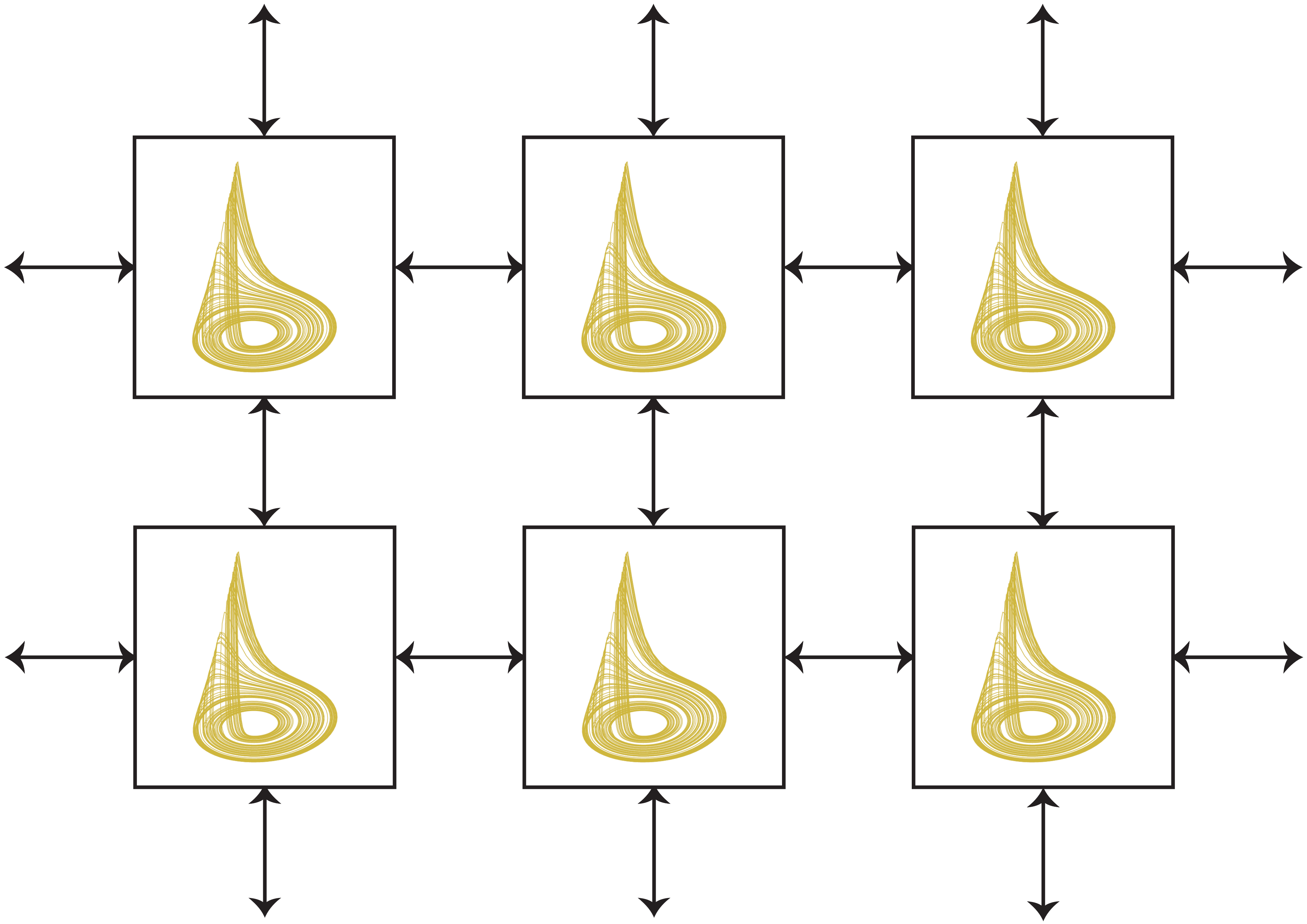
Pattern Formation I (Deterministic)

Reading for this lecture:

(These) *Lecture Notes*.

Pattern Formation I ...

Nature's not low-dimensional ... space ...



Pattern Formation I ...

Deterministic Spatially Extended Dynamical Systems:

System	Local State	Time	Space
Cellular Automata	Discrete	Discrete	Discrete
Map Lattice	Continuous	Discrete	Discrete
Oscillator Chain	Continuous	Continuous	Discrete
Partial Differential Equations	Continuous	Continuous	Continuous
...			

Pattern Formation I ...

Deterministic Spatially Extended Dynamical Systems ...

Space: Lattice in d-dimensions $\mathcal{L} = \mathbb{Z}^d$

Of cells i with local state: $x^i \in M, \; i \in \mathcal{L}$

Local state space

Global state $\vec{x}$ is a configuration of local states:

$$\vec{x} = (\ldots, x^{i_1}, x^{i_2}, x^{i_3}, \ldots), \; i_k \in \mathcal{L}$$

State space: $\mathcal{X} = \cdots \times M \times M \times M \times \cdots = M^{\mathcal{L}}$

State is a point: $\vec{x} \in M^{\mathcal{L}}$

State space dimension: ∞

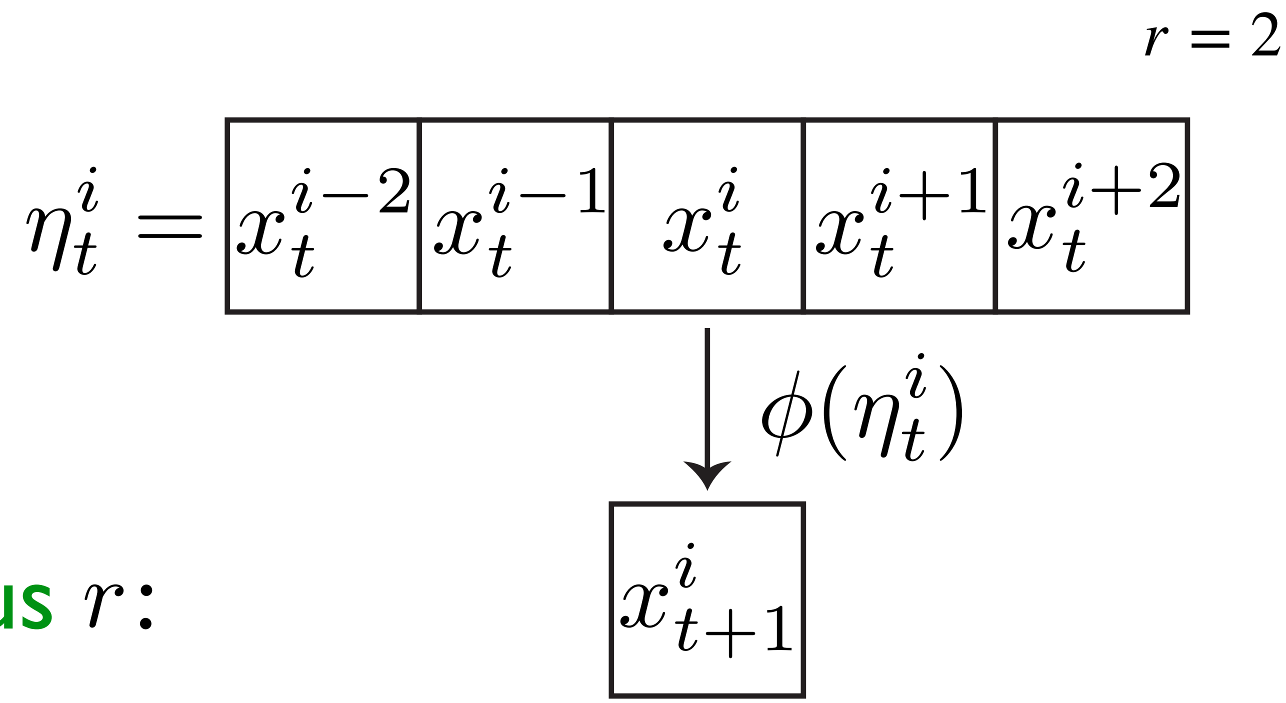
Two independent coordinates:

time + space

spacetime

Pattern Formation I ...

Deterministic Spatially Extended Dynamical Systems ...



Local Dynamic:

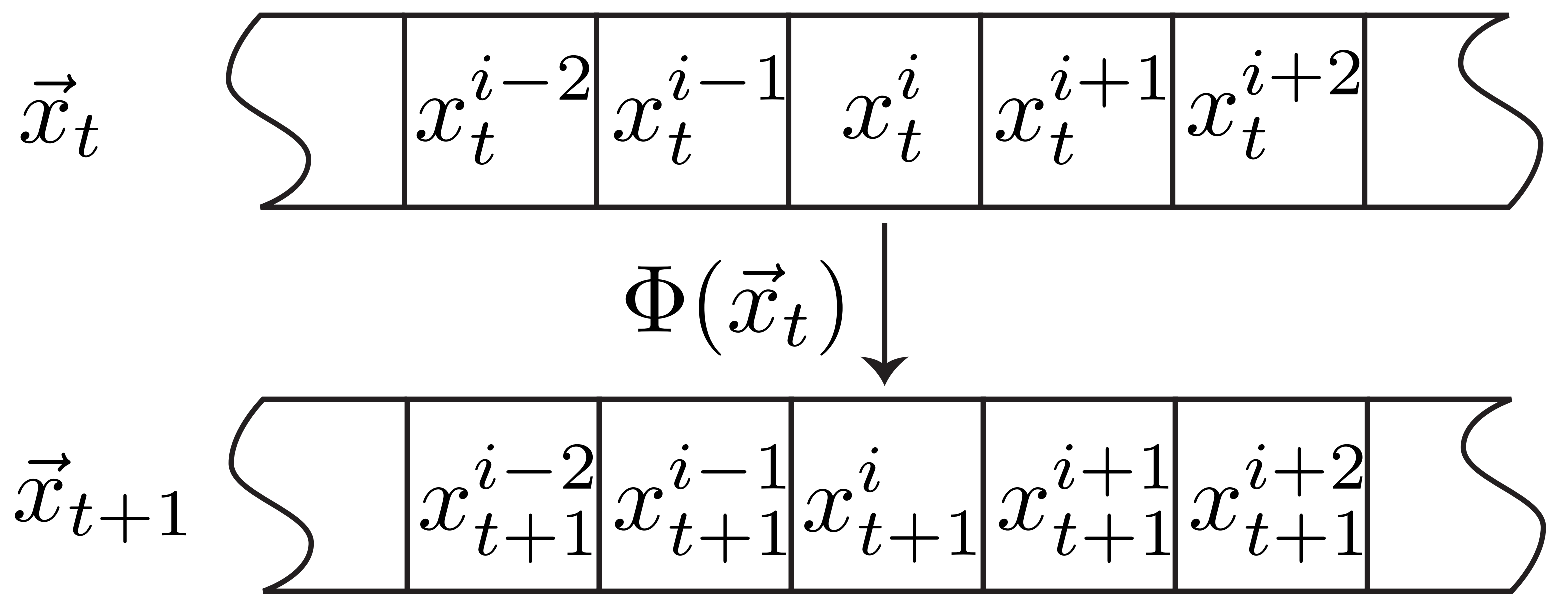
Cell's neighborhood of radius r:

$$\eta^i \in M^{2r+1}$$

Map neighborhood to next state:

$$\phi : \eta_t^i \rightarrow x_{t+1}^i$$

Pattern Formation I ...
Deterministic Spatially Extended Dynamical Systems ...



Global Dynamic:

Map configurations to configurations: $\Phi : \mathcal{X} \rightarrow \mathcal{X}$

Discrete time: $\vec{x}_{t+1} = \Phi(\vec{x}_t)$

Continuous time: $\dot{\vec{x}} = \Phi(\vec{x})$

Pattern Formation I ...

Deterministic Spatially Extended Dynamical Systems ...

What does dynamical systems theory have to say?

- Not clear: Historically, developed in low dimensions
- Point in state space: No reference to spatial structure
- Key objects:
  - Invariant Sets
  - Attractors: Fixed Points, Limit Cycles, Chaos
- Key concepts:
  - Instability-stability
  - Bifurcation

Pattern Formation I ...

Deterministic Spatially Extended Dynamical Systems ...

Do these objects and concepts exist in spacetime?

(Yes) But how to analyze? Visualize?

Also, new questions:

Spatial structure?

Two+ independent coordinates

How does time interact with space?

Pattern Formation I ...

Deterministic Spatially Extended Dynamical Systems ...

What's new in space, via examples of

Cellular Automata
Map Lattices

Pattern Formation I ...

Pattern formation in cellular automata:

Local state, space, and time: Discrete

Lattice: $\mathcal{L} = \mathbb{Z}^d$

Site: $i \in \mathcal{L}$

Local state: $s^i \in \Sigma = \{0, 1, \ldots, k-1\}$

State (global configuration): $\mathbf{s} = (\ldots, s_1, s_2, s_3, \ldots) \in \Sigma^{\mathcal{L}}$

State space

Neighborhood of radius r: $\eta^i = (s^{i-\vec{r}}, \ldots, s^i, \ldots, s^{i+\vec{r}})$

template

Local dynamic: $s^i_{t+1} = \phi(\eta^i_t)$

Synchronous: Local rule applied simultaneously across lattice

Global map: $\mathbf{s}_{t+1} = \Phi(\mathbf{s}_t)$

Initial configuration: $\mathbf{s}_0$

Pattern Formation I ...

Pattern formation in cellular automata ...

Observations:

Local dynamic ϕ is a function ...

Many-to-one global map Φ : Finite- or ∞-to-One

Dissipation: Sets cannot grow (shrink or stay the same)

Garden of Eden states:

ICs that cannot be evolved to

$$\Phi^{-1}(\mathbf{s}) \notin \Sigma^{\mathcal{L}}$$

Pattern Formation I ...

Pattern formation in cellular automata ...

Observations ...

How many?

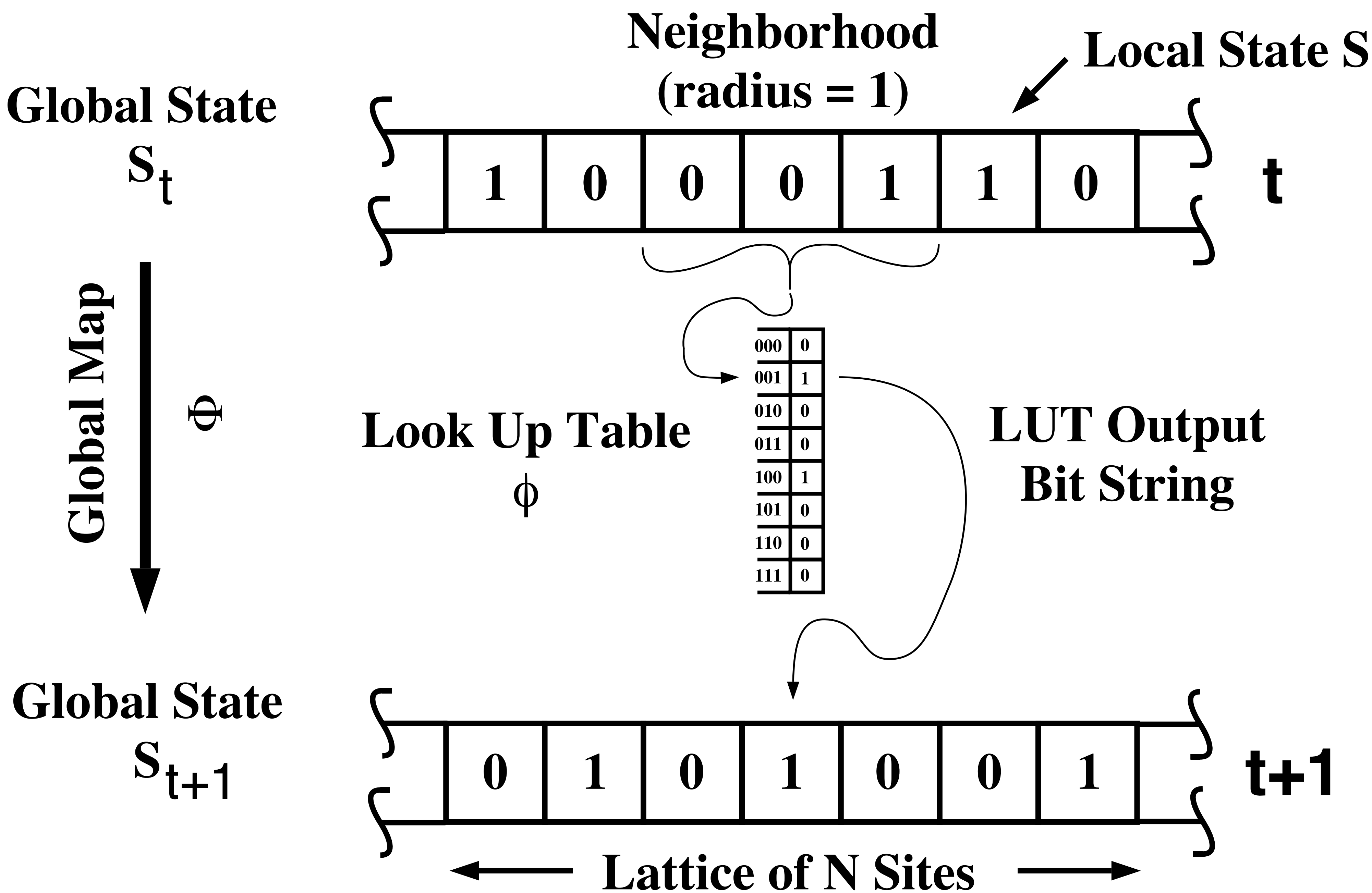
Number of Neighborhoods $\propto k^{r^d}$

Number of CA ϕs $\propto k^{k^{r^d}}$

Pattern Formation I ...

Pattern formation in cellular automata:

Cellular automata in one spatial dimension (d = 1):



Pattern Formation I ...

Pattern formation in cellular automata ...

1D CAs:

Elementary CAs (ECAs):

- One spatial dimension: N cells
- Binary local state: k = 2
- Nearest-neighbor coupling: r = 1
- Neighborhood: $\eta^i = (s^{i-1}, s^i, s^{i+1})$
- Local dynamic: $s^i_{t+1} = \phi(\eta^i_t)$ $i = i \mod N$

Number of CAs: $k^{k^{2r+1}}$

ECAs:

- Neighborhoods: $8 = 2^3$
- ECAs: $256 = 2^{2^3}$
- But: 88 classes equivalent under

$$0 \leftrightarrow 1$$
$$i \leftrightarrow -i$$

Pattern Formation I ...

Pattern formation in cellular automata:

Example: ECA 18

Local rule

$$\eta_t^i \longrightarrow s_{t+1}^i$$

$$000 \longrightarrow 0$$

$$001 \longrightarrow 1$$

$$010 \longrightarrow 0$$

$$011 \longrightarrow 0$$

$$100 \longrightarrow 1$$

$$101 \longrightarrow 0$$

$$110 \longrightarrow 0$$

$$111 \longrightarrow 0$$

↓

Integer

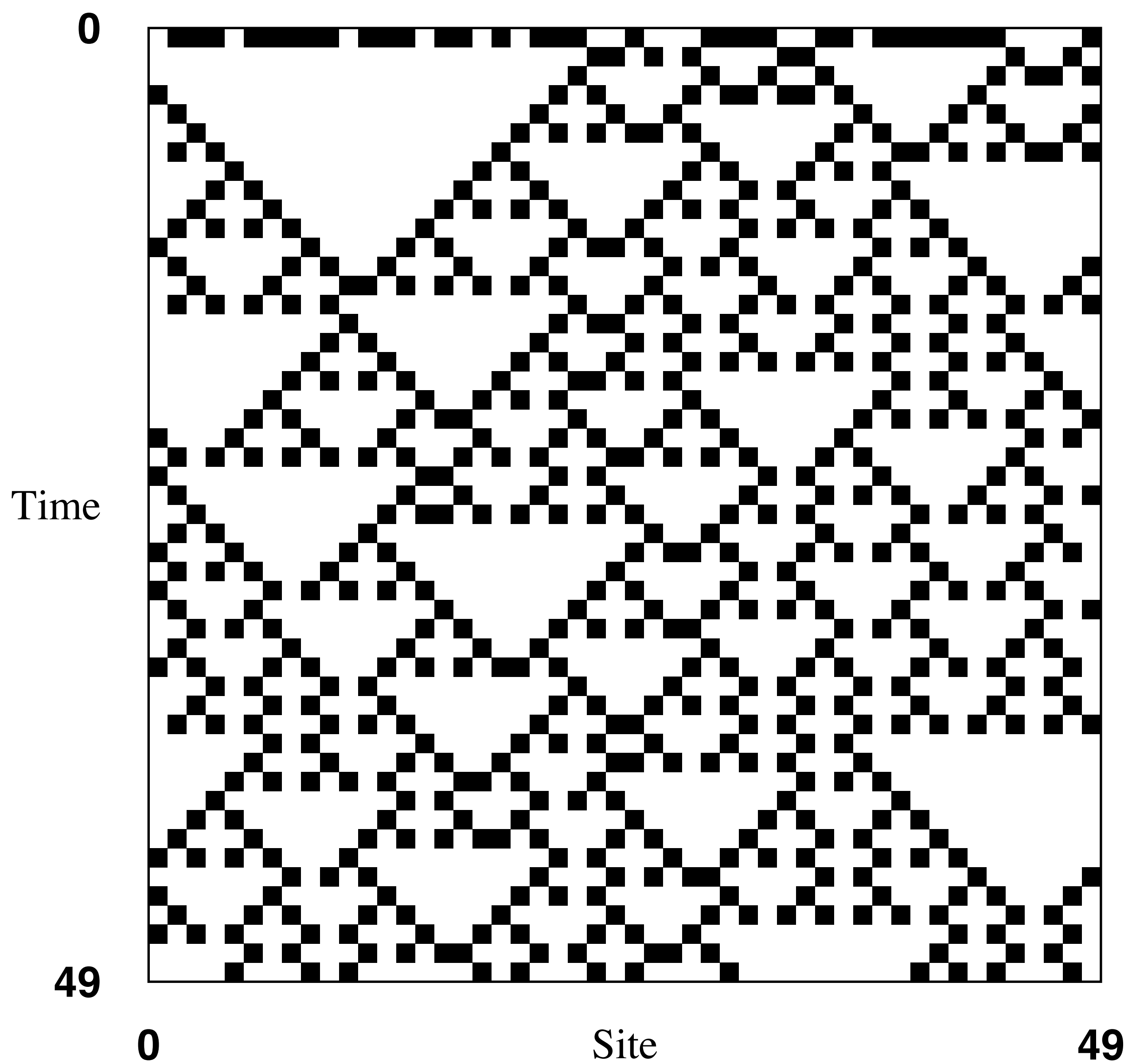
↓

18

Pattern Formation I ...

Pattern formation in cellular automata:

Spacetime diagram:



Initial Configuration (IC) s_0

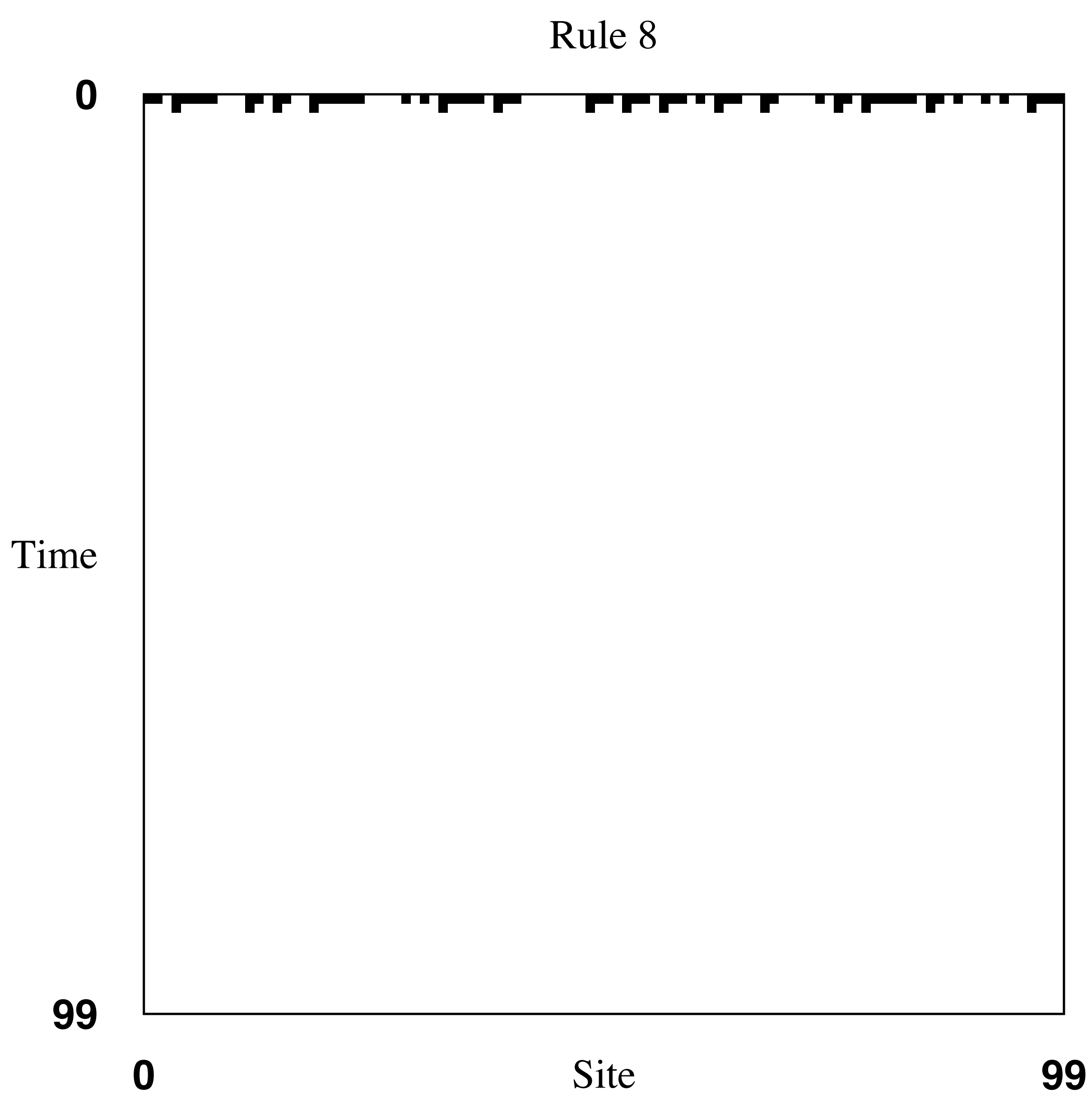
Periodic boundary conditions: $s^{i+N} = s^i$

Pattern Formation I ...

Pattern formation in cellular automata ...

1D CA Behavior Survey

Homogeneous fixed point:



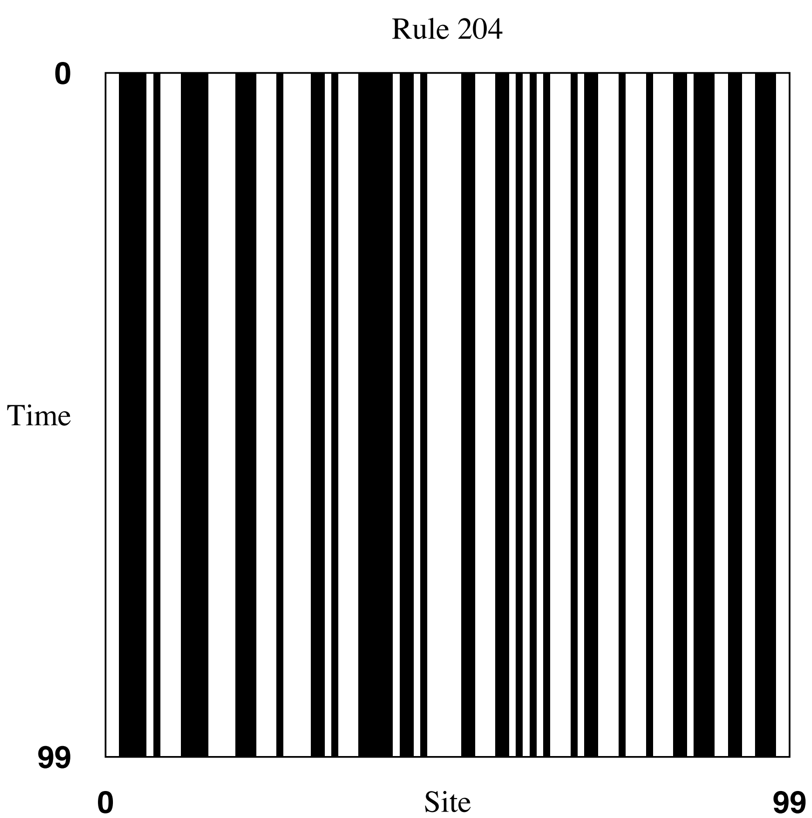
$$\Phi(0^N) = 0^N$$

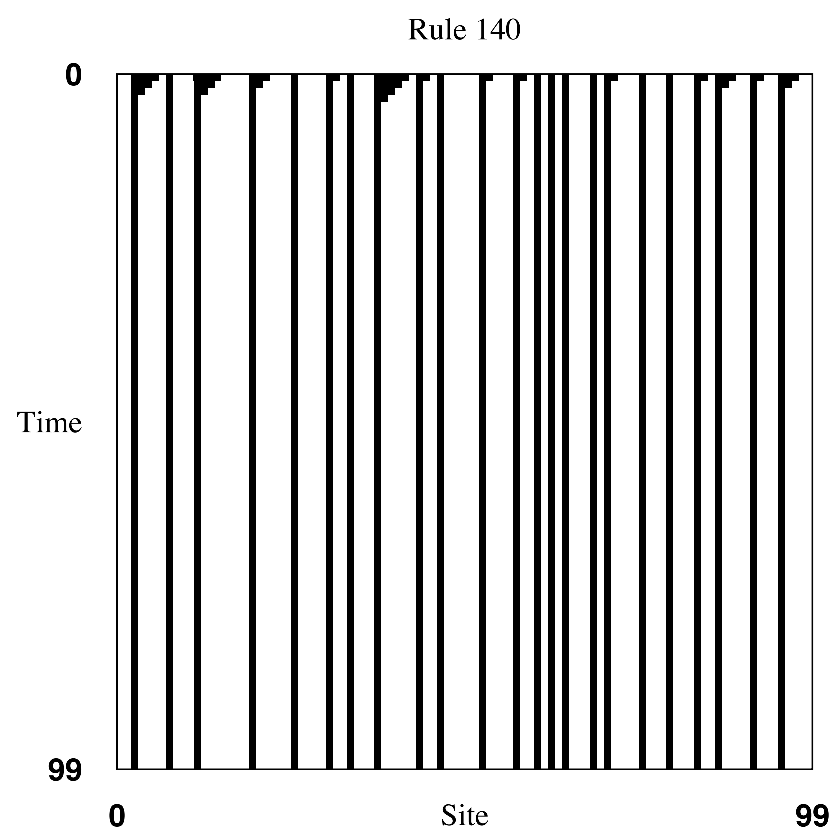
Pattern Formation I ...

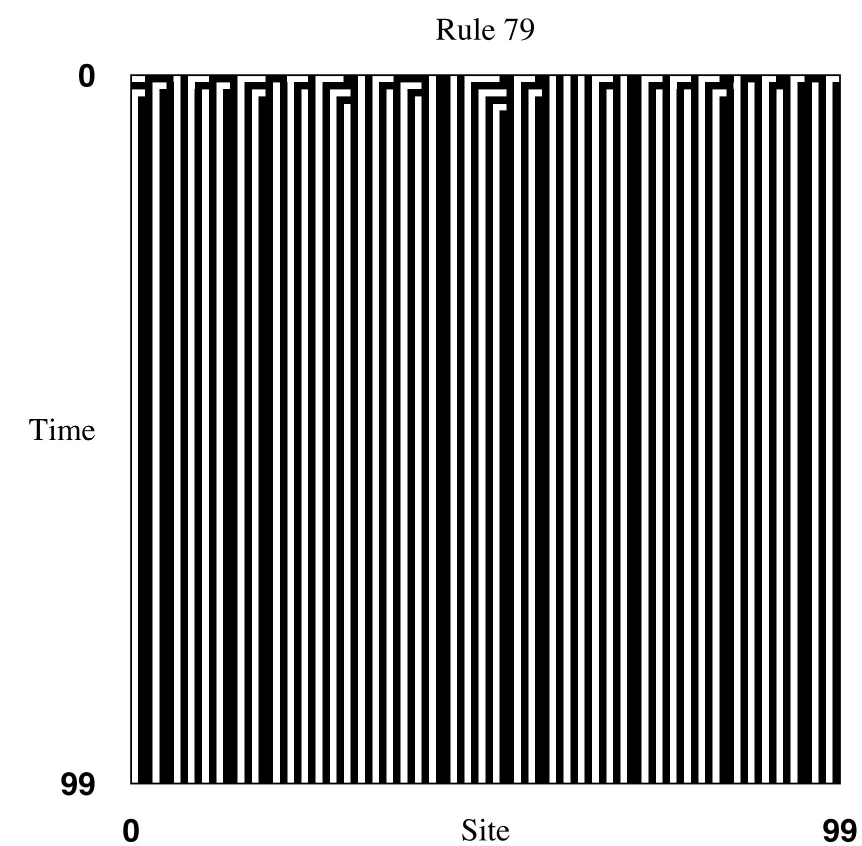
Pattern formation in cellular automata ...

1D CA Behavior Survey ...

Fixed points with spatial variation: $\Phi(\mathbf{s}) = \mathbf{s}$







Pattern Formation I ...

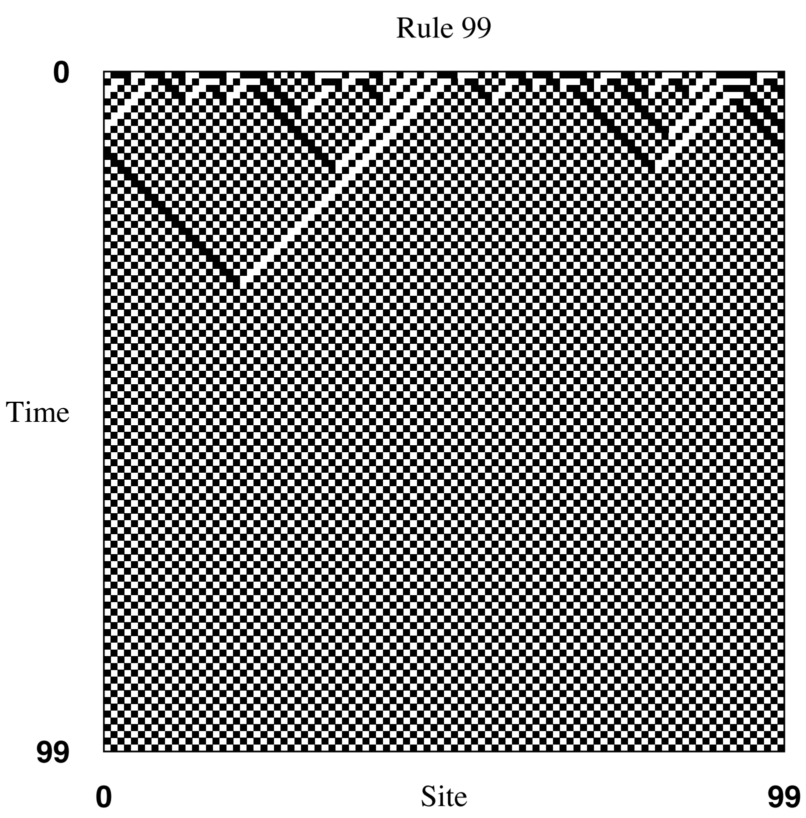
Pattern formation in cellular automata ...

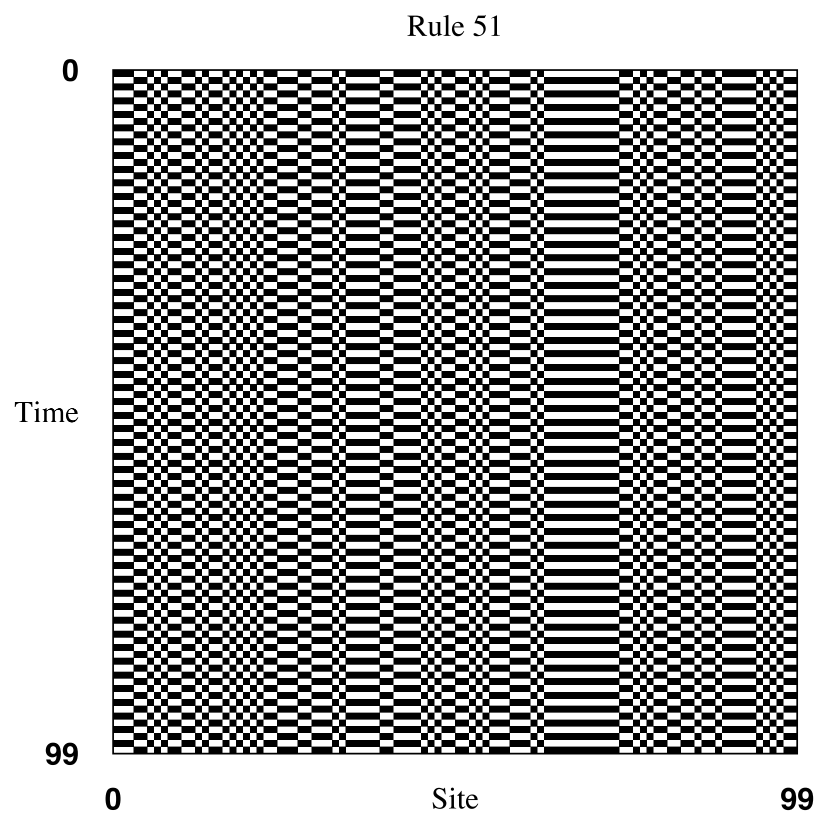
1D CA Behavior Survey ...

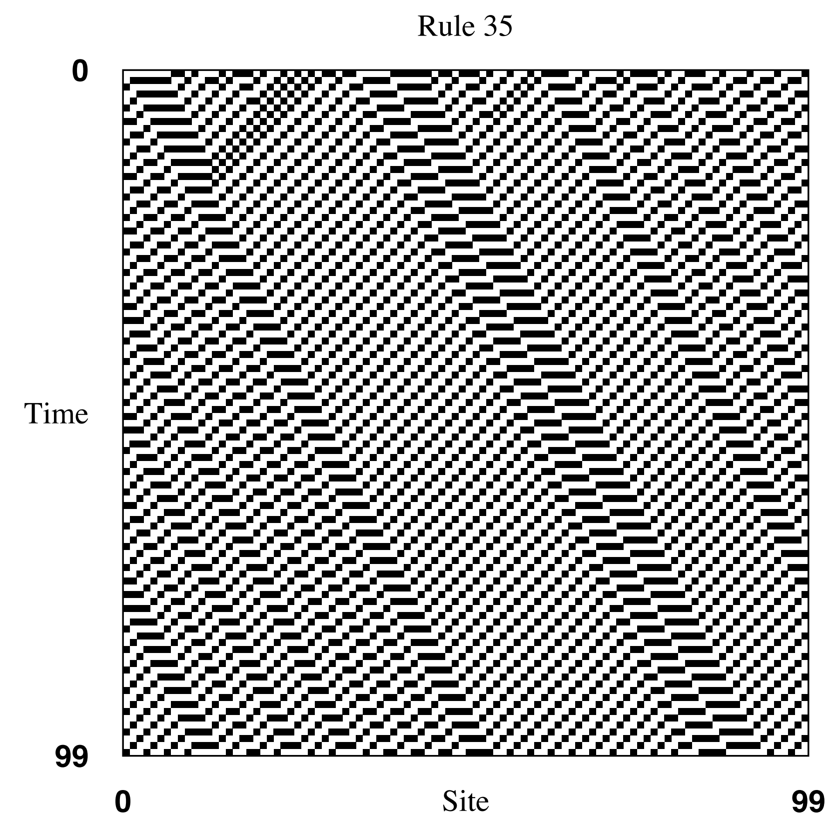
Periodic orbits and limit cycles: $\Phi^p(\mathbf{s}_t) = \mathbf{s}_t$

Homogeneous

Inhomogeneous





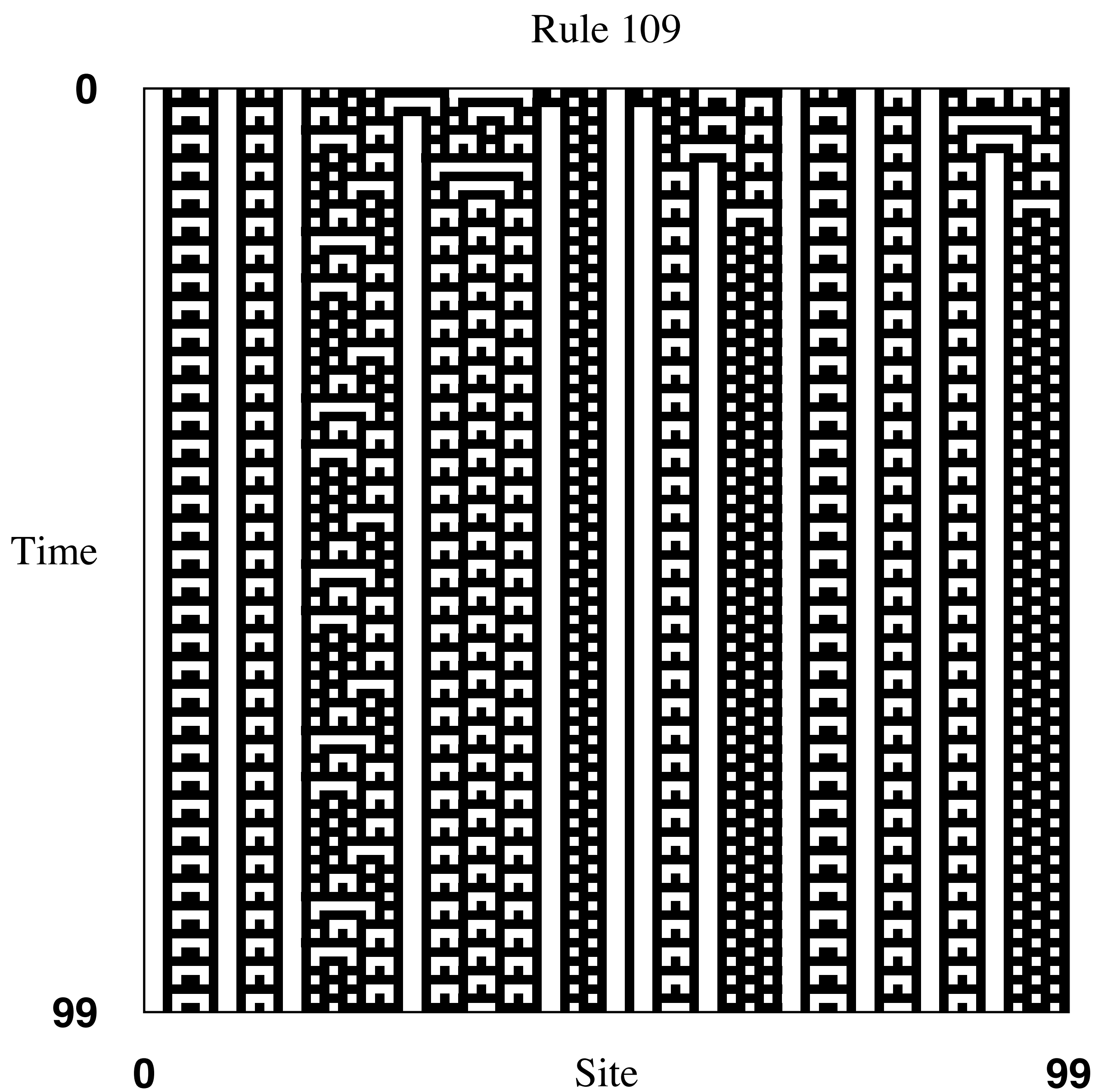


Pattern Formation I ...

Pattern formation in cellular automata ...

1D CA Behavior Survey ...

Mixture of local fixed-point and periodic "regions":



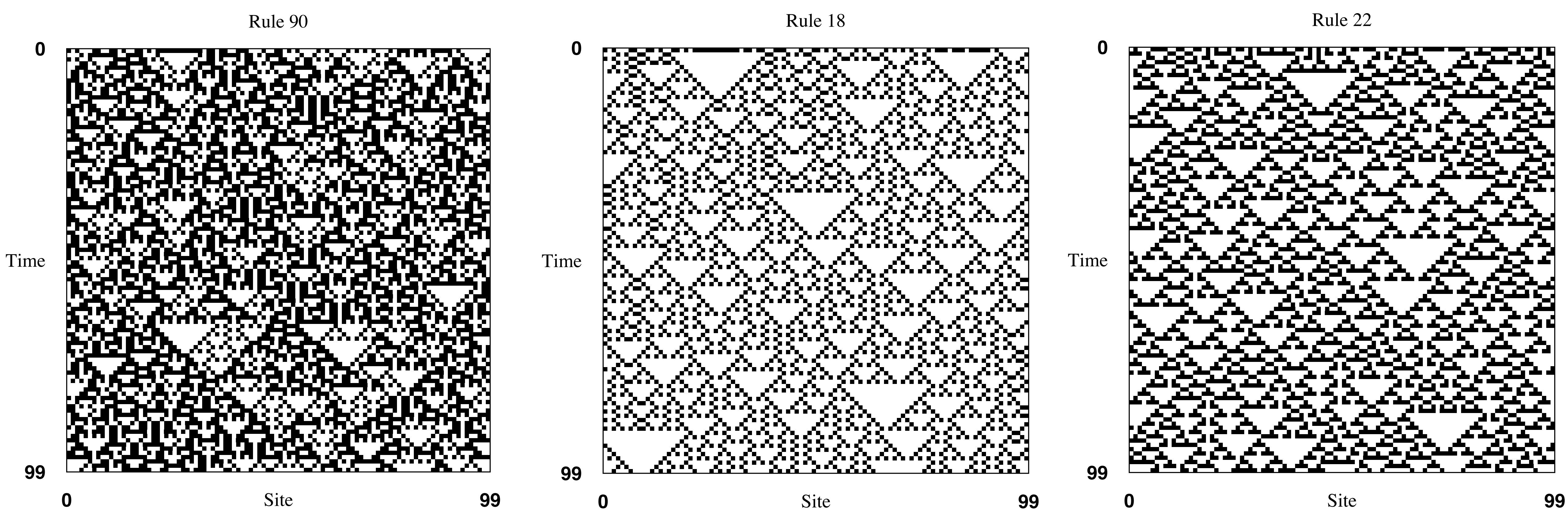
What kind of invariant set?

Pattern Formation I ...

Pattern formation in cellular automata ...

1D CA Behavior Survey ...

“Chaos”?



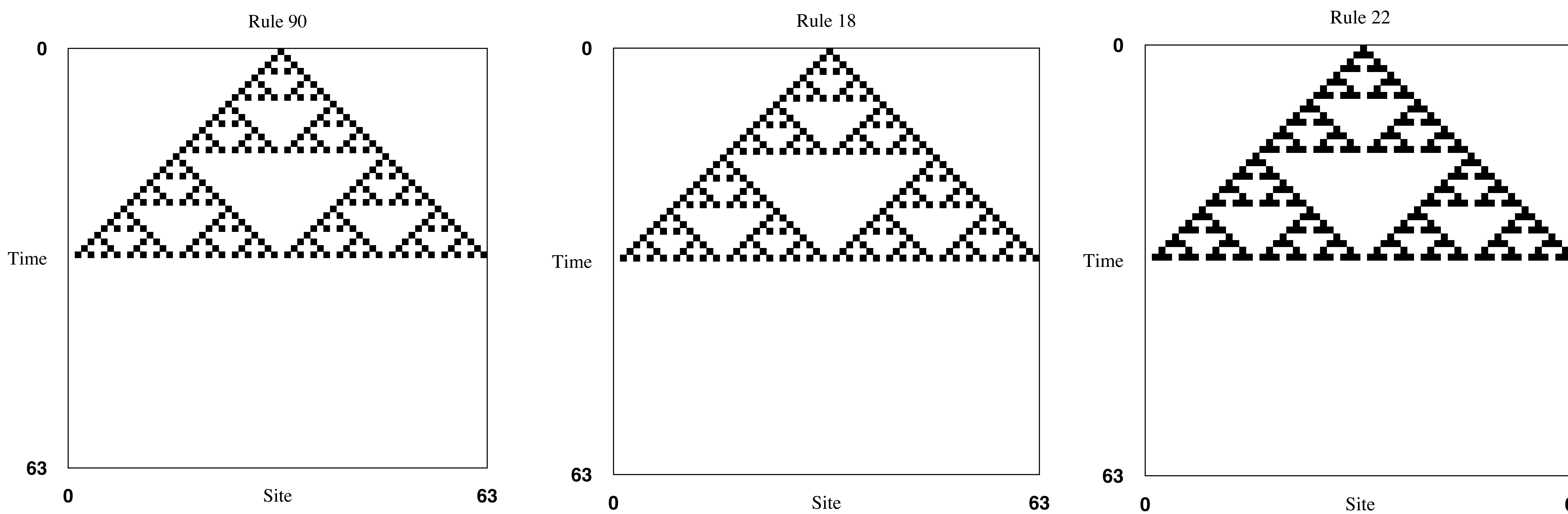
Pattern Formation I ...

Pattern formation in cellular automata ...

1D CA Behavior Survey?

"Chaos"? = Spreading of perturbations

IC: $0^N + \ldots 0001000 \ldots \mod 1$



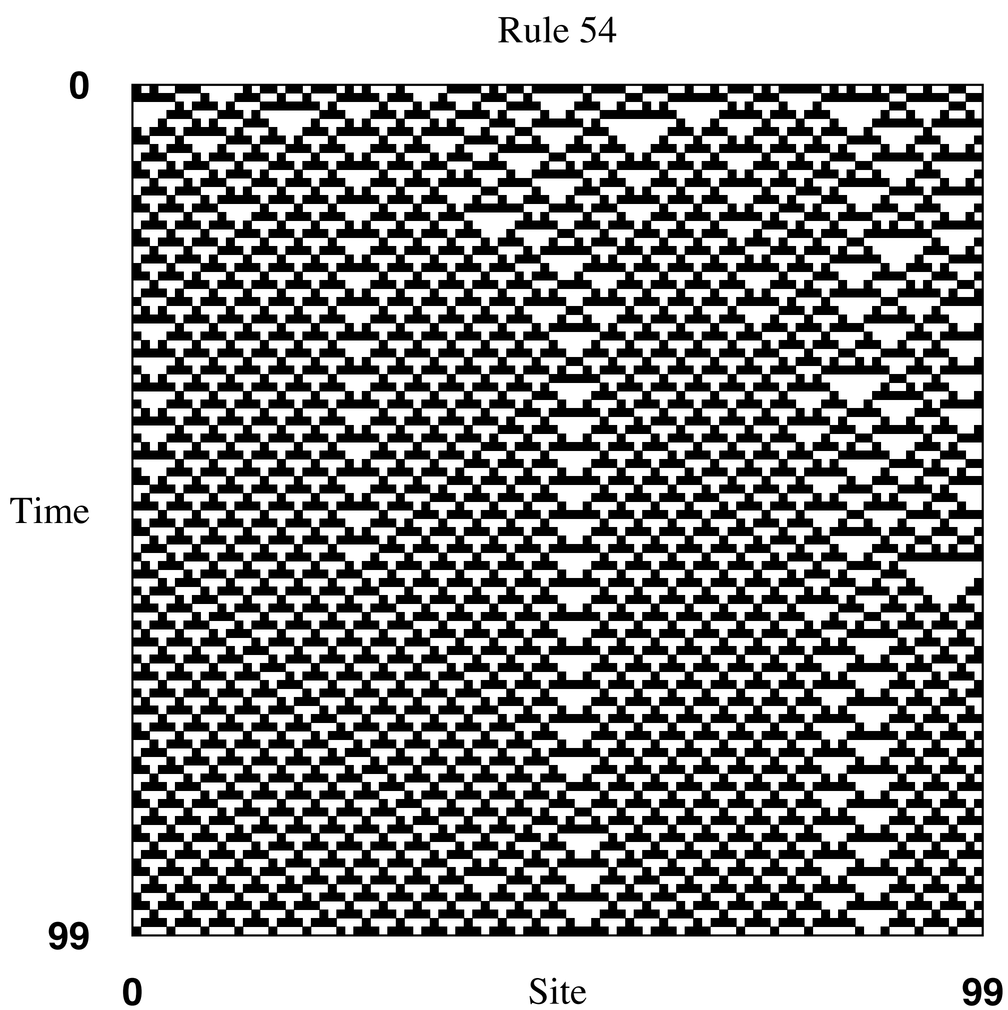
"Lyapunov" exponent = Spreading rate?

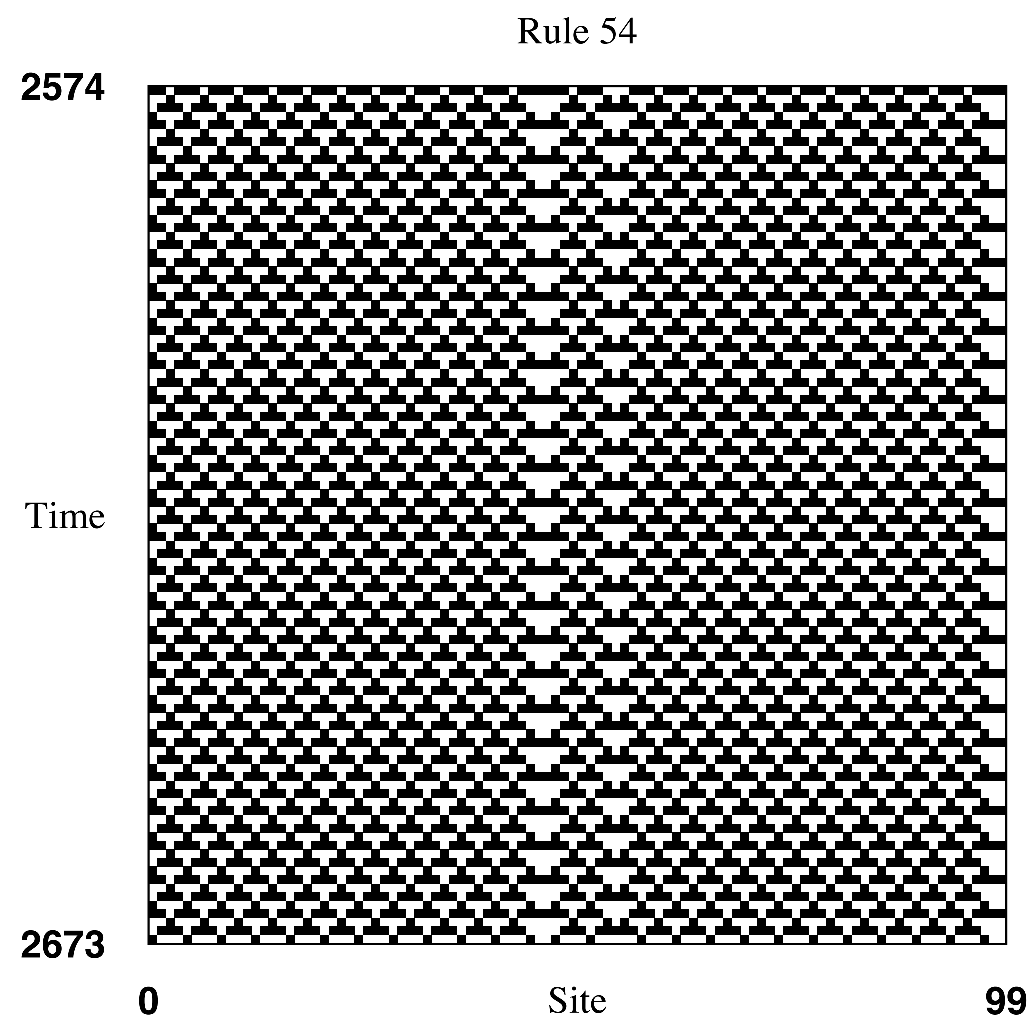
Pattern Formation I ...

Pattern formation in cellular automata ...

1D CA Behavior Survey ...

Long transients:



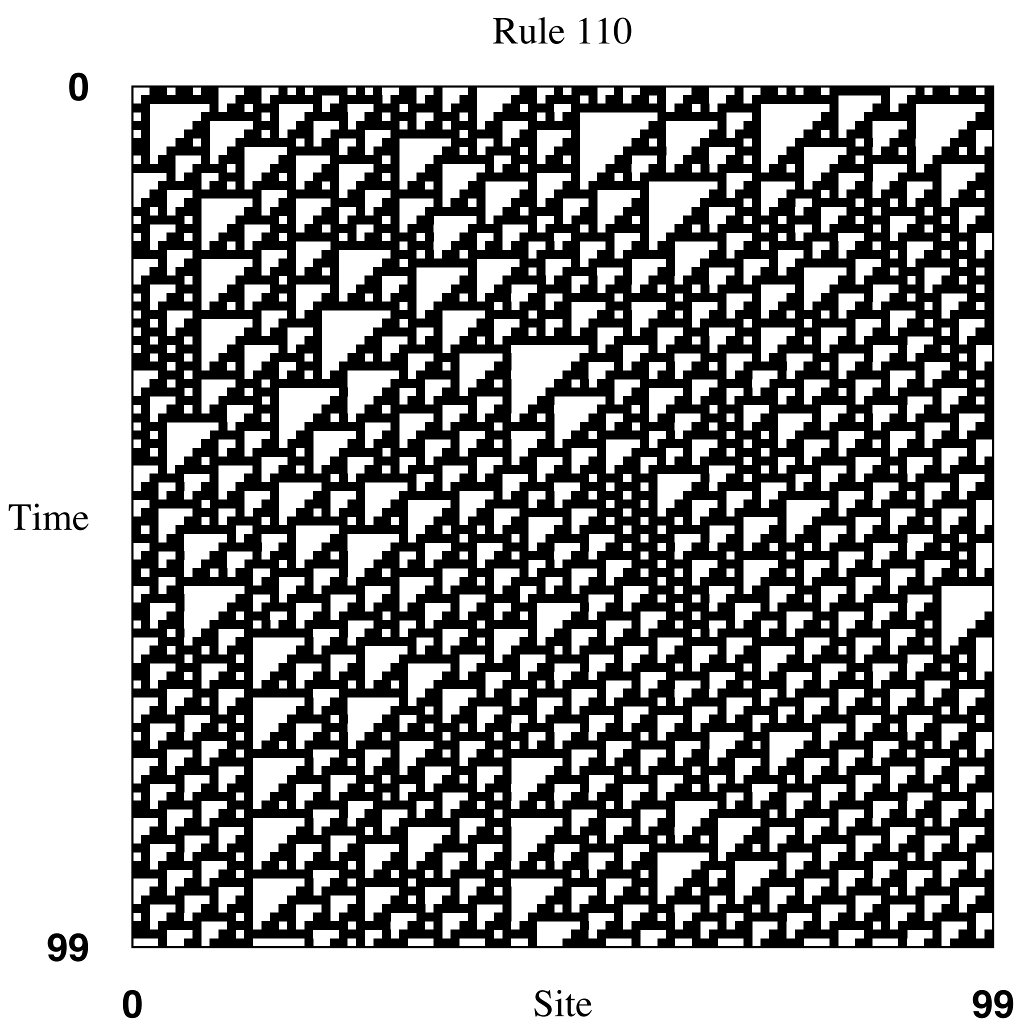


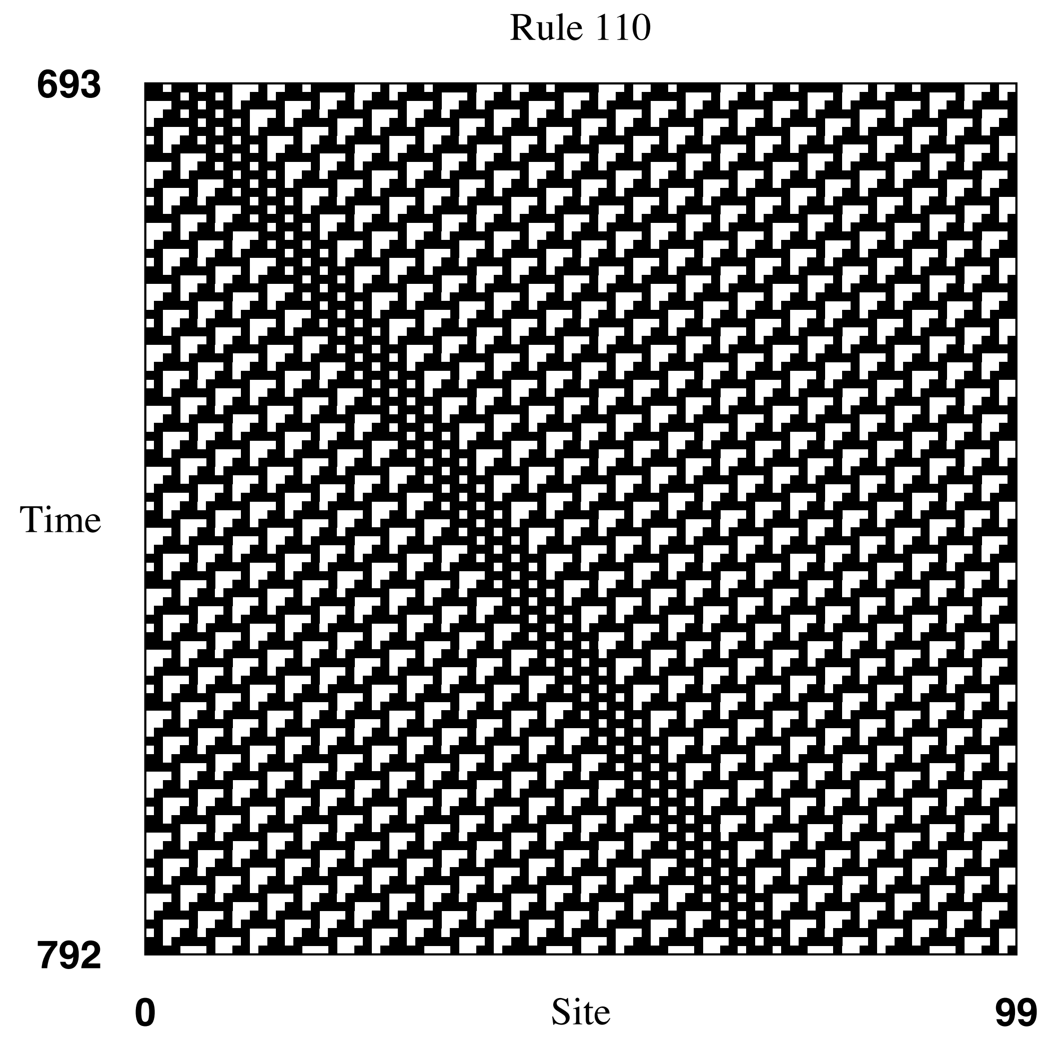
Pattern Formation I ...

Pattern formation in cellular automata ...

1D CA Behavior Survey ...

Long transients ...

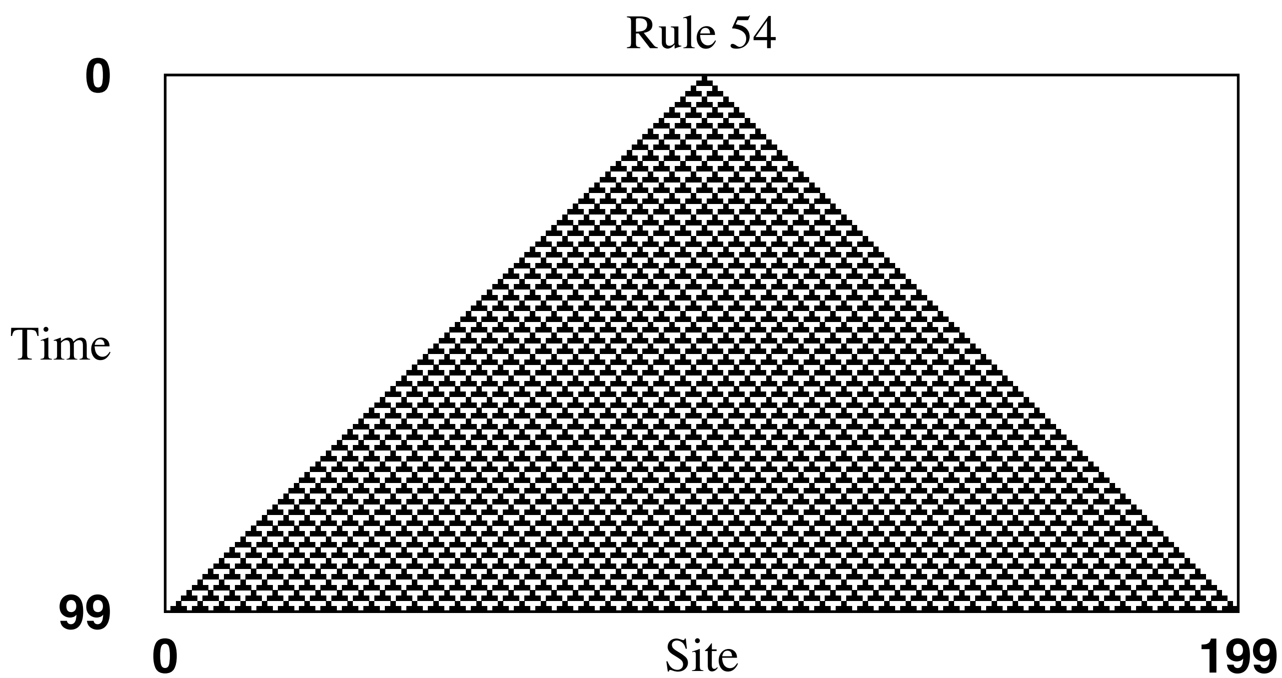


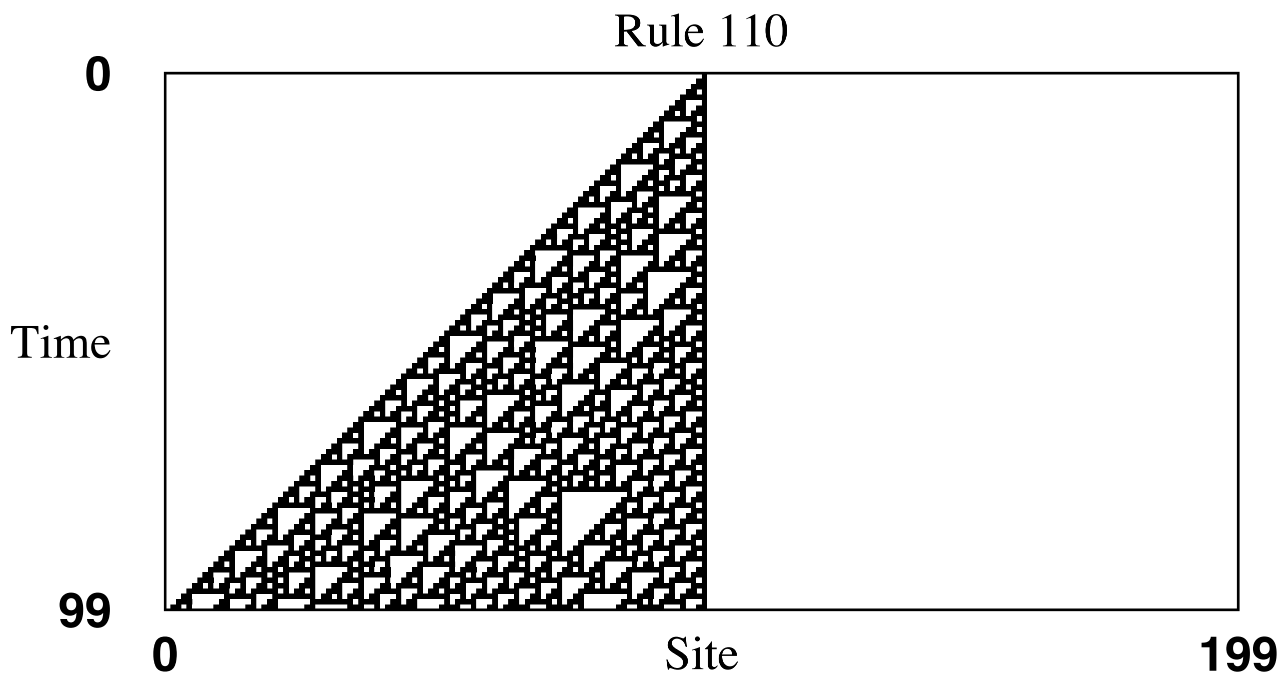


Pattern formation in cellular automata ...

1D CA Behavior Survey ...

Long transients ... IC: $0^N + \ldots 0001000\ldots \mod 1$





Pattern Formation I ...

Pattern formation in cellular automata ...

1D CAs ...

Simulation demo: la1d

r = 1 with Monte Carlo sampling

r = 2 with Monte Carlo sampling

Cylindrical view

As map of the Unit Square

$$\Phi : \mathrm{T}^2 \rightarrow \mathrm{T}^2$$

$$\mathbf{s} = (x, y) = (.s_1 s_2 s_3 \ldots, .s_0 s_{-1} s_{-2} \ldots)$$

Pattern Formation I ...

Pattern formation in map lattices:

Space and time: Discrete

Local State: Continuous

Map Lattice:

Local (k-dimensional) state: $x^i \in \mathbf{R} = \mathbb{R}^k$

Global state in d-dimensional space: $\mathbf{x} \in \mathcal{X} = \mathbf{R}^{\mathbb{Z}^d}$

Neighborhood: $\eta^i \in \overbrace{\mathbf{R} \times \mathbf{R} \cdots \times \mathbf{R}}^{r^d}$

Local dynamic: $x^i_{t+1} = \phi(\eta^i_t)$

Global dynamic: $\mathbf{x}^i_{t+1} = \Phi(\mathbf{x}_t)$

Pattern Formation I ...

Pattern formation in map lattices ...

Map Lattice ...

Continuous state space allows familiar tools:
- Qualitative dynamics and bifurcation analysis
- LCE spectrum

And some new ones:
- Spatial return maps
- Spatial propagation of perturbations (!= LCEs)

Pattern Formation I ...

Pattern formation in map lattices ...

Logistic Map Lattice in 1D:

Lattice: $i \in \mathbb{Z}$

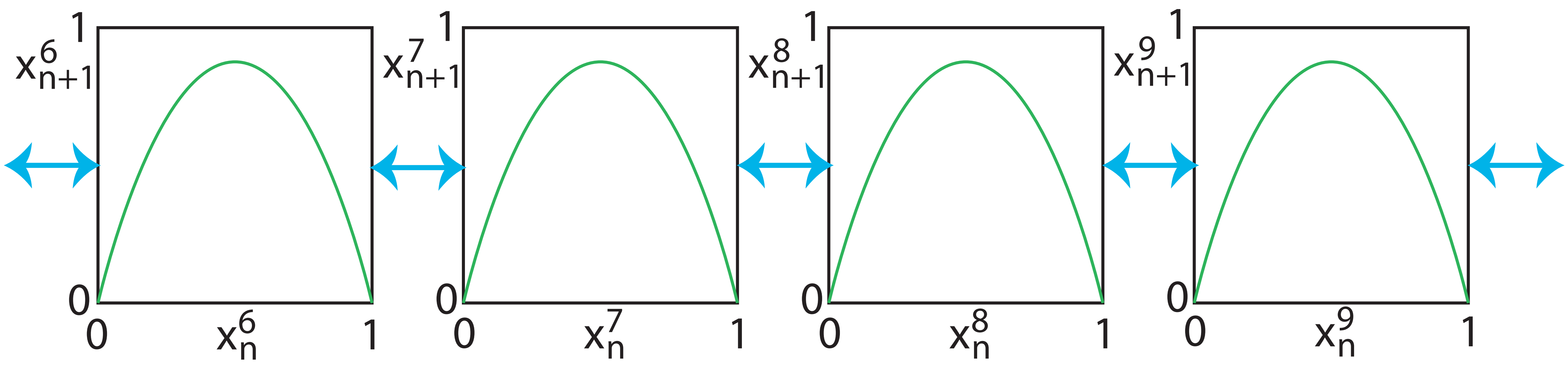
Local state: $x \in [0, 1]$

Local dynamic: $x^i_{t+1} = r x^i_t (1 - x^i_t) + c(x^{i-1}_t + x^{i+1}_t)$

Parameters:

Nonlinearity: $r \in [0, 4]$

Coupling strength: $c \in [0, 1]$



Pattern Formation I ...

Pattern formation in map lattices ...

Logistic Map Lattice in 1D ...

Infinite number of attractors

Consider two coupled maps:

$$x_{t+1} = r x_t (1 - x_t) + c y_t$$

$$y_{t+1} = r y_t (1 - y_t) + c x_t$$

$$r = 3.56995$$

$$c = 0.005$$

Isolated: $r' \approx r + c$

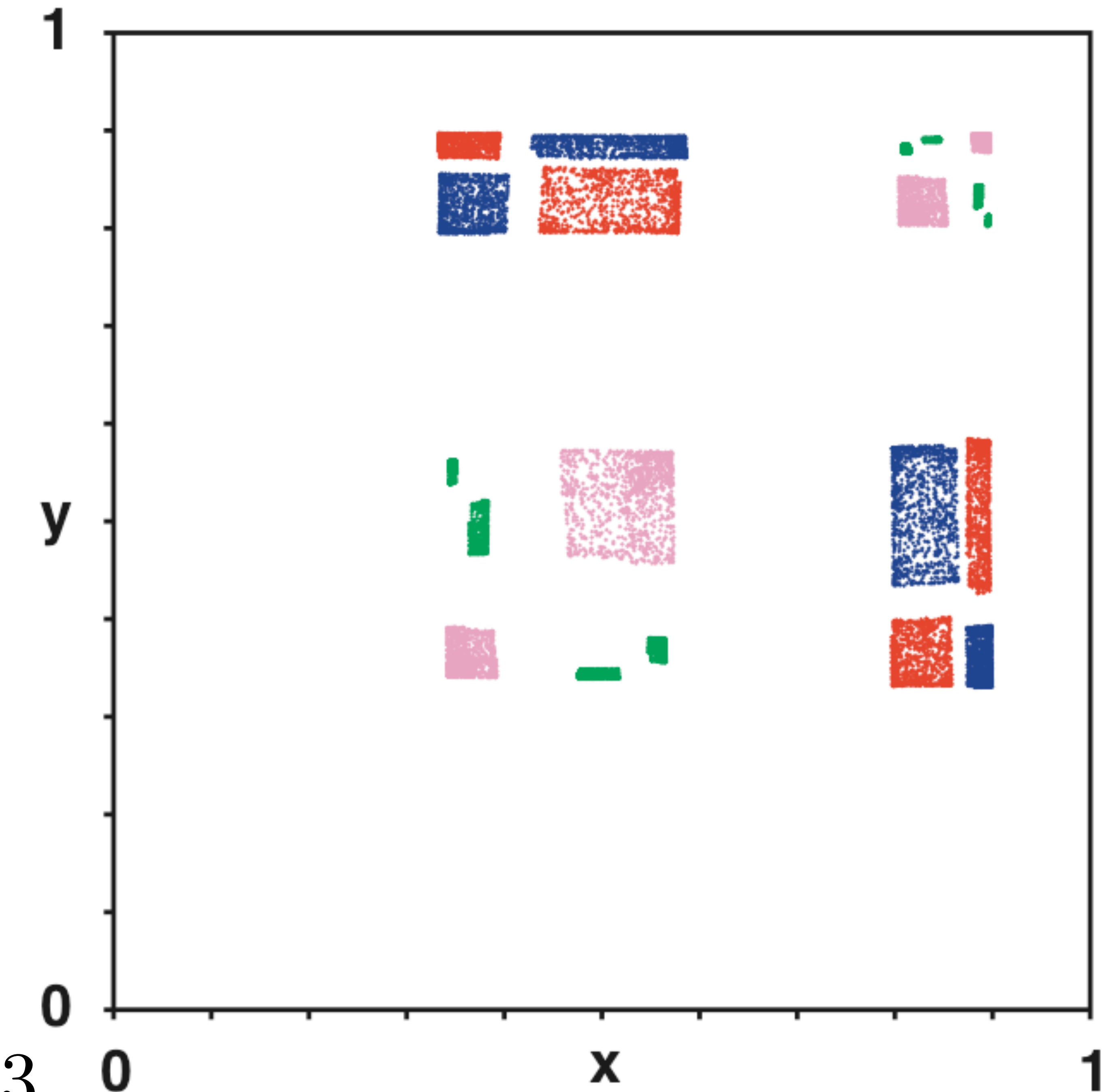
Each has one attractor

4-band chaos

Coupled:

Phase difference matters

$\Lambda^{\Delta\text{phase}} : \Delta\text{phase} = 0, 1, 2, 3$



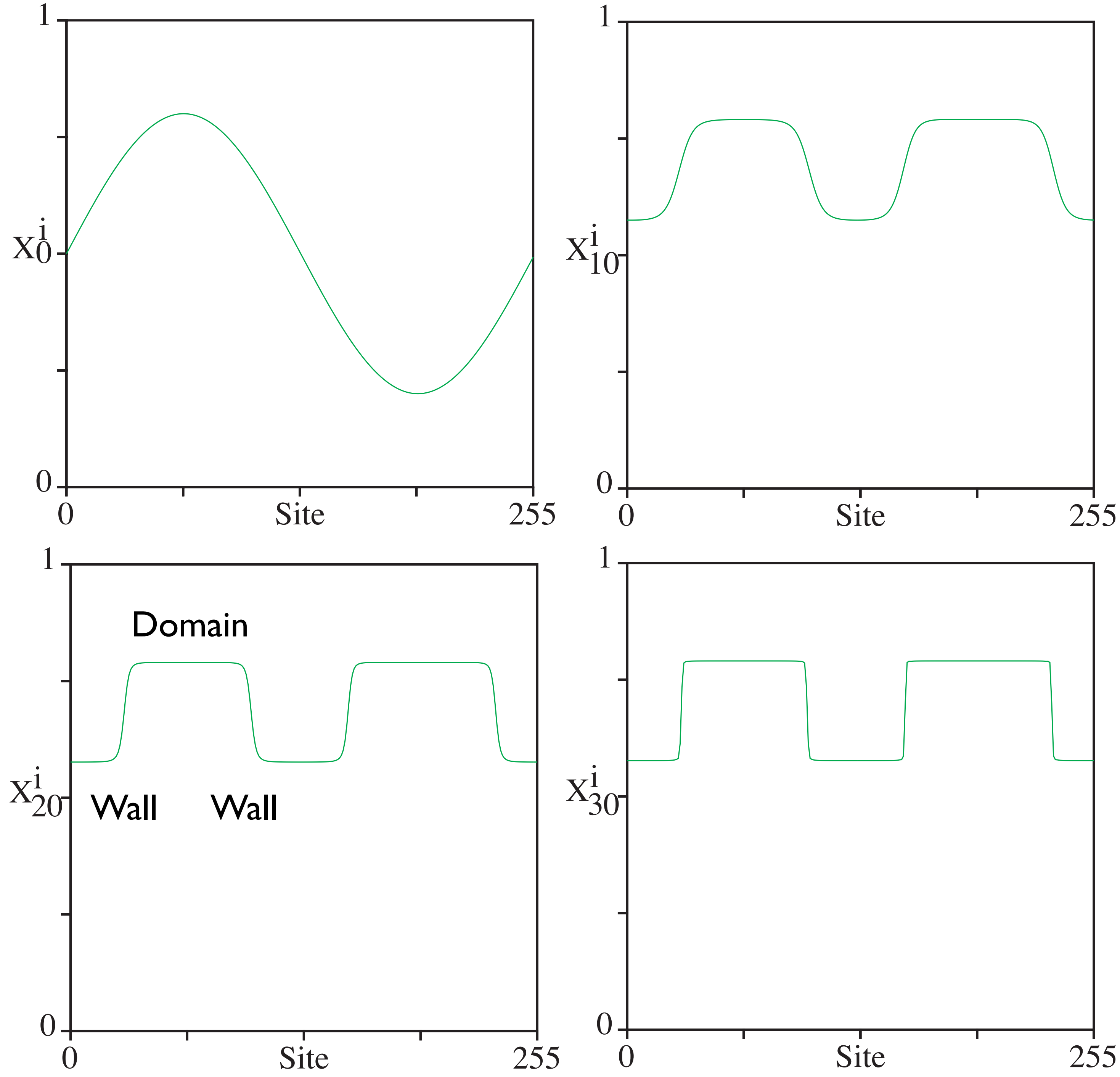
Pattern Formation I ...

Pattern formation in map lattices ...

Logistic Map Lattice in 1D ...

Spatiotemporal period-doubling:

r = 3.0
c = 0.05



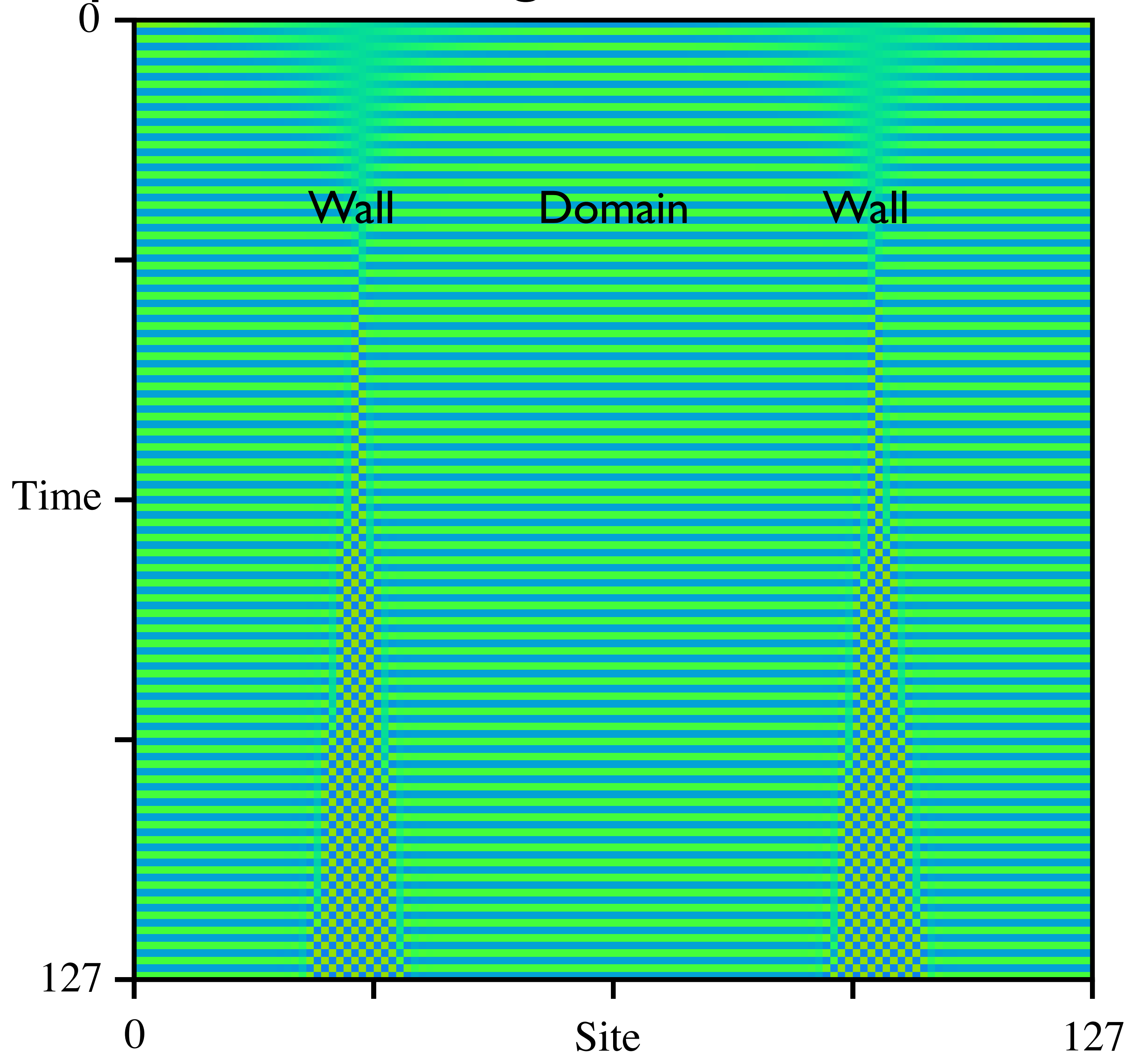
Pattern Formation I ...

Pattern formation in map lattices ...

Logistic Map Lattice in 1D ...

Spatiotemporal period-doubling:

r = 3.0
c = 0.05



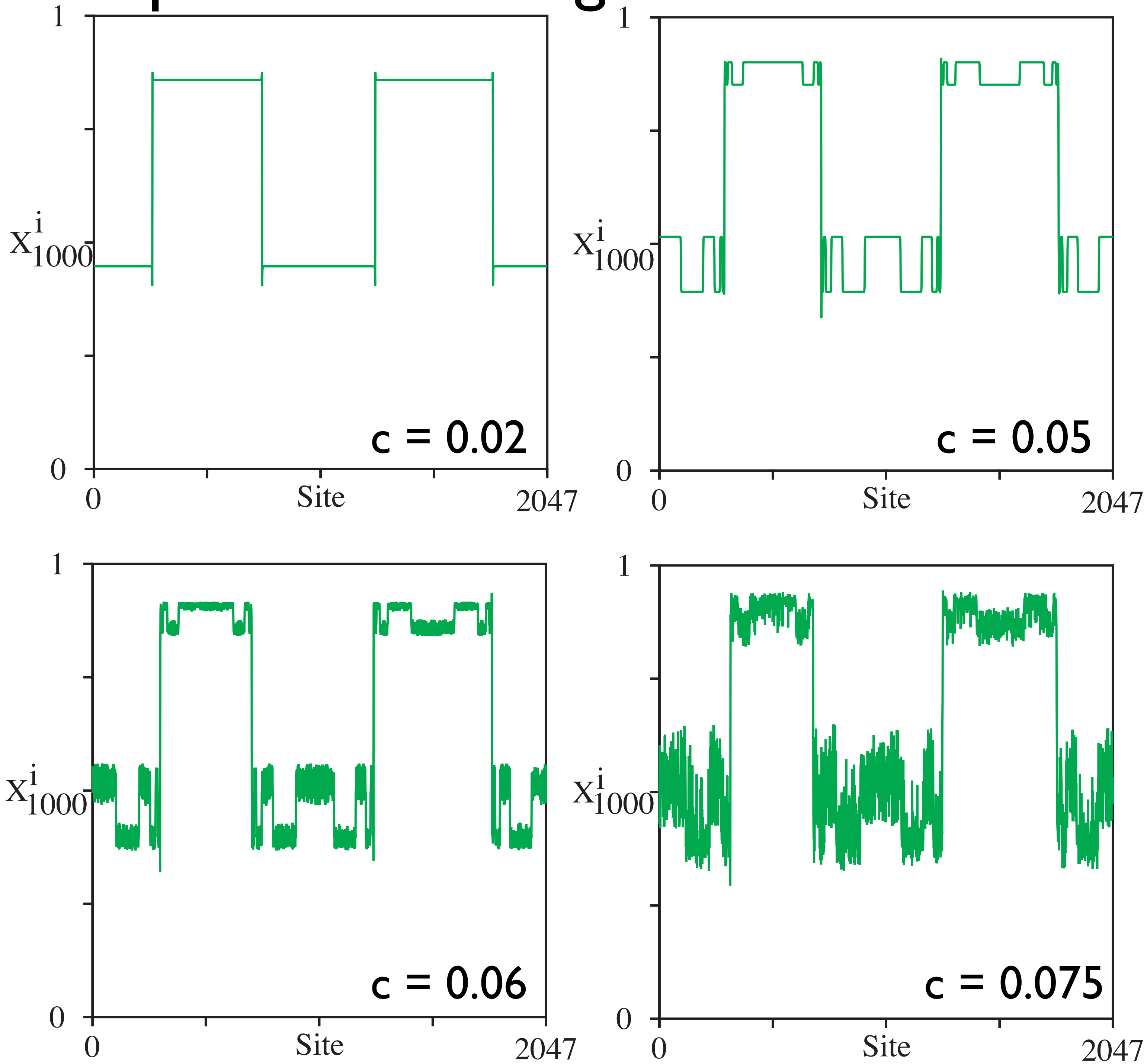
Pattern Formation I ...

Pattern formation in map lattices ...

Logistic Map Lattice in 1D ...

Spatiotemporal period-doubling:

r = 3.4



Pattern Formation I ...

Pattern formation in map lattices ...

Logistic Map Lattice in 1D ...

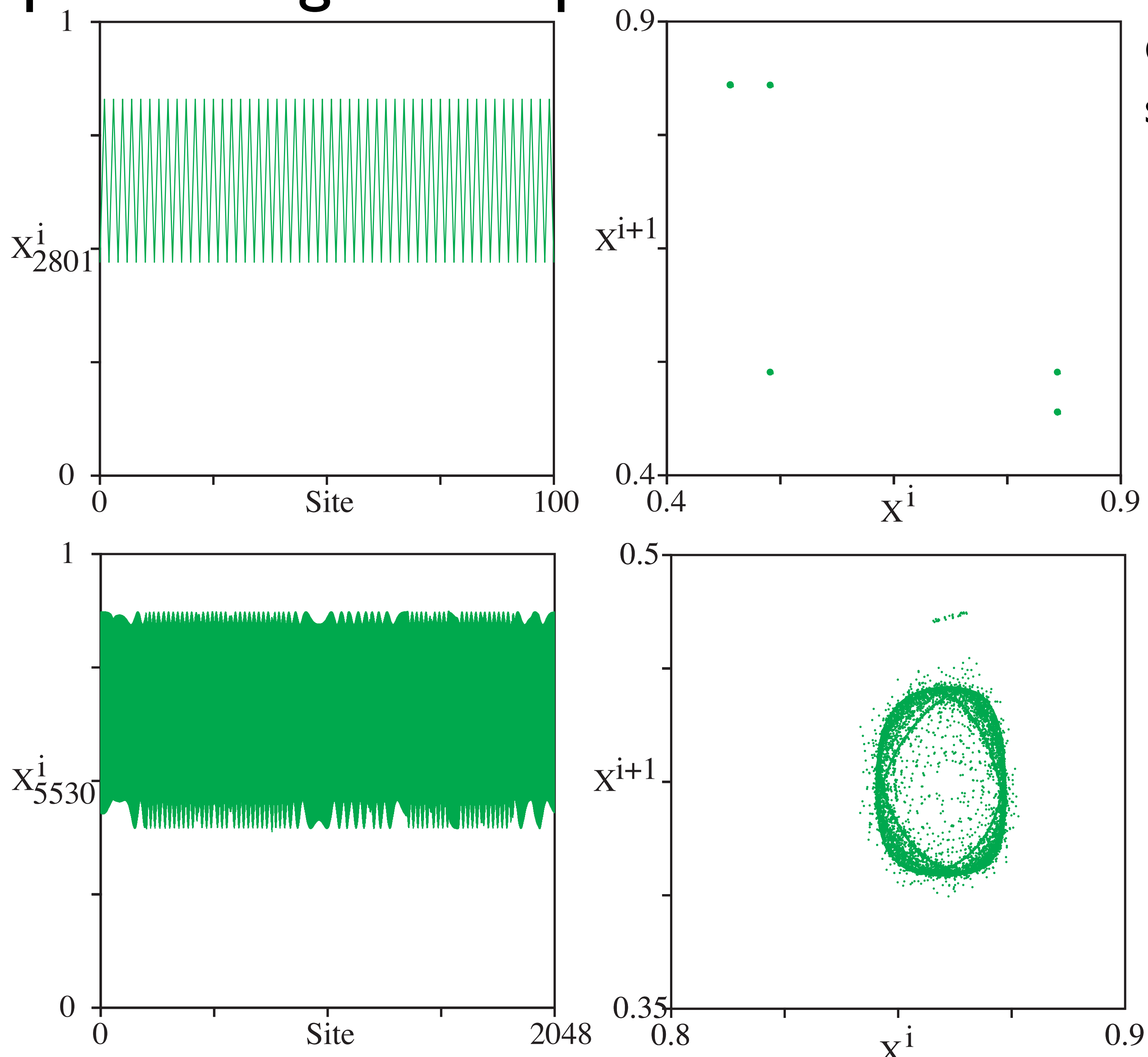
Bifurcation to quasiperiodicity: Emergent!

New, not part of logistic map

r = 3.0

c = 0.05

c = 0.075



Configuration v. spatial return map

Traveling waves!

Pattern Formation I ...

Pattern formation in map lattices ...

Logistic Map Lattice in 1D ...

Weak coupling:

- Local dynamic dominates
- Period-doubling in spacetime
- ∞ number of attractors
- Patterns: domains and walls

Strong coupling (relative to nonlinearity):

- Something new: Emergent
- Quasi-periodicity
- Spatiotemporal traveling waves
- Hopf "bifurcation" in spatial return map

Pattern Formation I ...

Pattern formation in map lattices ...

Dripping Handrail in 1D:

Lattice: $i \in \mathbb{Z}$

Local state: $x \in [0, 1]$

Local dynamic: $x^i_{t+1} = f(x^i_t) + c(x^{i-1}_t + x^{i+1}_t)$

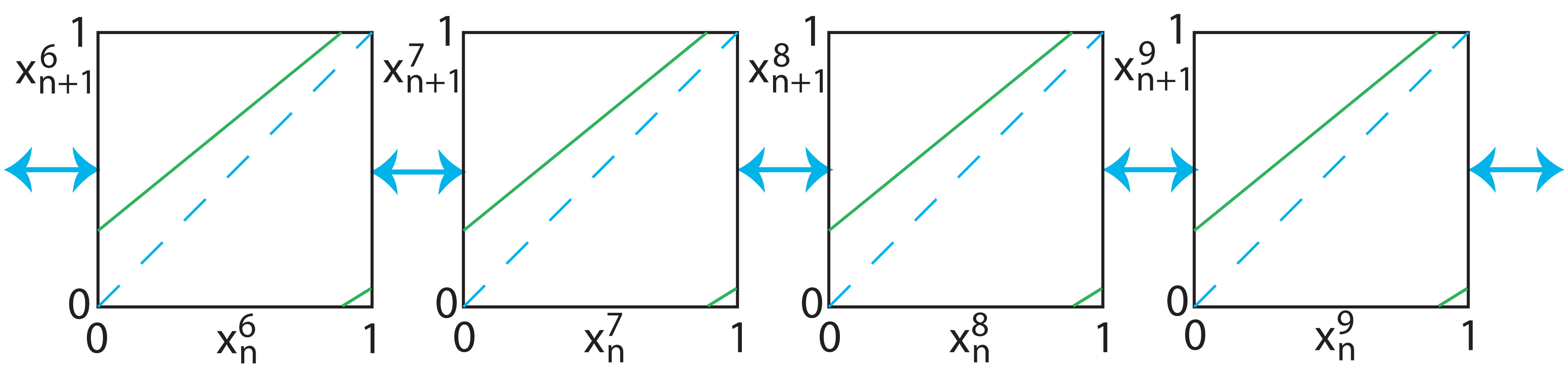
$$f(x) = w + sx \pmod 1$$

Parameters:

- Offset: $w \in [0, 1]$
- Rate: $s \in [0, 1]$
- Coupling strength: $c \in [0, 1]$

Model of:

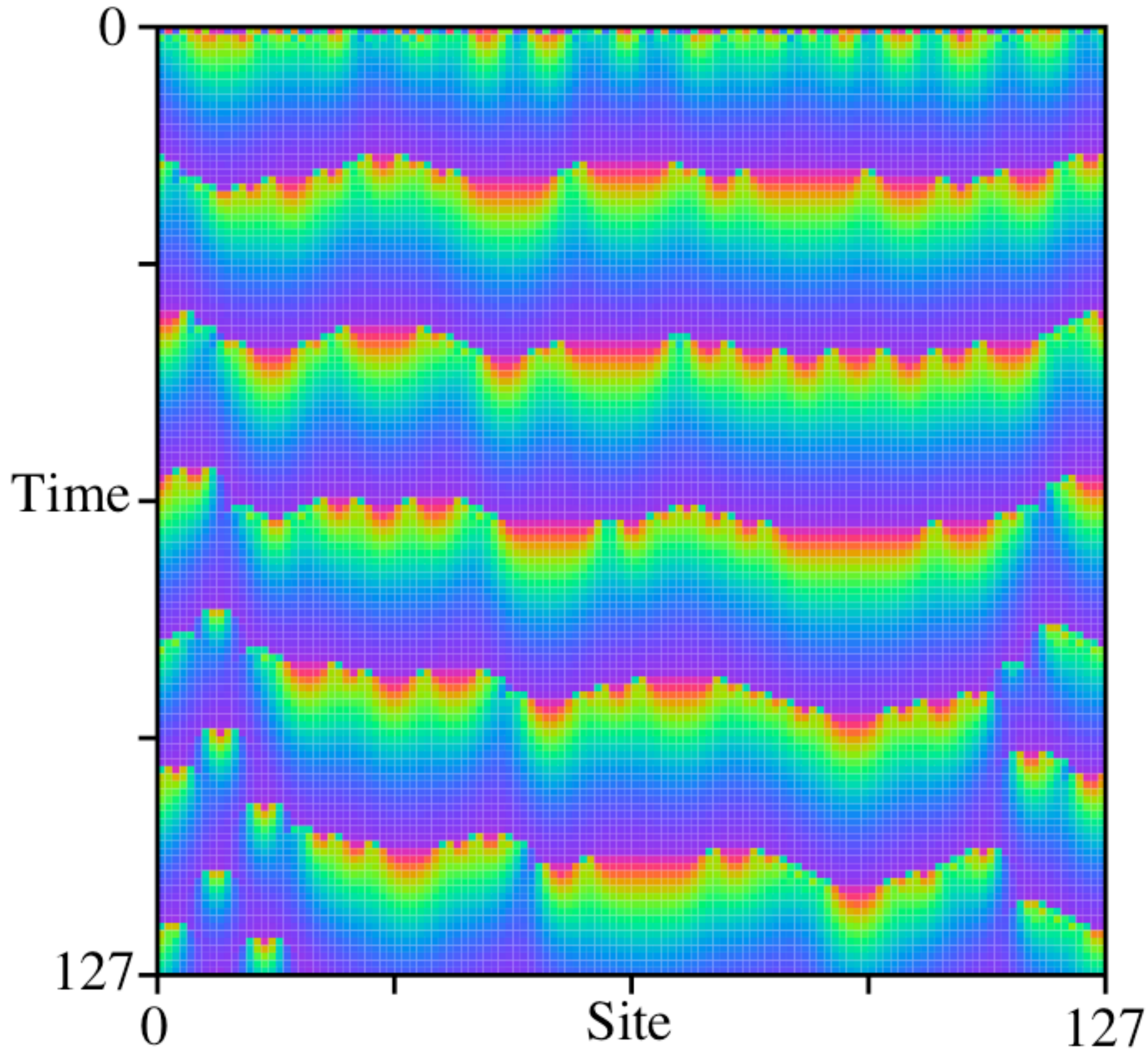
- Oscillator chains
- Integrate-and-fire neurons



Pattern Formation I ...

Pattern formation in map lattices ...

Dripping Handrail ...



Pattern Formation I ...

Pattern formation in map lattices ...

Dripping Handrail ...

Isolated map:

- Stable limit cycle

Small lattices (N small):

- All ICs ultimately periodic (domains & walls)
- Transient time: $T(N) \propto e^{e^N}$
- Never get to the attractor
- Transients dominate dynamics
- Subbasin-portal structure of DHR attractor-basin portrait

Pattern Formation I ...

Reading for next lecture:

Lecture Notes.