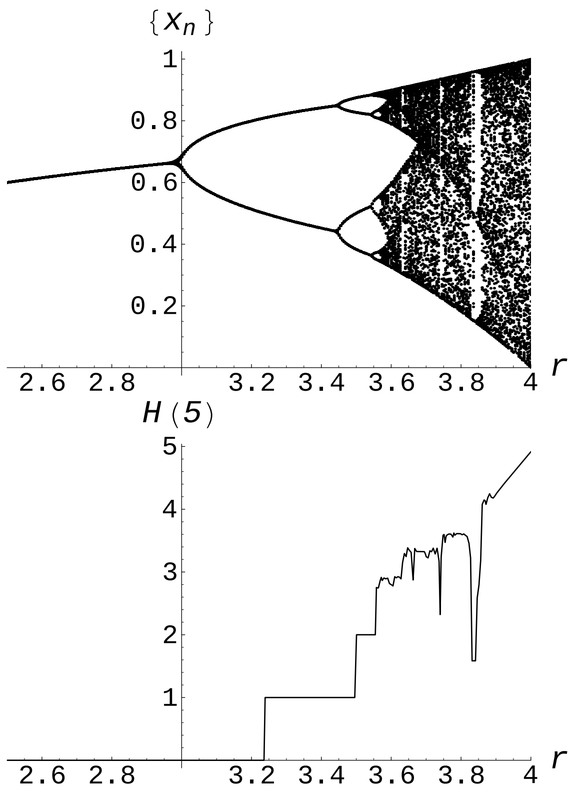

Figure 4: Bifurcation plot above and $H(5)$ below using partition $P(\mathbf{x}_2^g)$

Figure 5 shows the entropy of the logistic map above the lyapunov exponent. Here there are some immediately identifiable features. The superstable points where λ goes to negative infinity seem to correspond to spikes in the $H(5)$ graph. Where do the spikes in $H(L)$ come from? Examining Figure 4, one can see for $r < 3.2$ that $H(5) = 0$ even though the sequence bifurcated into two orbits near $r = 2$. What accounts for this? It is partly because of the partitioning. So long as the two orbits are confined to the same partition, in this case $(\frac{1}{2}, 1]$, they are invisible to the measure of entropy. $H(5)$ peaks near $r = 3.2$ when one of those orbits crosses into the other partition, crosses over the $\frac{1}{2}$ boundary, and the entropy measure is finally able to "see" it.

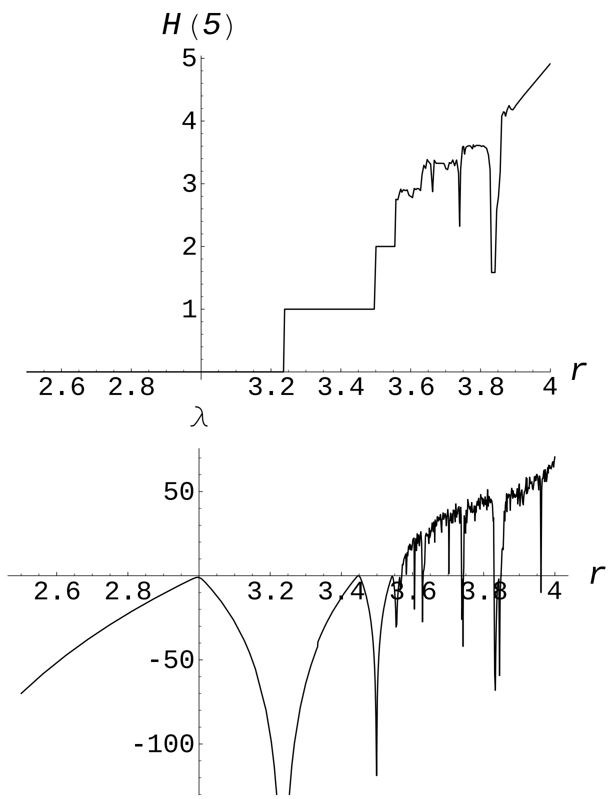

Figure 5: $H(5)$ of the logistic map using partition $P(\mathbf{x}_2^g)$ and the lyapunov exponent

Still unexplained is why the lyapunov exponent shows a point of superstability where $H(5)$ peaks. Is this a consequence of an intrinsic property of entropy, a happenstance of the partitioning? Perhaps it is a worthwhile property of good partitioning. Consider that the construction of the generating partitions solves $\frac{\partial f}{\partial x} = 0$ and the lyapunov exponent goes to negative infinity when that is the case. By using generating partitions, the places where orbits cross into other partitions coincides with points of superstability. There is probably a lot to recommend placing the division between partitions on places of superstability.

To rule out the coinciding of $H(5)$ peaks and the superstable points as an intrinsic property of entropy, consider what the entropy would look like with a different partition. Figure 6 shows $H(5)$ with the generating partition of two above and an equidistant partition of three below. Notice that all the features that correspond with the superstability points in the lyapunov exponent have vanished, with the exception of the superstable period three. However, it does appear that the entropy “sees” the bifurcations “sooner” for $r < 3$ on account of the three equidistant partitions.

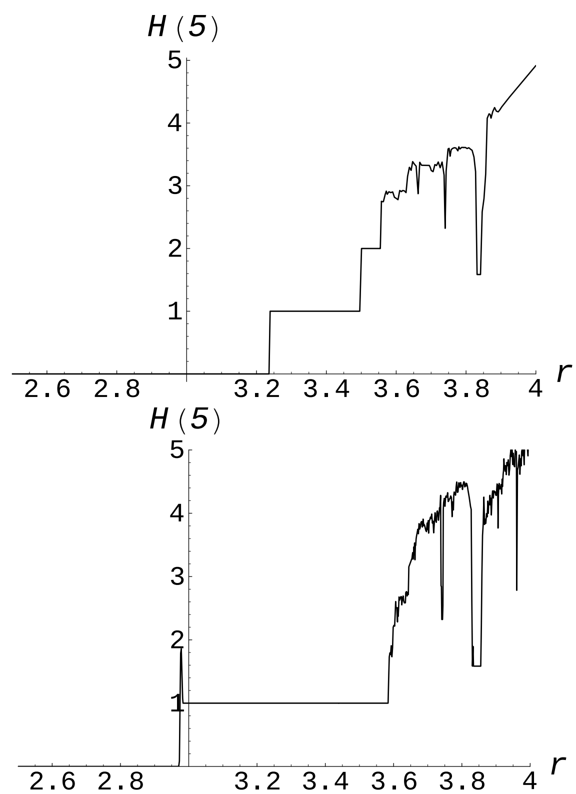

Figure 6: Above $H(5)$ with partition $P(\mathbf{x}_2^g)$ and below $H(5)$ with partition $P(\mathbf{x}_3^e)$

3.1 Partition Studies

One question to contend with is, how important is the partitioning one chooses? This paper will compare generating partitions for the logistic map with equidistant partitions by using interpolating partitions. Equidistant partitioning seems like a reasonable assumption to make, especially if one is building an apparatus to measure a physical system. As one changes the partitioning slightly, does it create drastic differences with respect to the entropy and entropy rate? Or is the entropy and entropy robust under partition variation?

To answer those questions, consider the following contour plots. The dark areas indicate low values. The light areas indicate high values. One can think of the previous plots of $H(L)$ vs. r as mere slices that we are now looking down upon. The number of partitions are noted as n and block length is noted as L. At the bottom of each graph, where $\alpha = 0$ that is the entropy for the equidistant partition, and as α increases from 0, it approximates the generating partition until α equals 1. At $\alpha = 1$, that is the entropy for generating partition. The case where $n = 2$ is not terribly interesting since the generating partition and equidistant partition are equivalent for that case, but it is shown in Figure 7 for completeness.

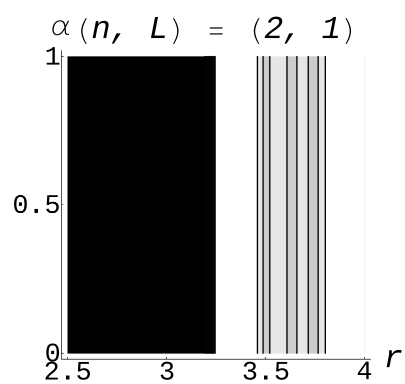

Figure 7: This contour plot shows $H(L)$ for interpolation parameter α vs. the logistic map parameter r. Dark areas indicate low values. Light areas indicate high values.

If the entropy and entropy rate are robust, the contour plots should have contiguous regions. If they are not robust, the contour plots should look very noisy. Figure 8 and Figure 9 are both contiguous plots, so the entropy and entropy rate do appear to be robust under partition variation.

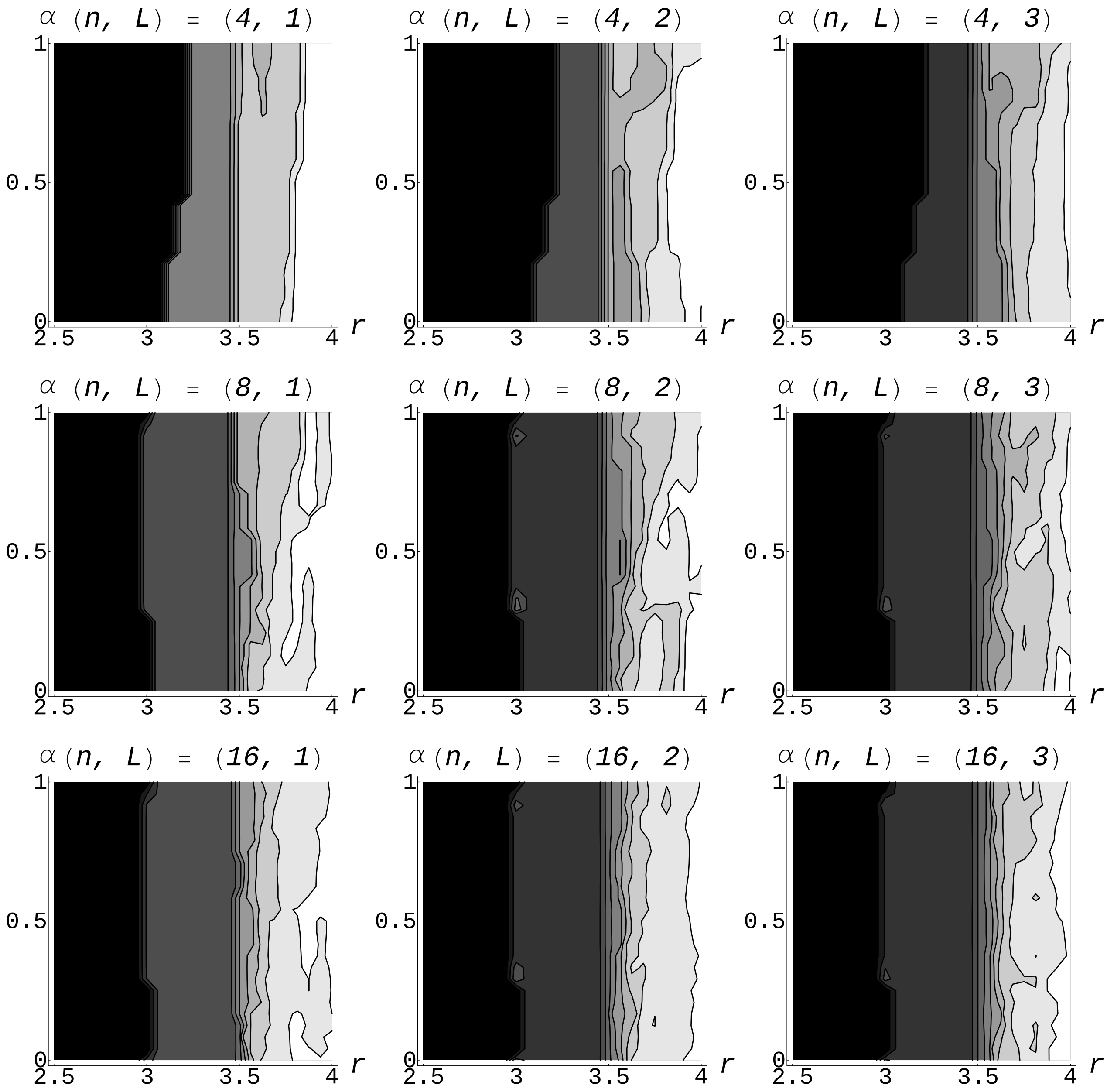

Figure 8: This contour plot shows $H(L)$ for interpolation parameter α vs. the logistic map parameter r. Dark areas indicate low values. Light areas indicate high values.

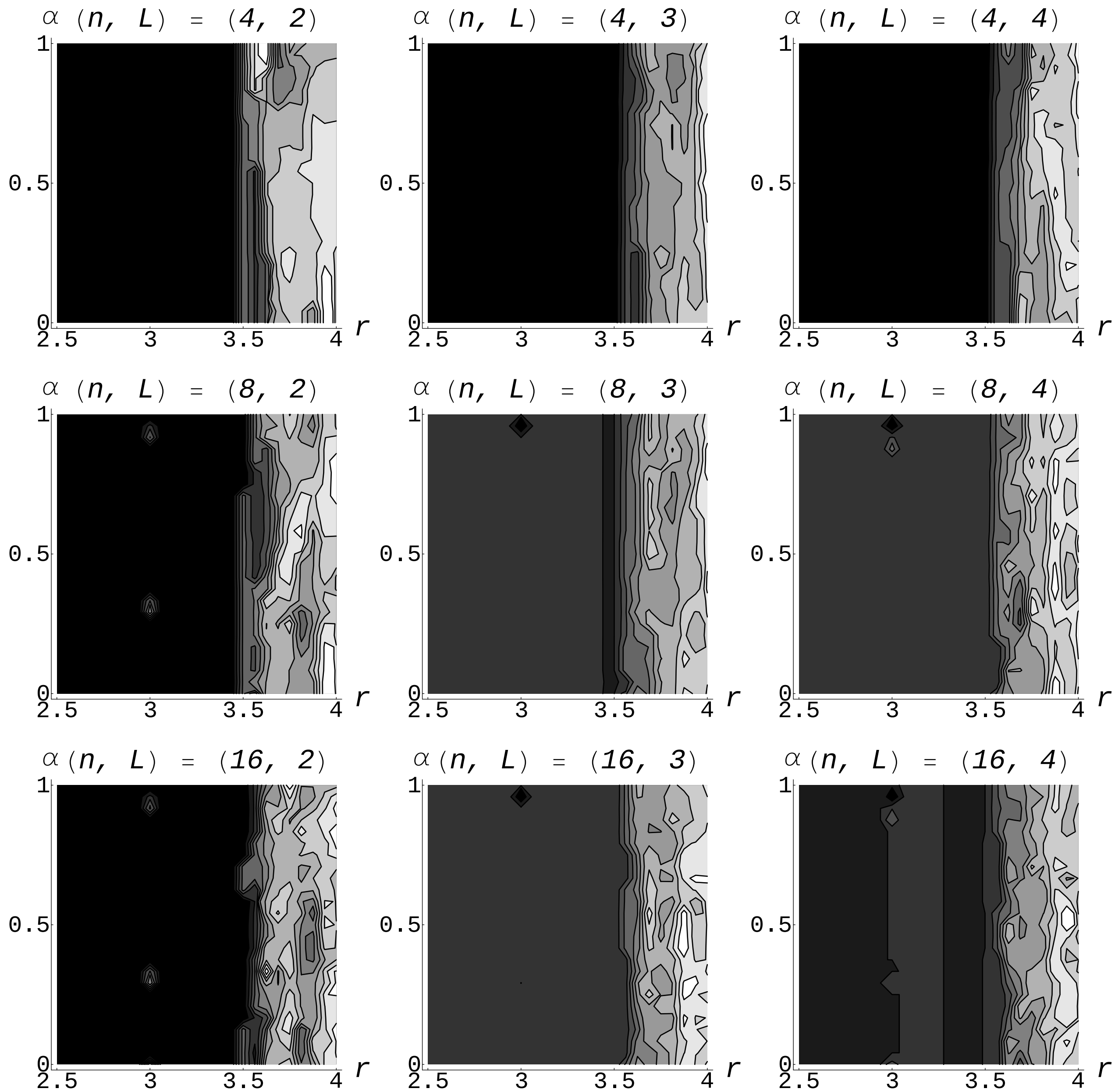

Figure 9: This contour plot shows $h_\mu(L)$ for interpolation parameter α vs. the logistic map parameter r. Dark areas indicate low values. Light areas indicate high values.

4 Conclusion

This paper has shown and explained some features of how the entropy and the lyapunov exponent relate, noting that the spikes in entropy relate to the points of superstability due to the choice of a generating partition. It has also shown how entropy and entropy rate behave under partition variation, and shown that they appear to be robust.

5 Future Work

The generating and equidistant partitions compared in this paper all had the same number of partitions. It may be of interest to define a means of morphing between partitions of different sizes and examining how the entropy and entropy rate behave. For direct comparisons, all the equidistant partitions had an even number of partitions, so in effect they had one part of the generating partition right: $x_i = \frac{1}{2}$ for some i. This merely serves to emphasize that there is much more variety in partitioning than this paper has addressed.