Example Chaotic Maps
(that you can analyze)

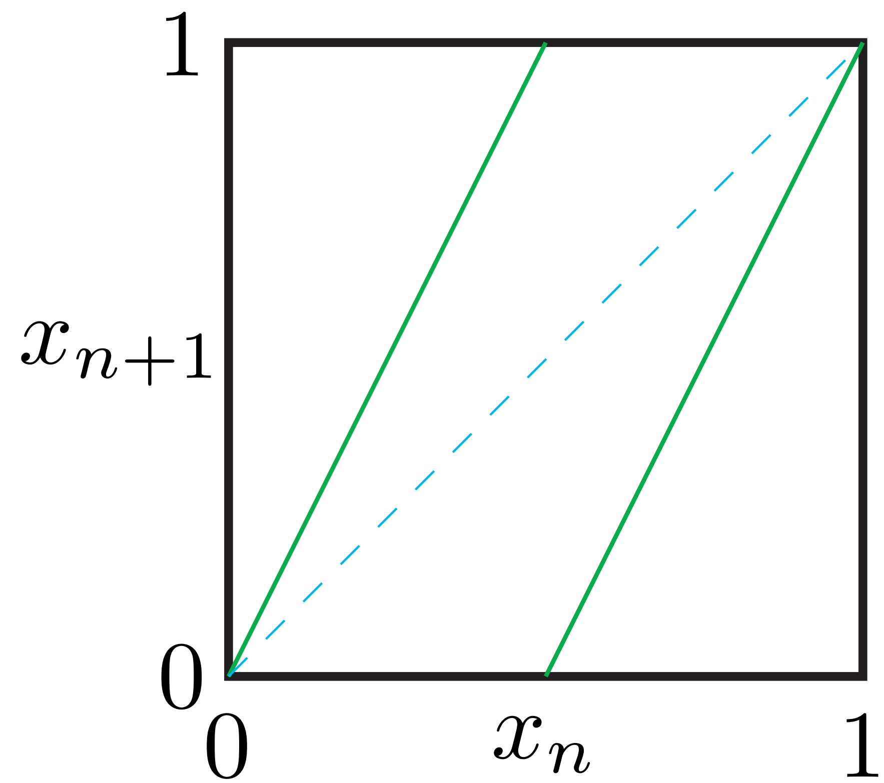
Reading for this lecture:

NDAC, Sections10.5-10.7.

Example 1D Maps ...

Shift Map: $x_n \in [0, 1]$

$$x_{n+1} = f(x_n) = 2x_n \pmod 1$$



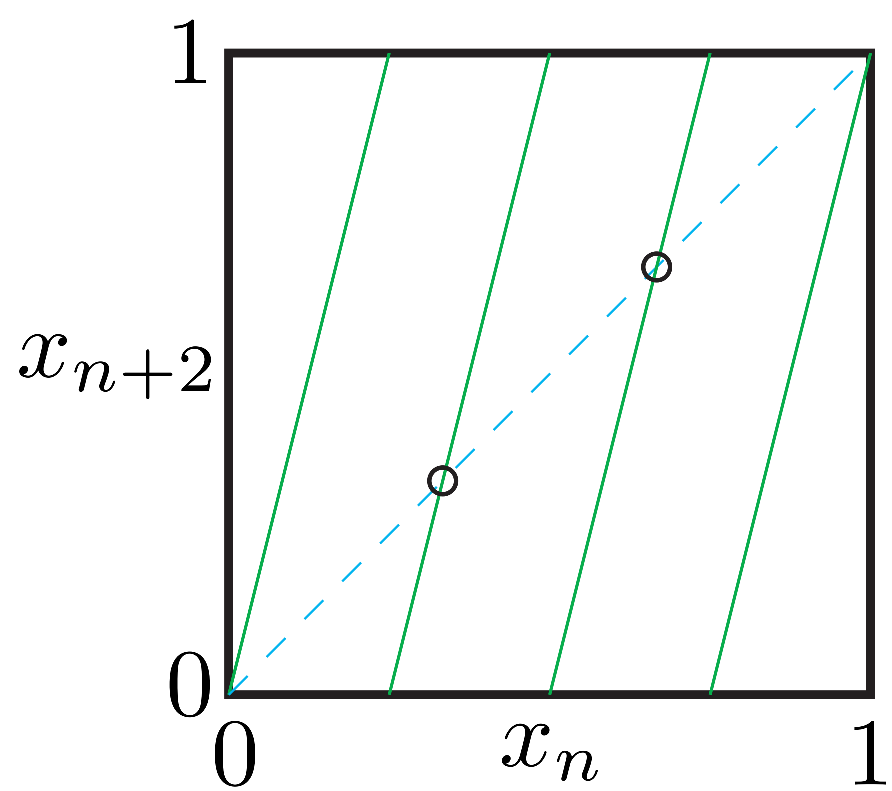
Fixed Point: $x^* = 0$

Unstable: $f'(x^*) = 2 > 1$

Period-2 Orbit: $\{x^*\} = \{1/3, 2/3\}$

Unstable: $(f^2)'(x^*) = 4 > 1$

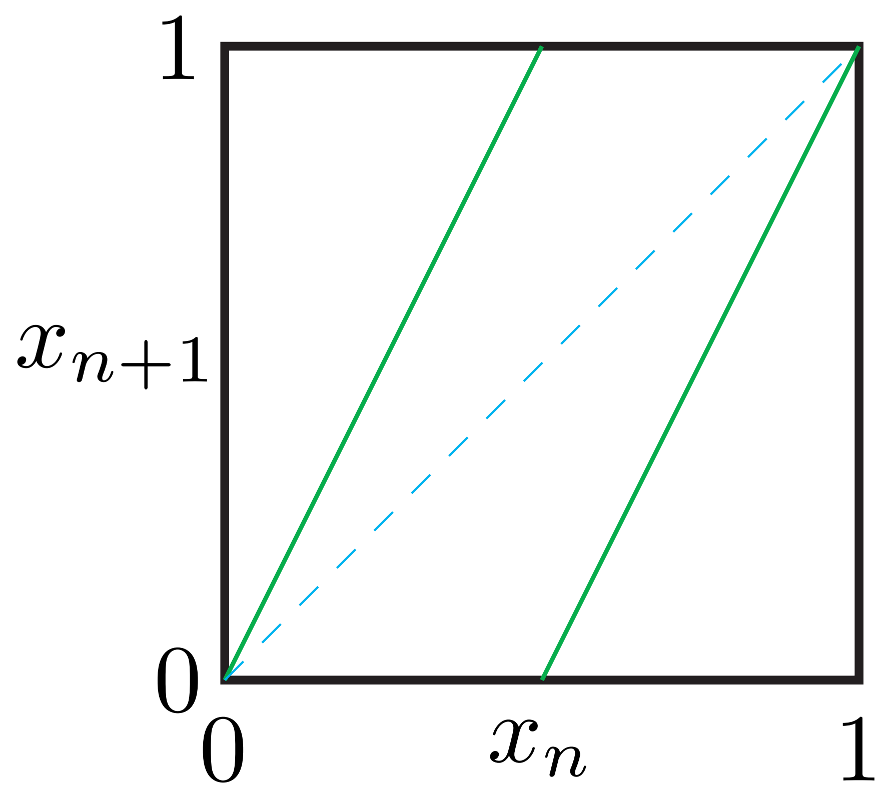
All periodic orbits unstable



Example 1D Maps ...

Shift Map: $x_n \in [0,1]$

$$x_{n+1} = 2x_n \pmod 1$$



Solvable! $x_n = 2^n x_0 \pmod 1$

Chaotic mechanism:

shift up least significant digits

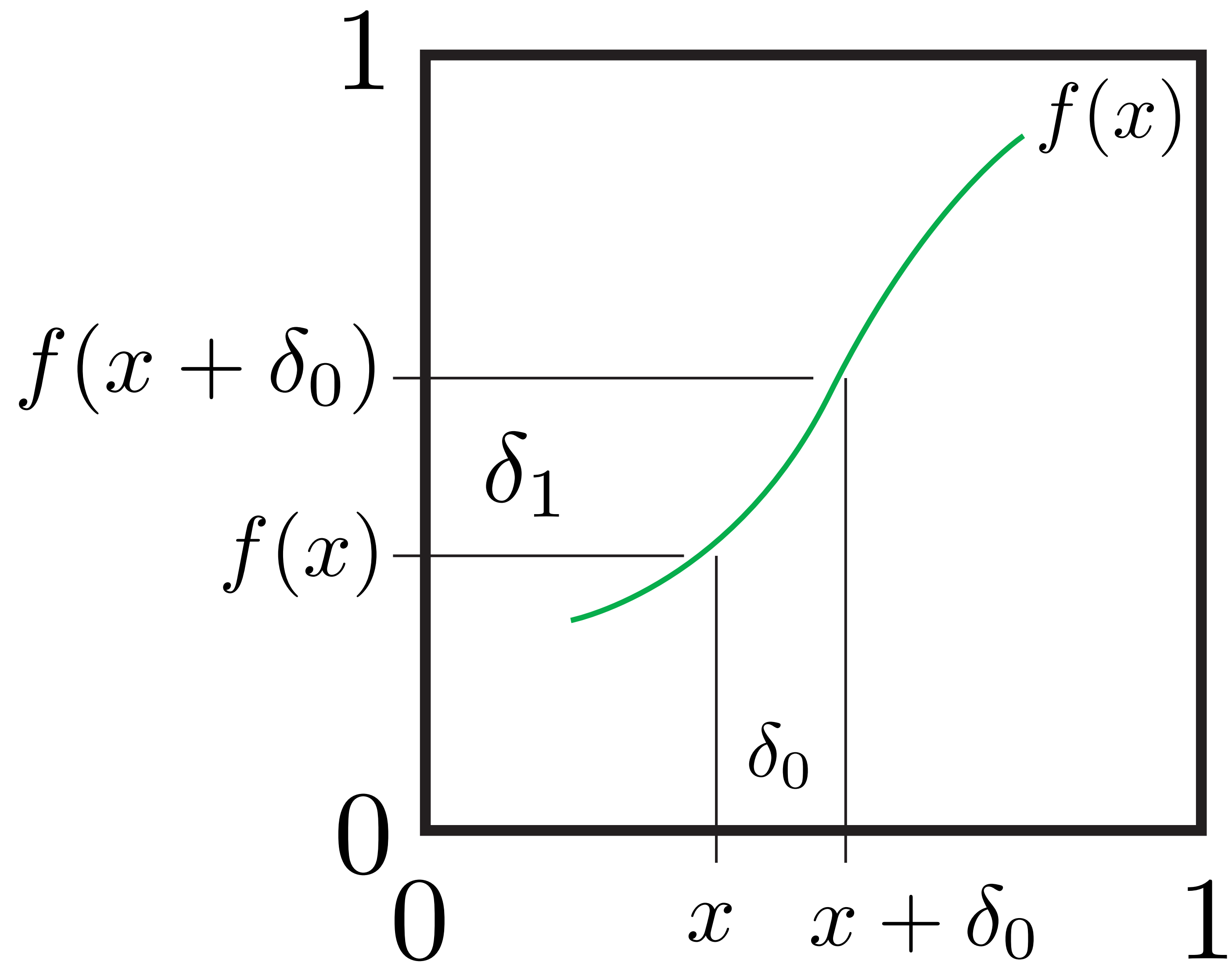
$$x_0 = 0.1\boxed{101010111.}..$$

$$x_1 = 0.\boxed{1010101110}...$$

Example 1D Maps ...

Lyapunov Characteristic Exponent for 1D Maps:

$$x_{n+1} = f(x_n)$$



$$\delta_1 \approx f'(x_0)\delta_0$$
$$\delta_2 \approx f'(x_1)\delta_1$$
$$\delta_2 \approx f'(x_1)f'(x_0)\delta_0$$
$$\vdots$$
$$|\delta_N| \sim |\delta_0| e^{\lambda \cdot N}$$

or, Definition: LCE

$$\lambda = \lim_{\substack{N \to \infty \\ ||\delta_0|| \to 0}} \log_2 \left| \frac{\delta_N}{\delta_0} \right|$$

Example 1D Maps ...

Lyapunov Characteristic Exponent for 1D Maps ...

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \log_2 \left| \frac{f^N(x_0 + \delta_0) - f^N(x_0)}{(x_0 + \delta_0) - x_0} \right|$$

$\delta_0 \to 0$

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \log_2 \left| (f^N)'(x_0) \right|$$

$$(f^N)'(x_0) = f'(x_{N-1})(f^{N-1})'(x_0) = f'(x_0) f'(x_1) \cdots f'(x_{N-1})$$

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \log_2 |f'(x_n)|$$

$\lambda < 0$ stable

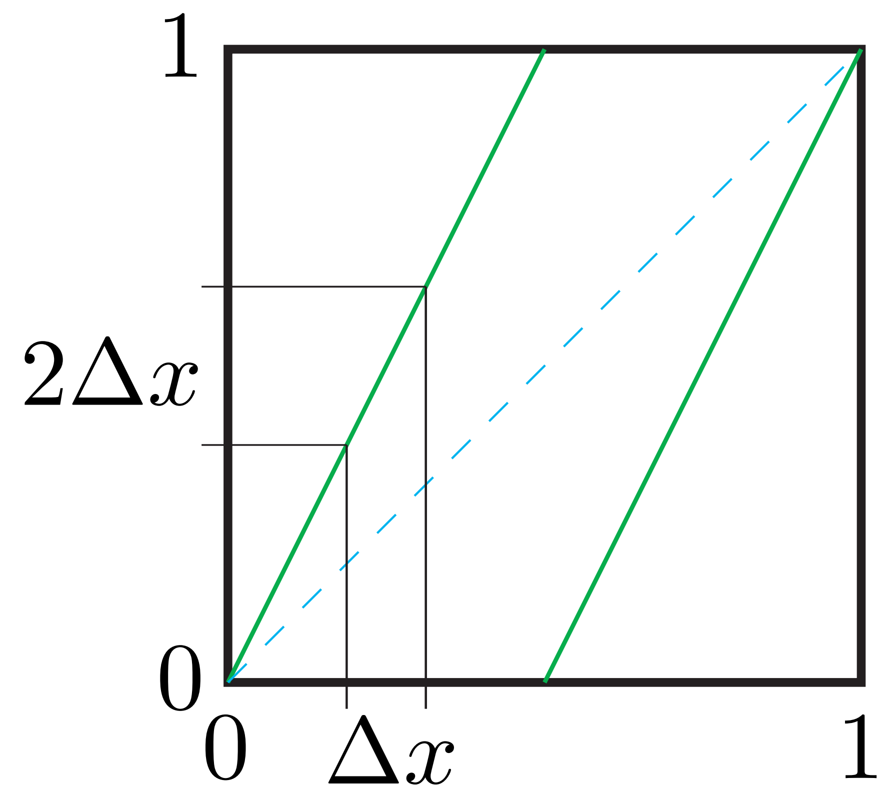
$\lambda > 0$ unstable

Example 1D Maps ...

Back to Shift Map: Its LCE ...

$$x_{n+1} = f(x_n) = 2x_n \pmod 1$$

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \log_2 |f'(x_n)|$$



Independent of state:

$$f'(x) = 2$$

Amplification per step
(or bits of resolution *lost*):

$$\lambda = 1$$

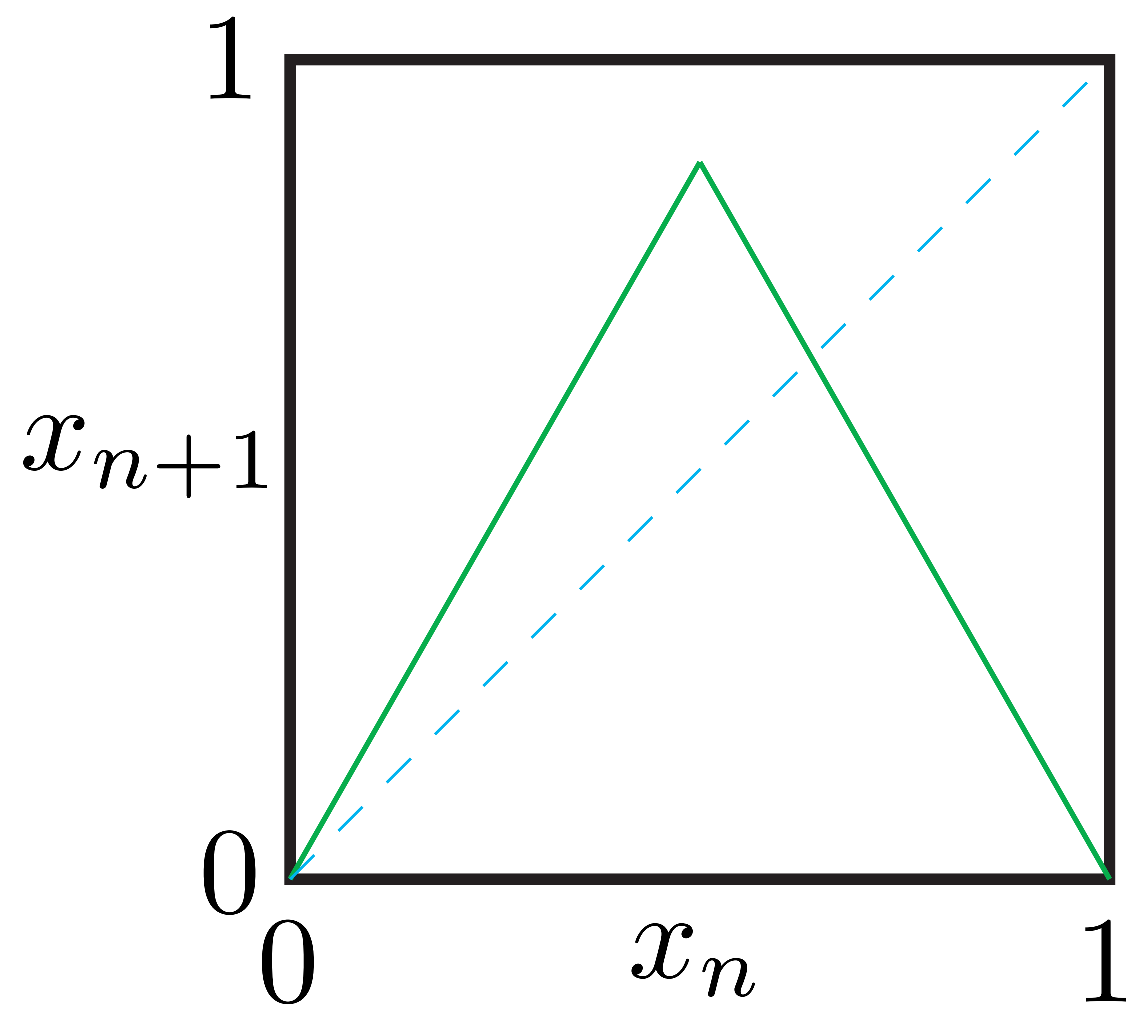
Example 1D Maps ...

Tent Map: $x_n \in [0, 1]$

$$x_{n+1} = \begin{cases} a x_n, & 0 \leq x_n \leq \frac{1}{2} \\ a(1 - x_n), & \frac{1}{2} < x_n \leq 1 \end{cases}$$

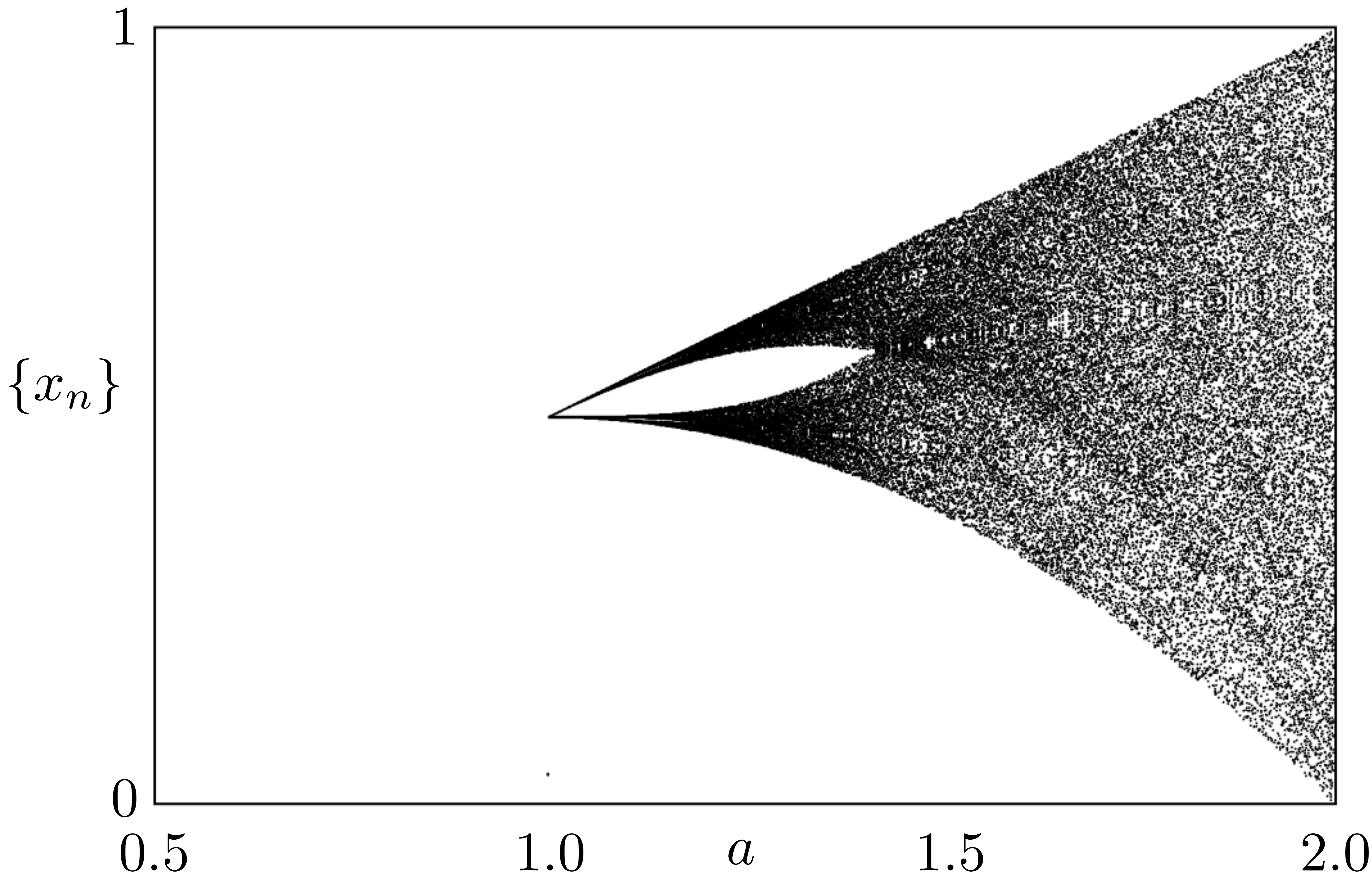
Slope: $a \in [0, 2]$

Height at max: $\dfrac{a}{2}$



Example 1D Maps ...

Tent Map Bifurcation Diagram:



Example 1D Maps ...

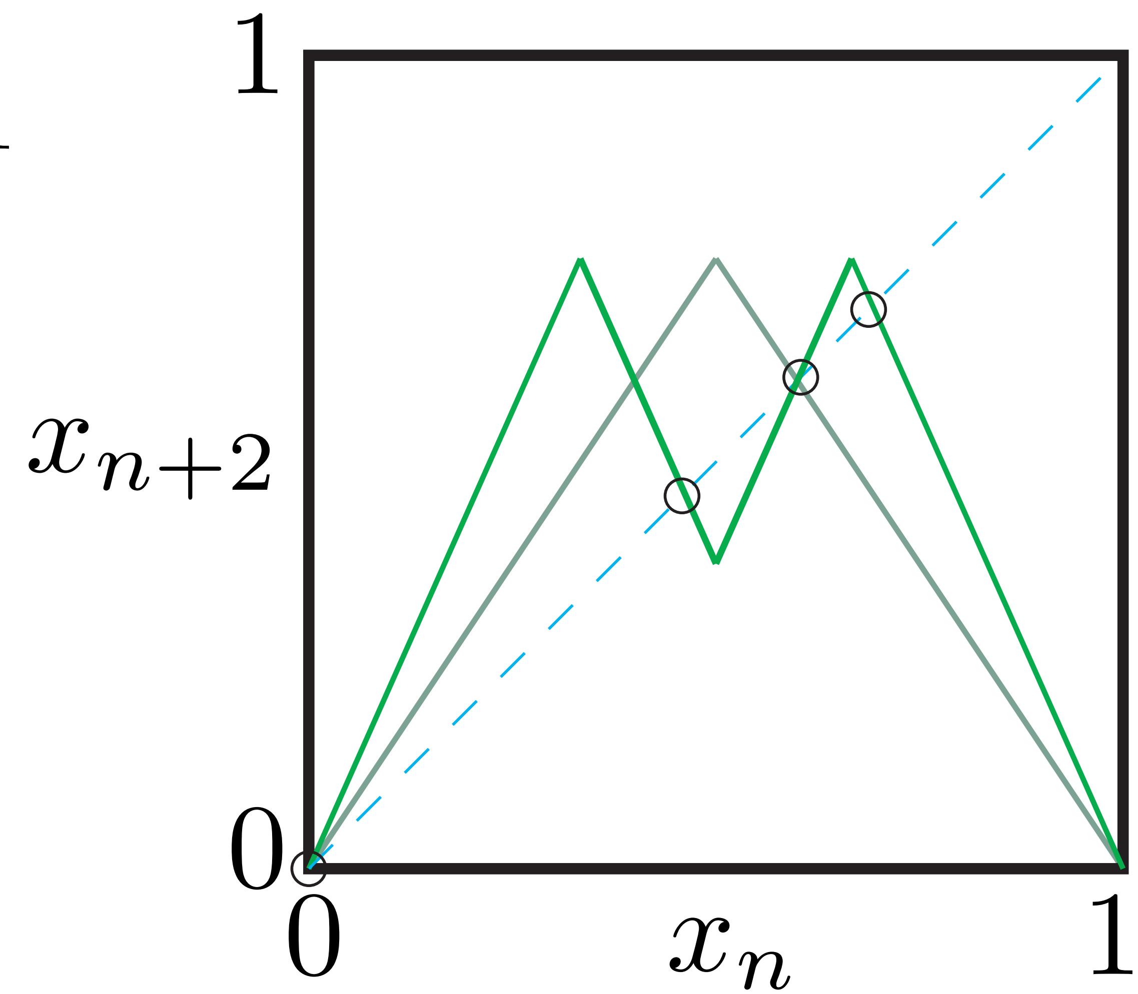
Tent Map ...

Stable fixed point: $x^* = 0,\ 0 \leq a < 1$

Unstable fixed points: $\{0, \frac{a}{1+a}\},\ 1 \leq a \leq 2$

All periodic orbits unstable: $p > 1$

No periodic windows



Example 1D Maps ...

Tent Map LCE:

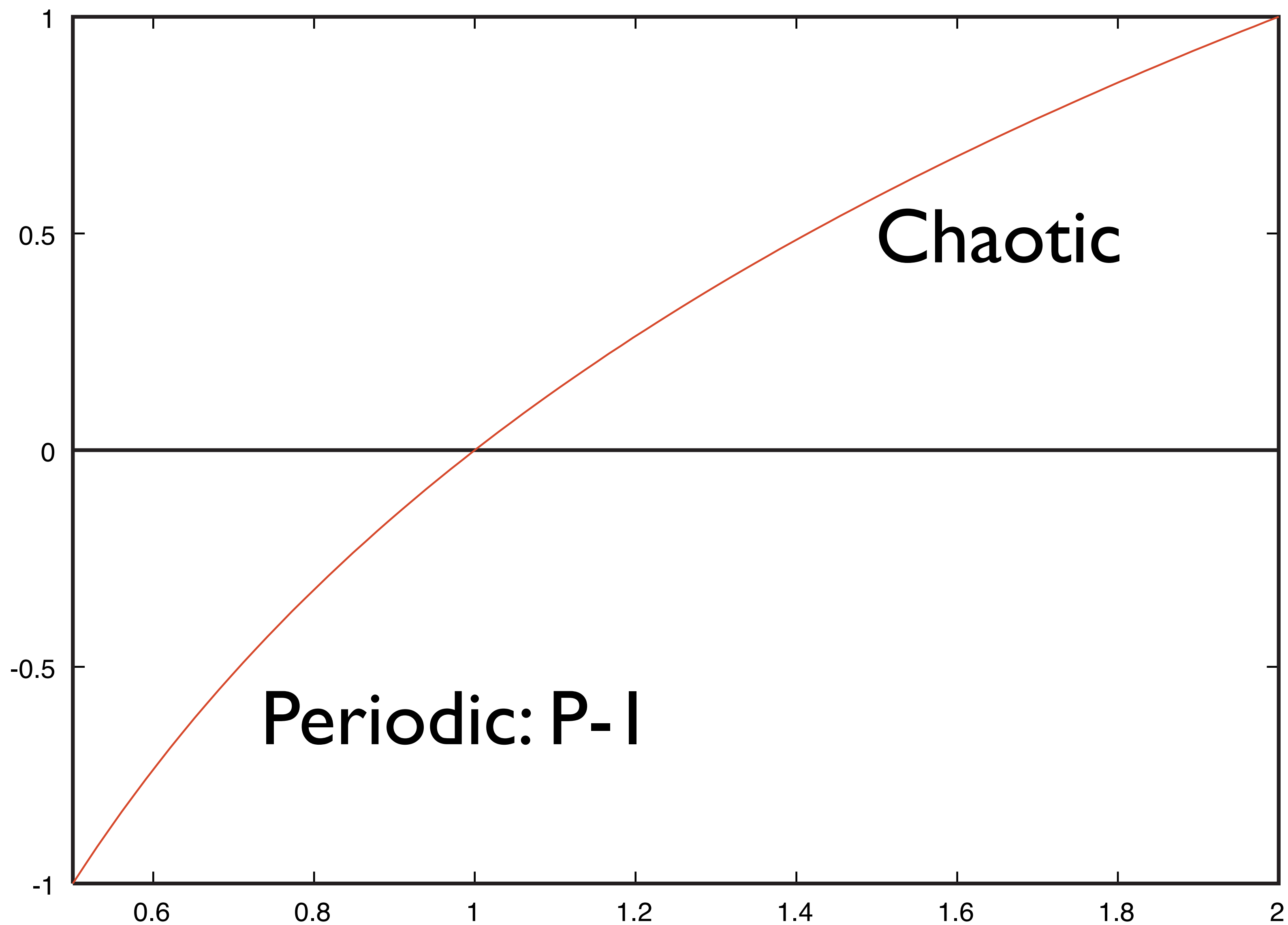
$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \log_2 |f'(x_n)|$$

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \log_2 |\pm a| = \log_2 a$$

Example 1D Maps ...

Tent Map LCE:

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \log_2 |\pm a| = \log_2 a$$

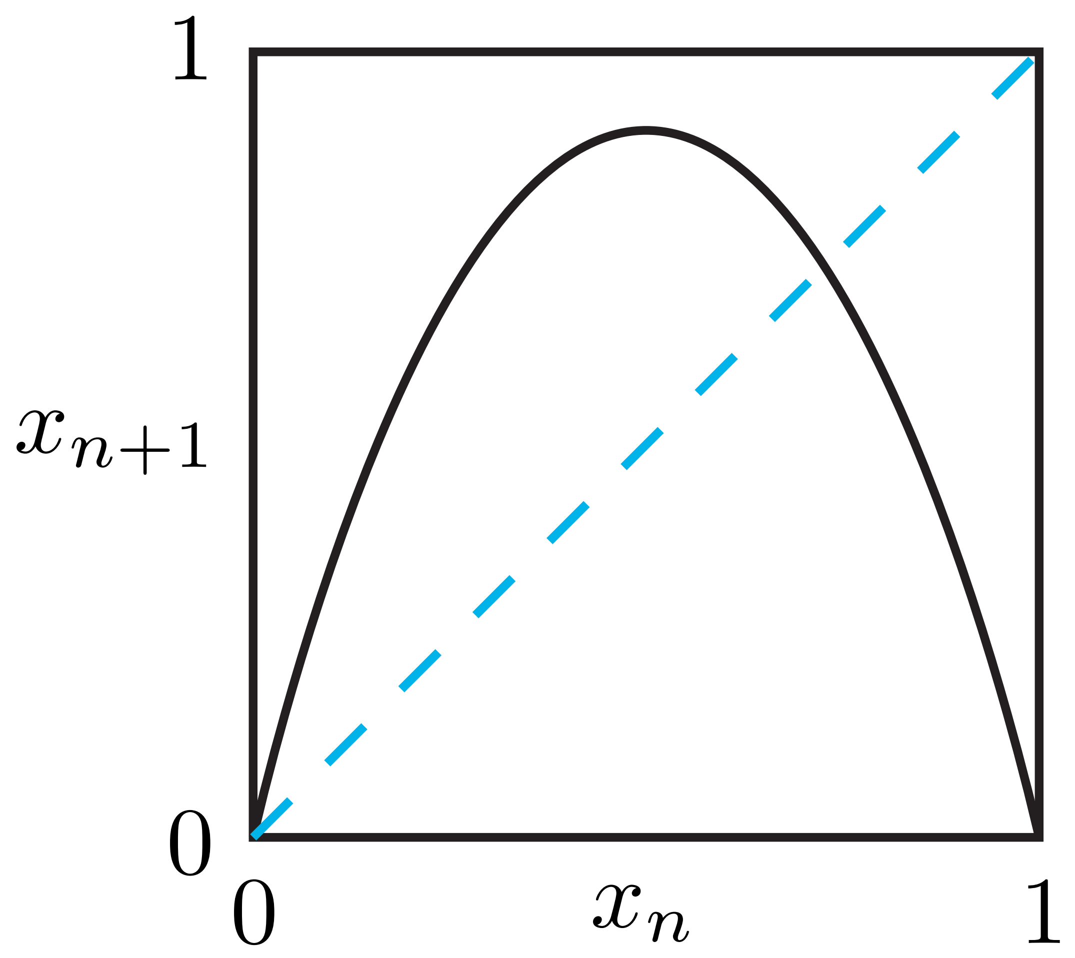


Example 1D Maps ...

Logistic map: $x_{n+1} = r x_n (1 - x_n)$

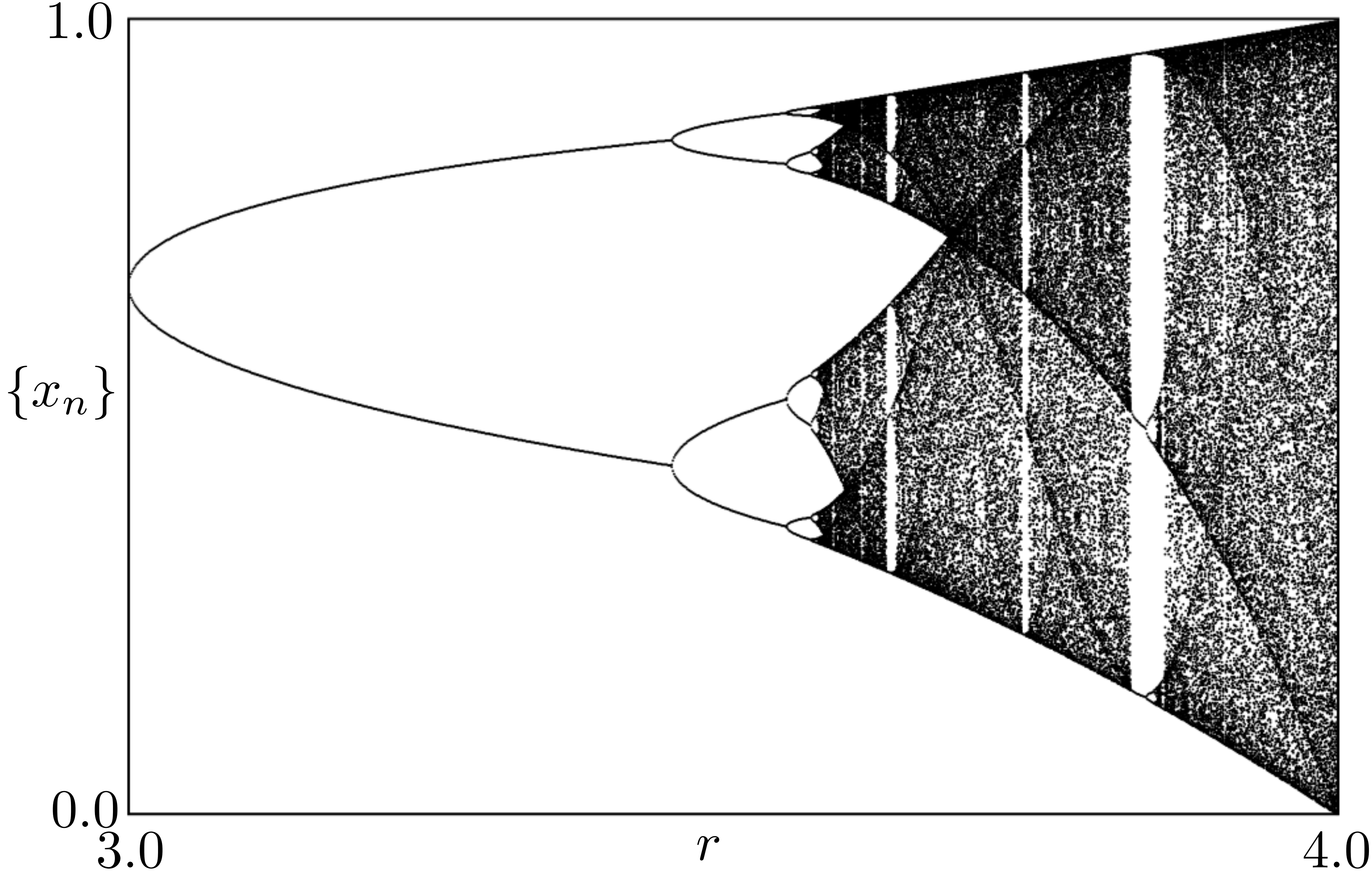
State space: $x_n \in [0, 1]$

Parameter (height): $r \in [0, 4]$



Example 1D Maps ...

Logistic map bifurcation diagram ...



Example 1D Maps ...

Logistic map LCE:

Local stability depends on state: $f'(x) = r(1-2x)$

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \log_2 |f'(x_n)|$$

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \log_2 |r(1-2x_n)|$$

Period 1: $x^* = 0,\ 0 \leq r \leq 1$ $\quad f'(x^*) = r$ $\quad \lambda = \log_2 r$

Period 1: $x^* = \frac{r-1}{r},\ 1 \leq r \leq 3$ $\quad \lambda = \log_2 |2-r|$

Superstable: $f'(x_i) = 0$ $\quad \lambda \to -\infty$ $\quad r = 2$

Bifurcations: $\lambda = 0$

Onset of chaos: $\lambda = 0$

Example 1D Maps ...

LCE for 1D Maps ... an aside on the Ergodic Theorem

Rather than time average:

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \log_2 |f'(x_n)|$$

Average over attractor's distribution: $\Pr(x), \;\; x \in \Lambda$

Invariant distribution: $\Pr(x) = f \text{ ``} \circ \text{''} \Pr(x)$

State-space averaged LCE:

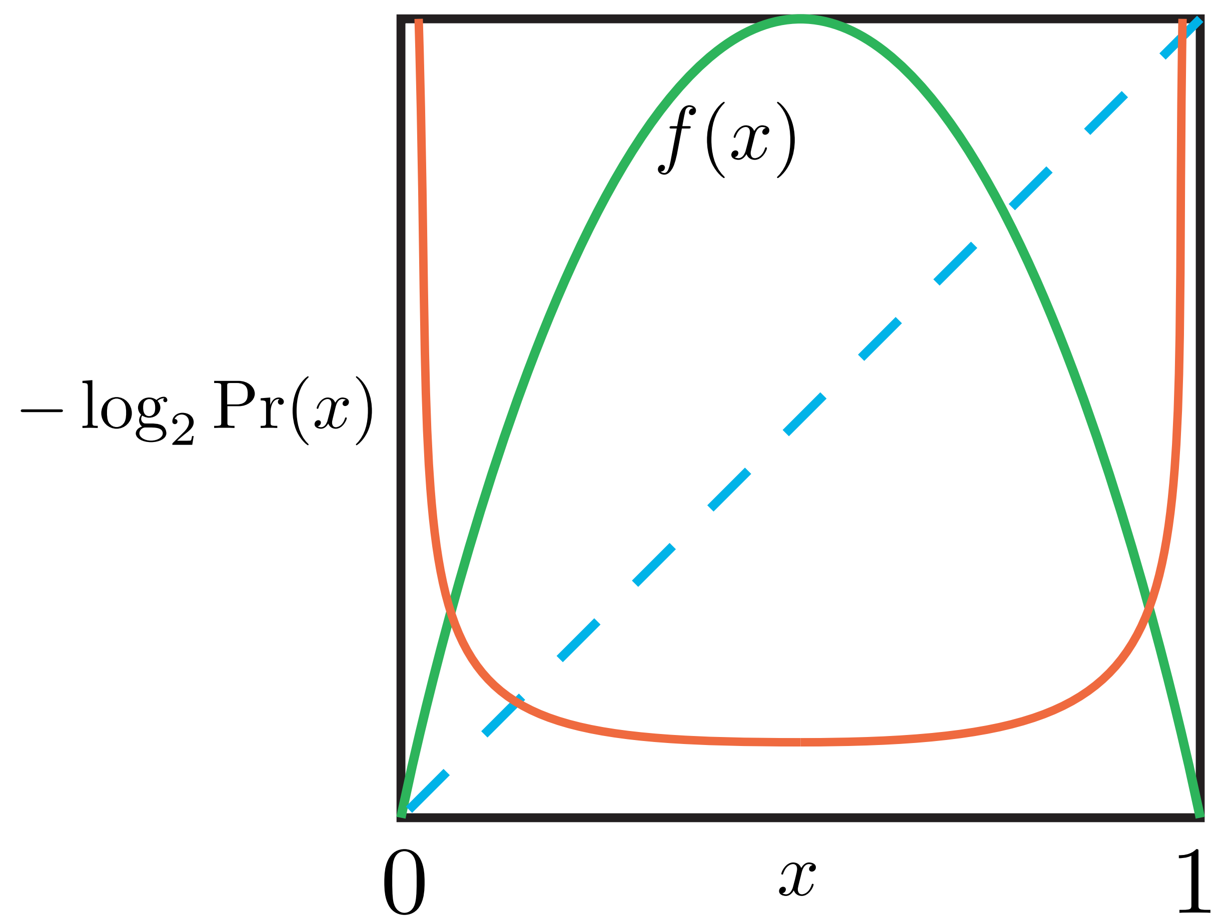
$$\lambda = \int_\Lambda dx \; \Pr(x) \log_2 |f'(x)|$$

Example 1D Maps ...

Logistic map LCE: $r = 4$

Invariant distribution:

$$\Pr(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

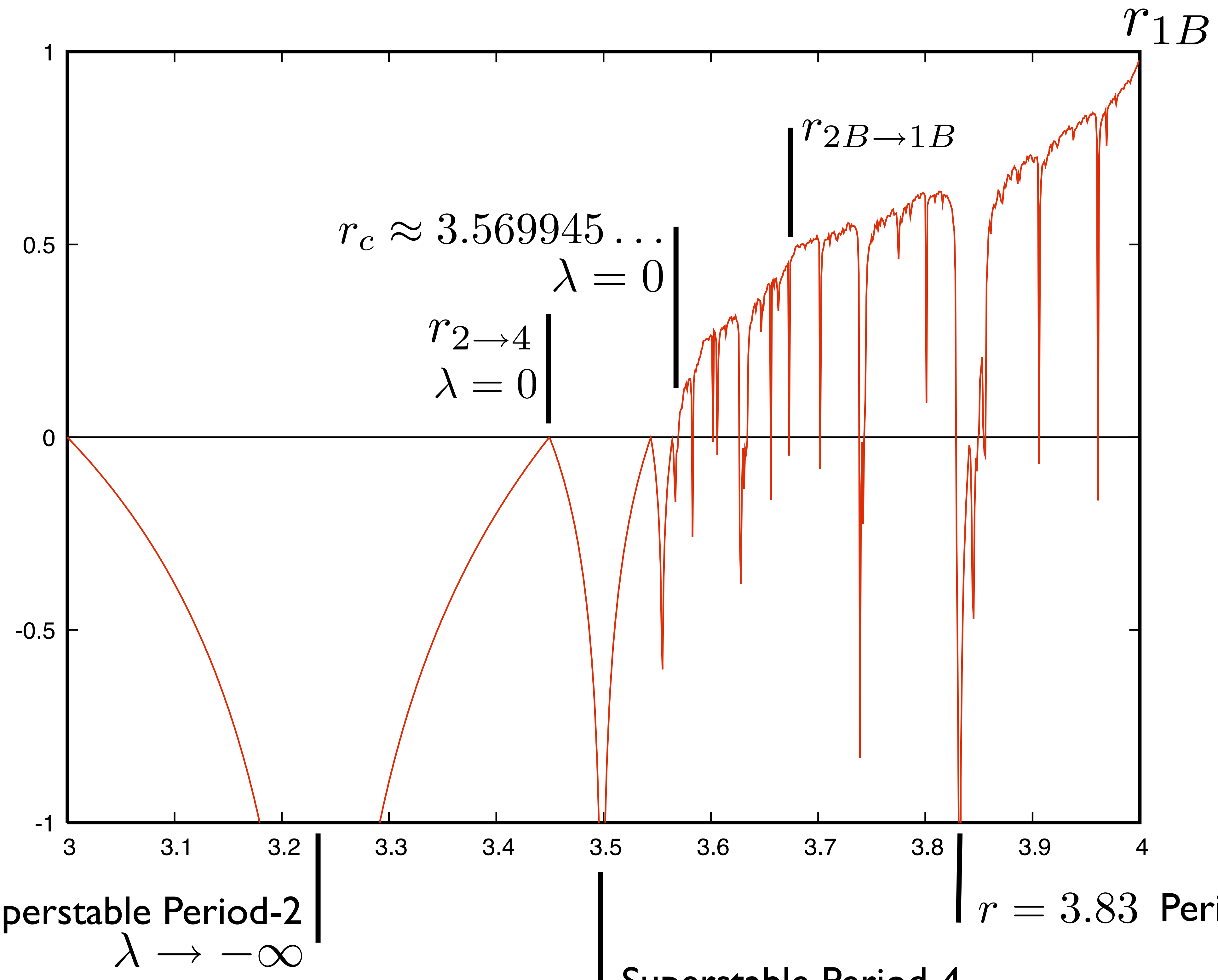


$$\lambda = \int_0^1 dx \frac{\log_2 |4 - 8x|}{\pi\sqrt{x(1-x)}}$$

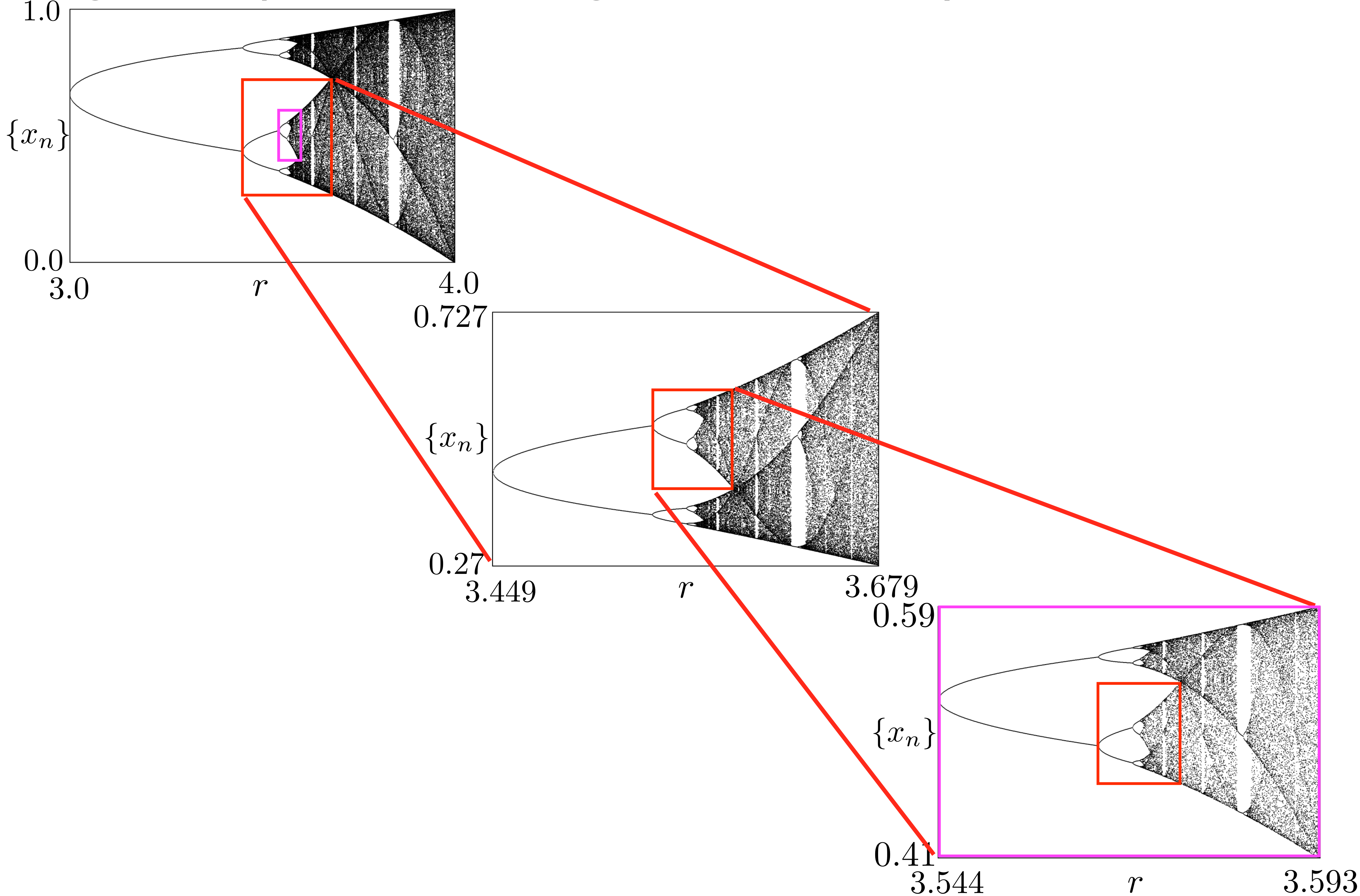
$\lambda = 1$ bit per step

Example 1D Maps ...

LCE view of period-doubling route to chaos:



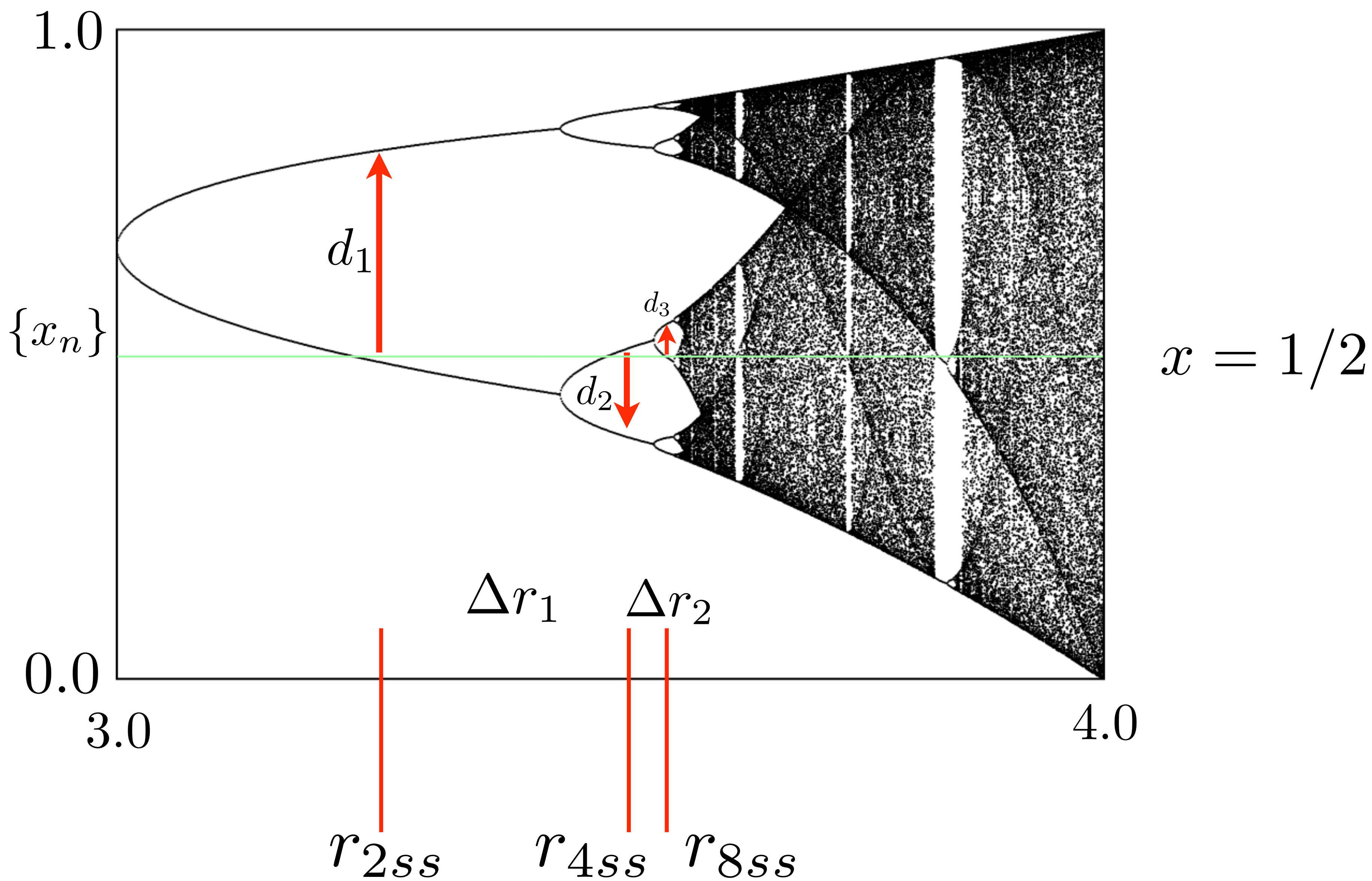
Example 1D Maps ...
Logistic map bifurcation diagram self-similarity



Example 1D Maps ...

Bifurcation Theory of 1D Maps ...

Scaling analysis of period-doubling cascade:



Universal constants:

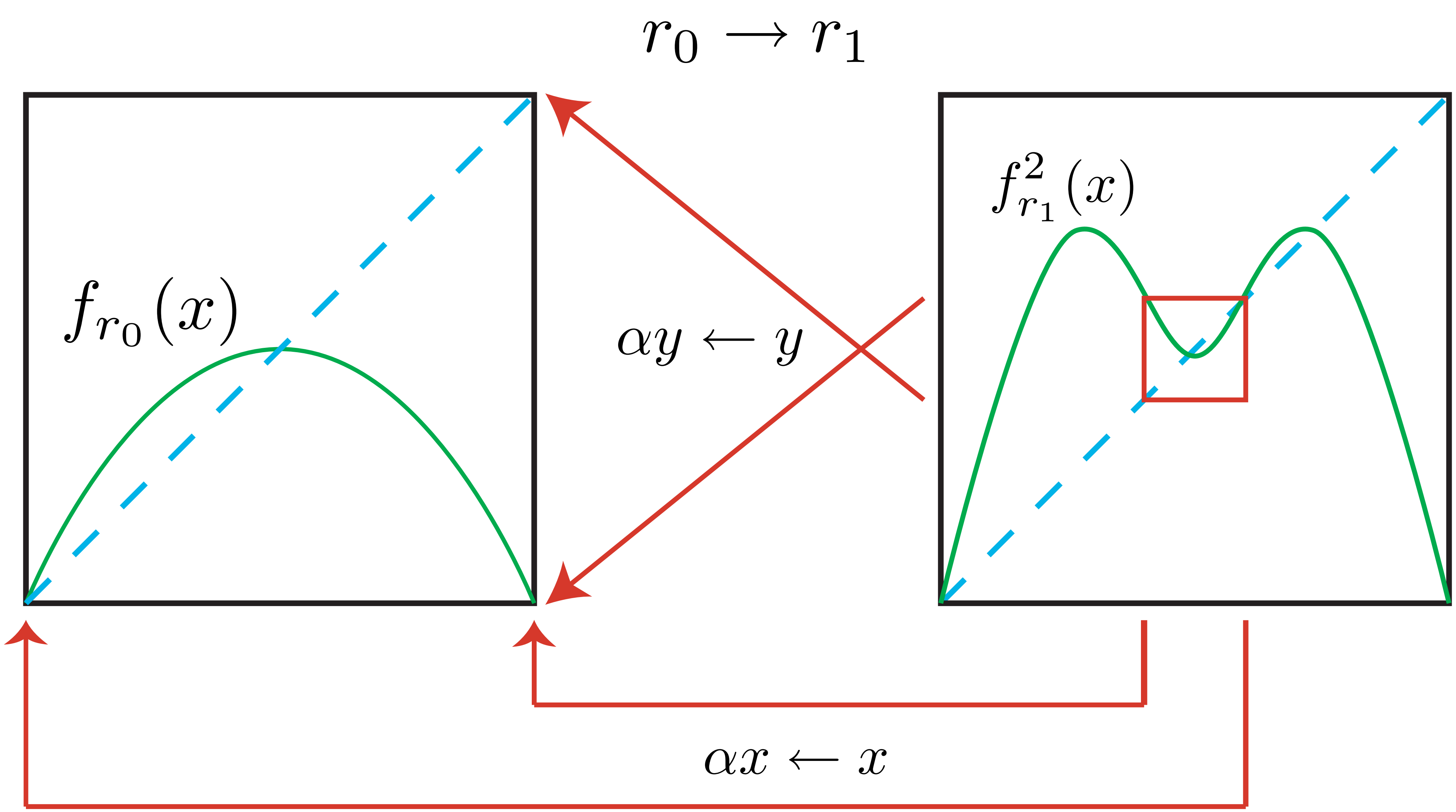
$$\delta = \lim_{n\to\infty} \frac{\Delta r_n}{\Delta r_{n+1}} = 4.669\ldots$$

$$\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}} = -2.5029\ldots$$

Example 1D Maps ...

Bifurcation Theory of 1D Maps ...

Renormalization group analysis of period-doubling:



$|\alpha| > 1$ $\quad\quad$ $\alpha < 0 \Leftrightarrow$ flip

Example 1D Maps ...

Bifurcation Theory of 1D Maps ...

Renormalization group analysis of period-doubling ...

$$f(x, r_0) \approx \alpha f^2(\frac{x}{\alpha}, r_1)$$

$$f^2(\frac{x}{\alpha}, r_1) \approx \alpha^2 f^4(\frac{x}{\alpha^2}, r_2)$$

$$\vdots$$

$$f(x, r_0) \approx \alpha^n f^{(2^n)}(\frac{x}{\alpha^n}, r_n)$$

Example 1D Maps ...

Bifurcation Theory of 1D Maps ...

Renormalization group analysis of period-doubling ...

Universal Map:

$$g_0(x) = \lim_{n \to \infty} \alpha^n f^{(2^n)}(\frac{x}{\alpha^n}, r_n)$$

for $x \sim x_{\max}$

Example 1D Maps ...

Bifurcation Theory of 1D Maps ...

Renormalization group analysis of period-doubling ...

$$r_\infty : \quad f(x, r_\infty) \approx \alpha f^2(\frac{x}{\alpha}, r_\infty) \qquad \text{for } x \sim x_{\max}$$

Limiting functional equation: ($x_{\max} = 0$)

$$g(x) = \alpha g^2\left(\frac{x}{\alpha}\right)$$

Boundary conditions: $g(0) = 0 \;\&\; g'(0) = 0$

Bifurcation Theory of 1D Maps ...

Renormalization group analysis of period-doubling ...

How to solve?

$$g(x) = \alpha g^2\left(\frac{x}{\alpha}\right) \quad \text{with} \quad g(0) = 0 \ \& \ g'(0) = 0$$

Taylor expansion: $g(x) = a + bx^2 + cx^4 + \cdots \quad g(x) \text{ even}$

Find: $\alpha = -2.5029\ldots$

Parameter rescaling: (more work)

Find: $\delta = 4.669\ldots$

Example 1D Maps ...

Reading for next lecture:

Lecture Notes.