Mechanisms of Chaos: Stable Instability

Reading for this lecture:

NDAC, Sec. 12.0-12.3, 9.3, and 10.5.

Mechanisms of Chaos ...

Unpredictability:

- Orbit complicated: difficult to follow
- Repeatedly convergent and divergent
- Net amplification of small variations

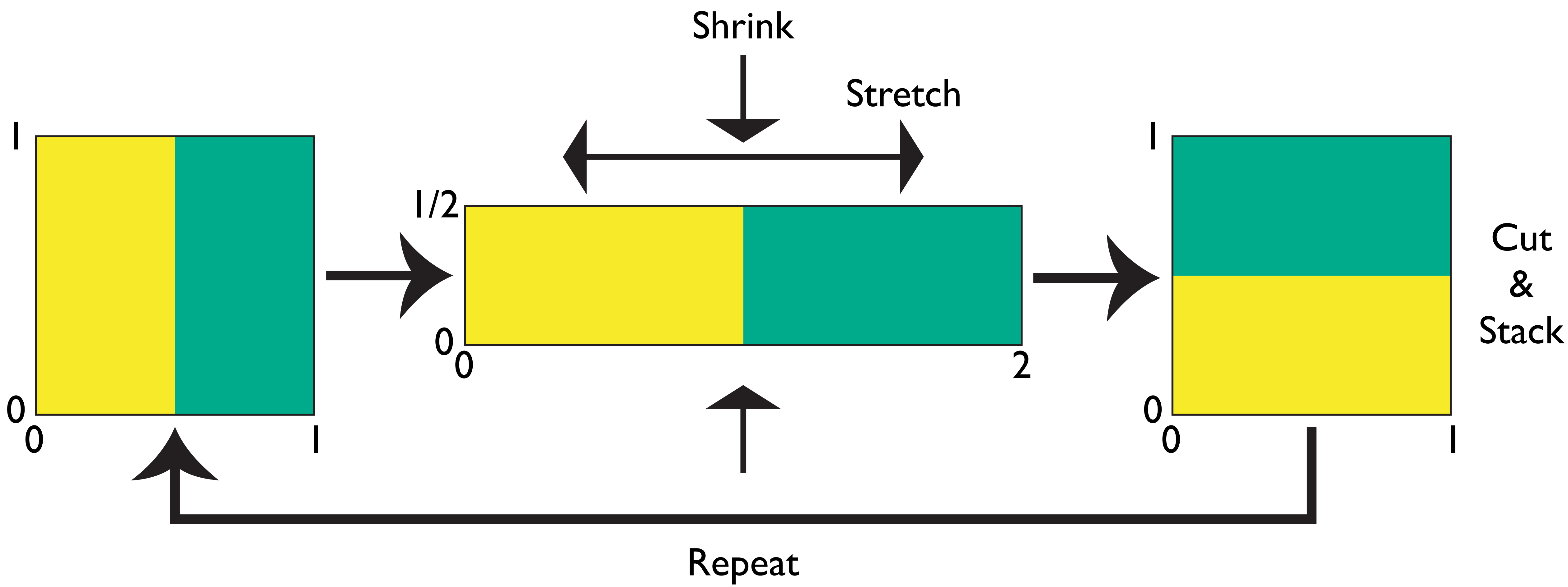
What geometry produces this?

Stretch and fold:

- Flow stretches state space
- But to be stable, must be done in a compact region
  - Must fold back into region

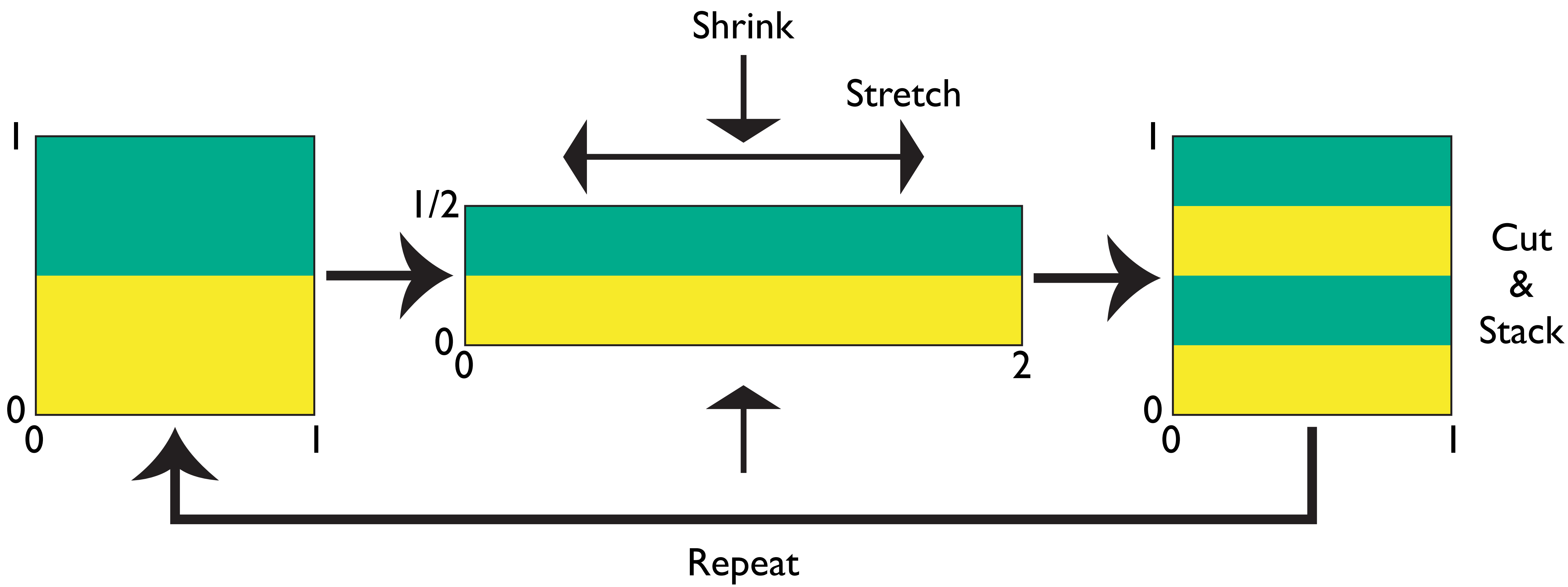
Mechanisms of Chaos ...

Baker's transformation:
kneading state space



Mechanisms of Chaos ...

Baker's transformation ...
kneading state space



Mechanisms of Chaos ...

Baker's transformation ...

2D Baker's Map:

$$(x_n, y_n) \in [0,1] \times [0,1]$$

$$\begin{aligned} x_{n+1} &= 2x_n (\text{mod } 1) \\ y_{n+1} &= \begin{cases} \frac{1}{2} y_n, & x_n \le \frac{1}{2} \\ \frac{1}{2} + \frac{1}{2} y_n, & x_n > \frac{1}{2} \end{cases} \end{aligned}$$

Mechanisms of Chaos ...

Baker's transformation ...

Stability? $A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

Calculate:

$\lambda_1 = 2$ Stretch $\vec{v}_1 = (1, 0)$ Only horizontal

$\lambda_2 = 1/2$ Shrink $\vec{v}_2 = (0, 1)$ Only vertical

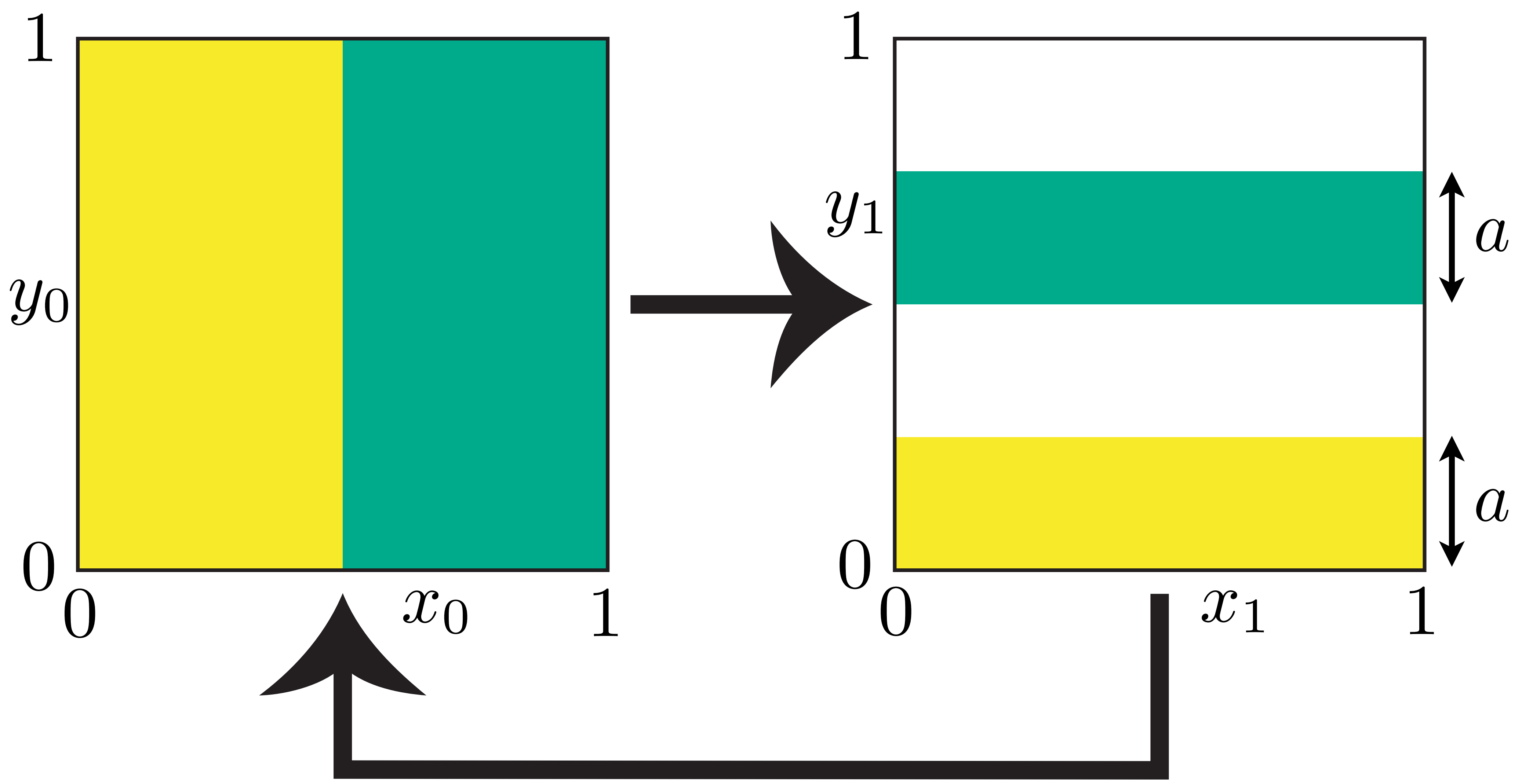
$\mathrm{Tr}(A) = 5/2$

$\mathrm{Det}(A) = 1$ Area preserving: No attractor per se

Independent of $\vec{x}$

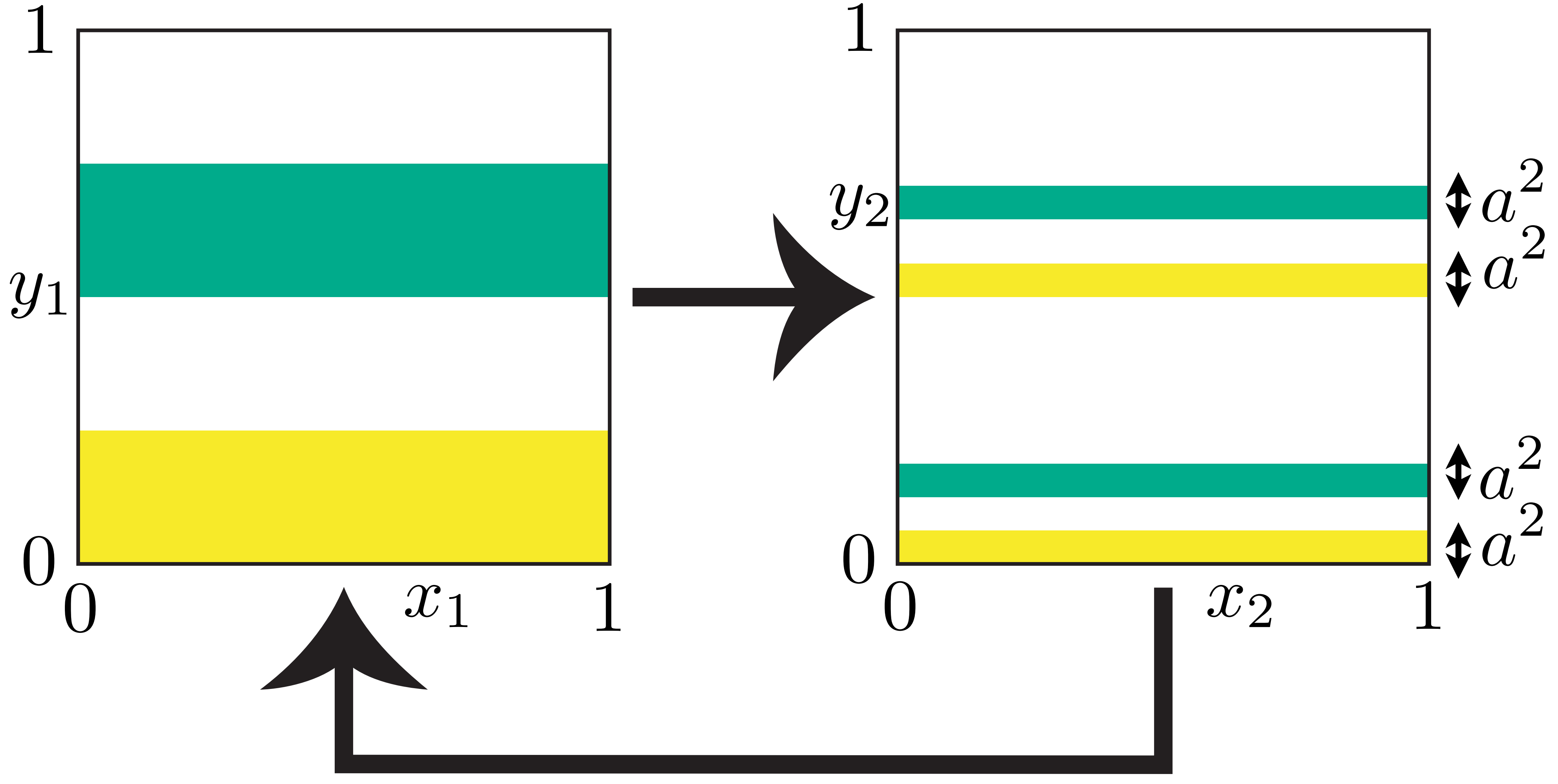
Mechanisms of Chaos ...

Dissipative Baker's Map:



Mechanisms of Chaos ...

Dissipative Baker's Map ... again!



Dissipative Baker's Map ...

$$x_{n+1} = 2x_n (\text{mod } 1)$$

$$y_{n+1} = \begin{cases} ay_n, & x_n \leq \frac{1}{2} \\ \frac{1}{2} + ay_n, & x_n > \frac{1}{2} \end{cases}$$

$$a \in \left[0, \frac{1}{2}\right]$$

Mechanisms of Chaos ...

Dissipative Baker's Map ...

Stability? $A = \begin{pmatrix} 2 & 0 \\ 0 & a \end{pmatrix}$

Calculate:

$$\lambda_1 = 2 \qquad \vec{v}_1 = (1, 0)$$
$$\lambda_2 = a \qquad \vec{v}_2 = (0, 1)$$

Independent of $\vec{x}$

$$\mathrm{Det}(A) = 2a$$

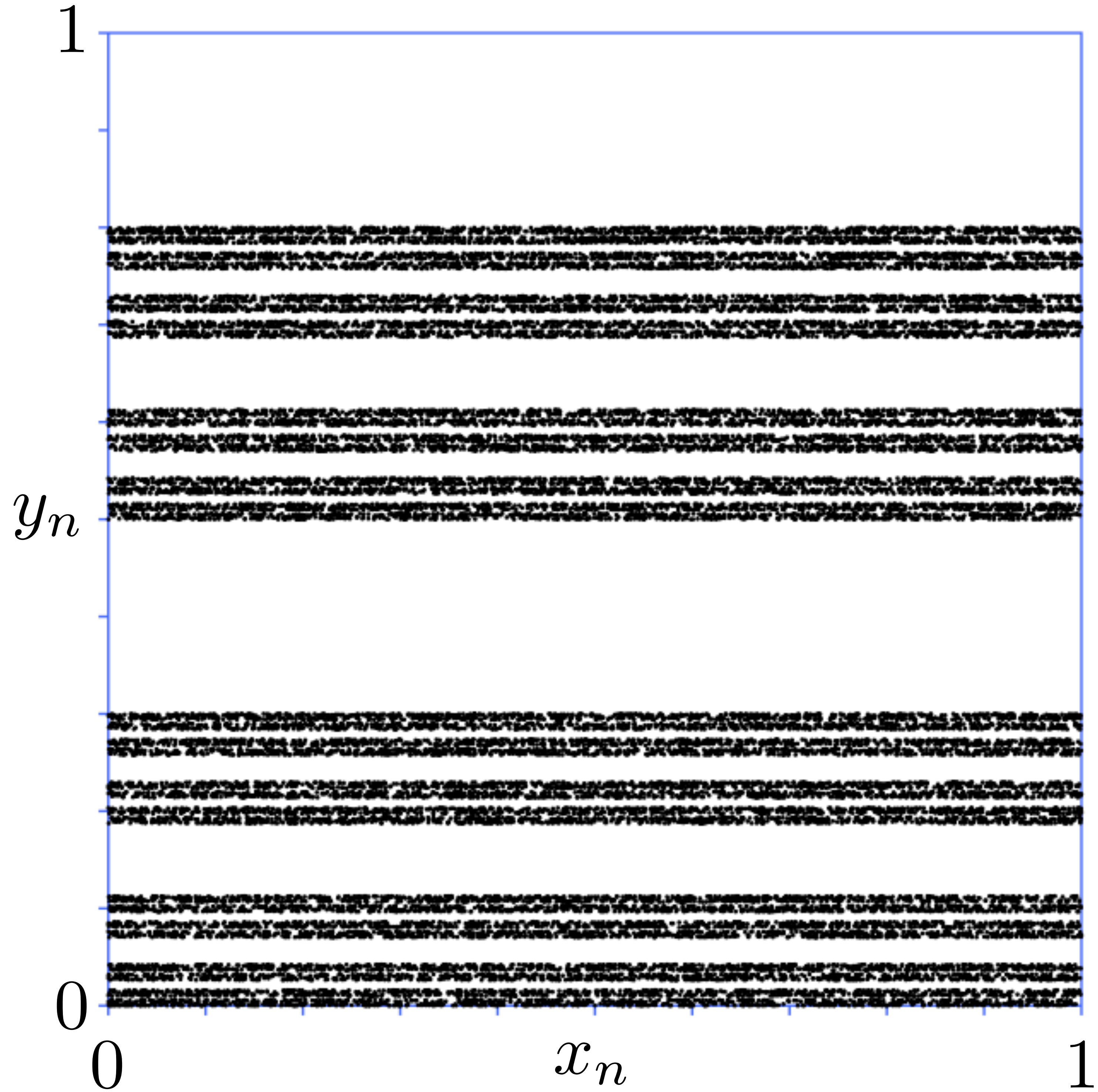
Dissipative: $a < 1/2$

Area contraction

Attractor!

Mechanisms of Chaos ...

Dissipative Baker's Map Simulation: $a = 0.3$



Mechanisms of Chaos ...

Dissipative Baker's Map ...

Stability? (x, y) versus $(x + \epsilon, y + \delta)$

$$\Delta x_1 = 2(x_0 + \epsilon) - 2x_0 = 2\epsilon$$
$$\Delta y_1 = a(y_0 + \delta) - ay_0 = a\delta$$

$\Delta x_n = 2^n \epsilon$ Exponential Growth of Errors

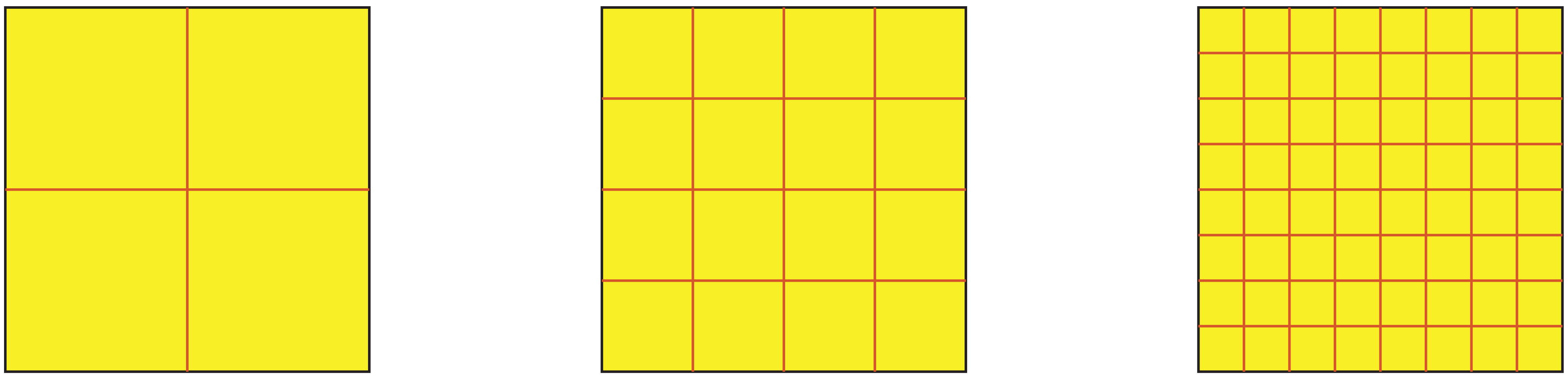
$\Delta y_n = a^n \delta$ Exponential Stability

Mechanisms of Chaos ...

Dimension of a Set:

Number of boxes to cover set at given measurement resolution:

$\epsilon = \frac{1}{2} \quad N = 4$ $\qquad \epsilon = \frac{1}{4} \quad N = 16$ $\qquad \epsilon = \frac{1}{8} \quad N = 64$



$$N(\epsilon = \tfrac{1}{2^n}) = \left(\tfrac{1}{2^n}\right)^{-2} = 2^{2n}$$

$$N(\epsilon) \propto \epsilon^{-2}$$

Generalizing

$$N(\epsilon) \propto \epsilon^{-d}$$

Or (Definition) dimension: $d = \lim_{\epsilon \rightarrow 0} - \frac{\log N(\epsilon)}{\log \epsilon}$

Mechanisms of Chaos ...

Dimension of Dissipative Baker's Attractor ...

At iteration n:

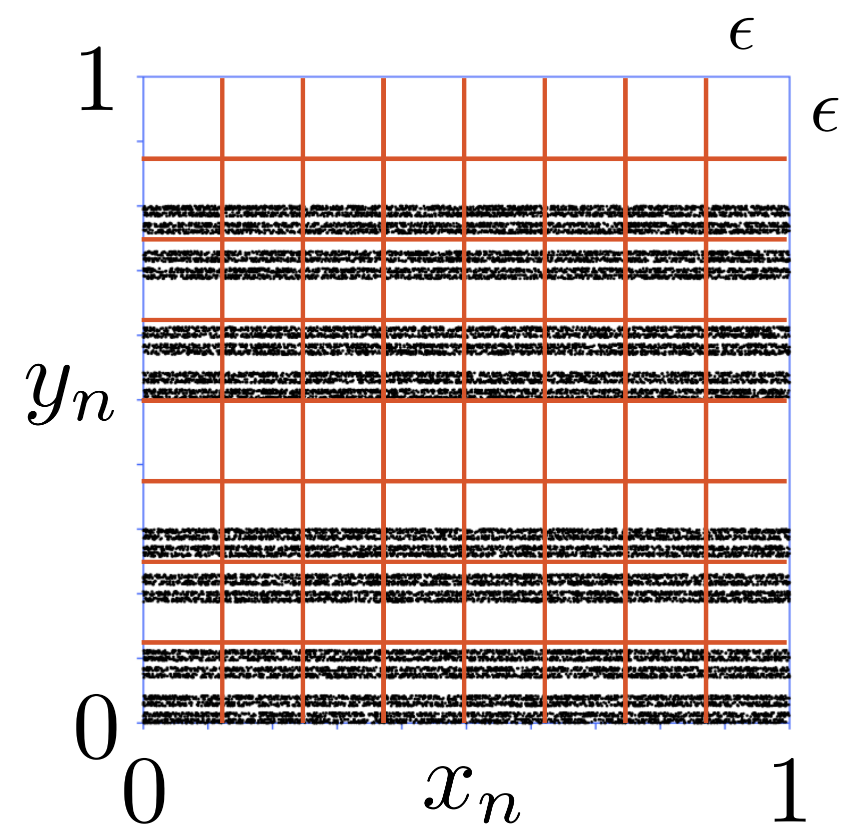
2^n strips of thickness a^n

How many boxes $N(\epsilon)$ to cover attractor at resolution ϵ?

Take: $\epsilon = a^n$

Number of boxes for each strip: a^{-n}

$$N(\epsilon) = a^{-n} \times 2^n = \left(\frac{a}{2}\right)^{-n}$$

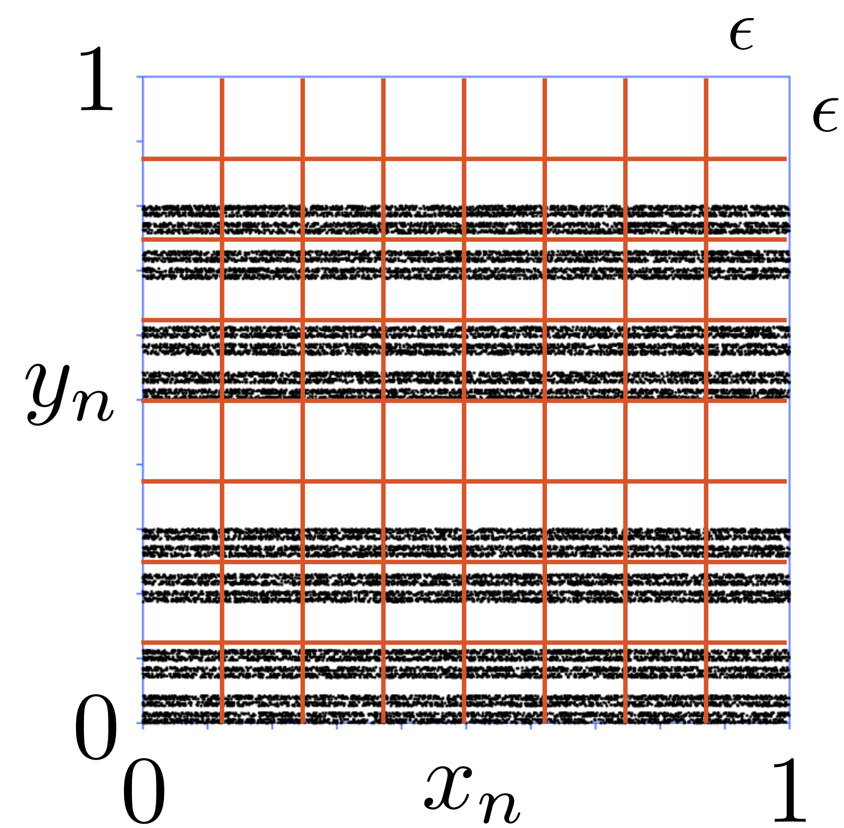


Mechanisms of Chaos ...

Dimension of Dissipative Baker's Attractor ...

Dimension:

$$
\begin{aligned}
d &= \lim_{\epsilon \to 0} -\frac{\log N(\epsilon)}{\log \epsilon} \\
&= \lim_{n \to \infty} -\frac{\log (a/2)^{-n}}{\log a^n} \\
&= 1 + \frac{\log \frac{1}{2}}{\log a}
\end{aligned}
$$



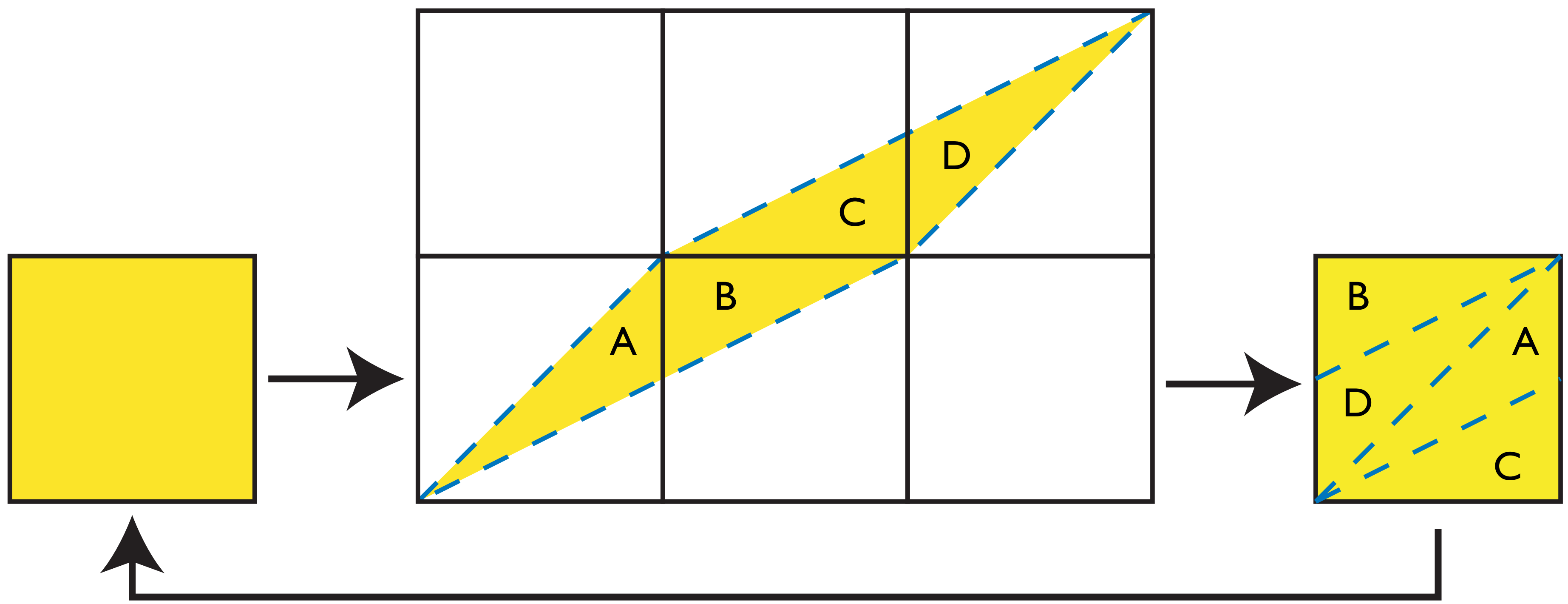
$$a = 0.3 \Rightarrow d = 1.576 \ldots < 2 \quad !$$

Area preserving: as $a \to \frac{1}{2}, \; d \to 2$

Mechanisms of Chaos ...

Cat map (aka toral automorphism): $(x, y) \in \mathbf{T}^2$

Intrinsic stretch/shrink directions



$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \pmod 1$$

Fixed point: $\vec{x}^* = (0, 0)$

Mechanisms of Chaos ...

Cat map (aka Toral automorphism) ...

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \pmod 1$$

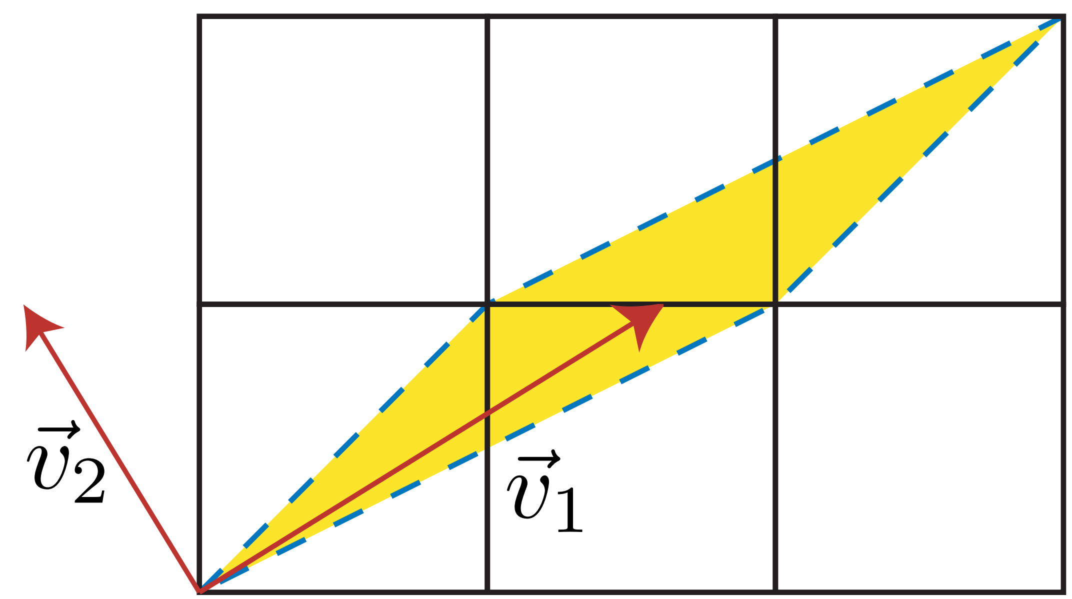
Calculate:

$$\lambda_1 = \frac{3+\sqrt{5}}{2} > 1 \quad \text{stretch} \qquad \vec{v}_1 = \left(\frac{1+\sqrt{5}}{2}, 1\right)$$

$$\lambda_2 = \frac{3-\sqrt{5}}{2} < 1 \quad \text{shrink} \qquad \vec{v}_2 = \left(\frac{1-\sqrt{5}}{2}, 1\right)$$

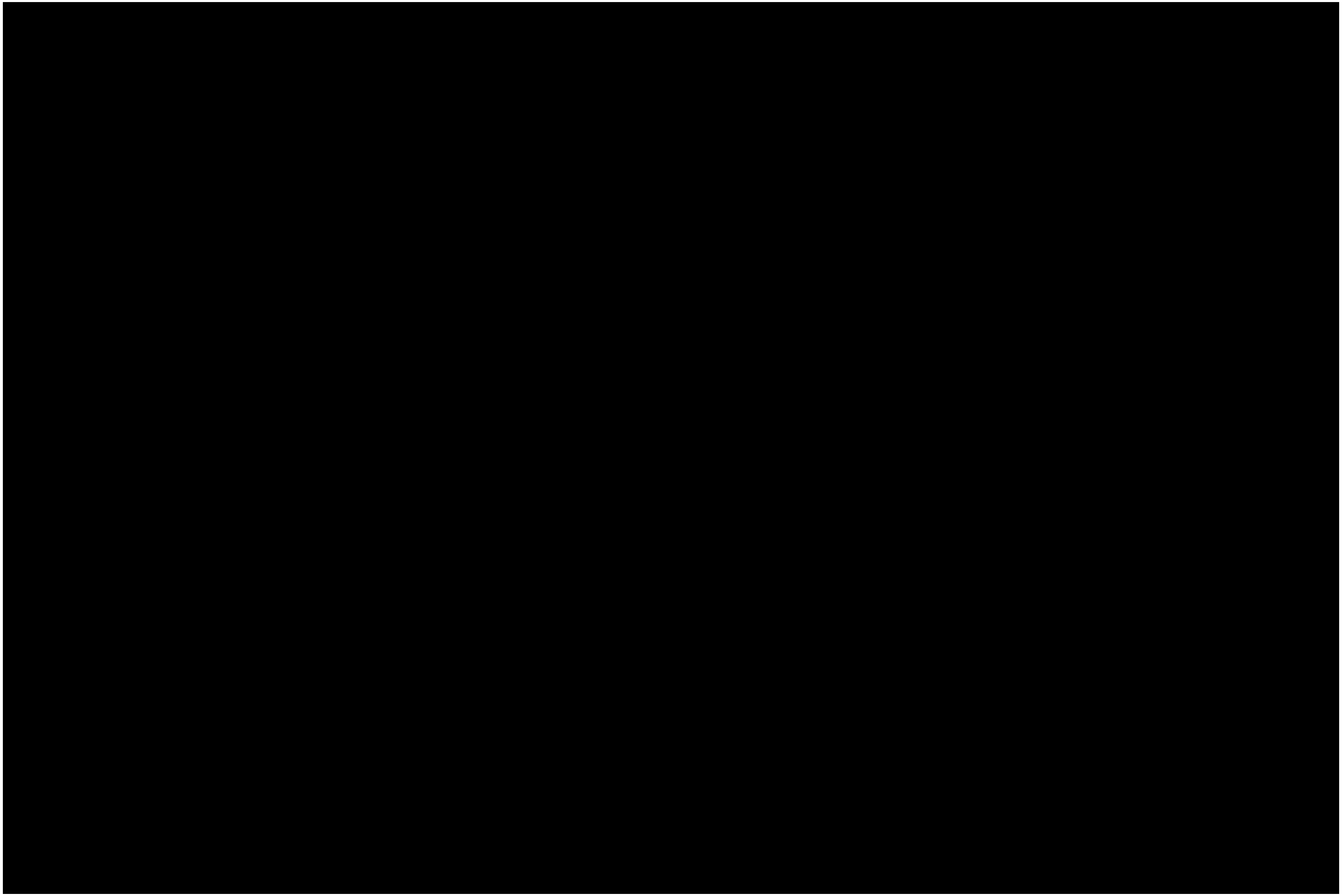
$$\mathrm{Tr}(A) = 3$$

$$\mathrm{Det}(A) = 1 \quad \text{area preserving}$$



Independent of $\vec{x}$

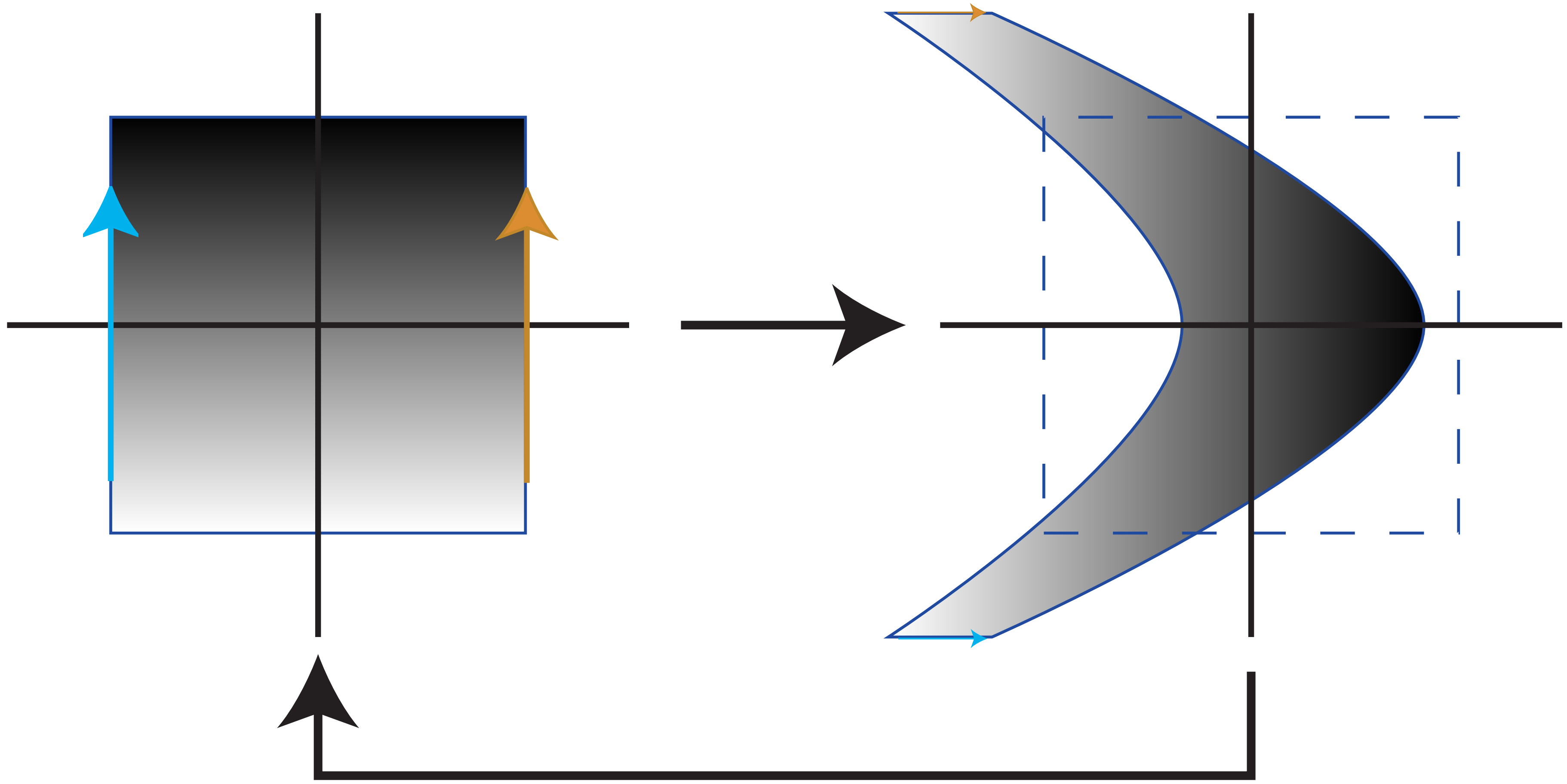
Mechanisms of Chaos ...



Mechanisms of Chaos ...

Hénon map: $(x, y) \in \mathbf{R}^2$

$$
\begin{aligned}
x_{n+1} &= y_n + 1 - a x_n^2 \\
y_{n+1} &= b x_n
\end{aligned}
$$



Stretch and fold depend on location

Mechanisms of Chaos ...

Hénon map ...

Stretch & fold depend on location:

Jacobian:

$$A = \begin{pmatrix} -2ax_n & 1 \\ b & 0 \end{pmatrix}$$

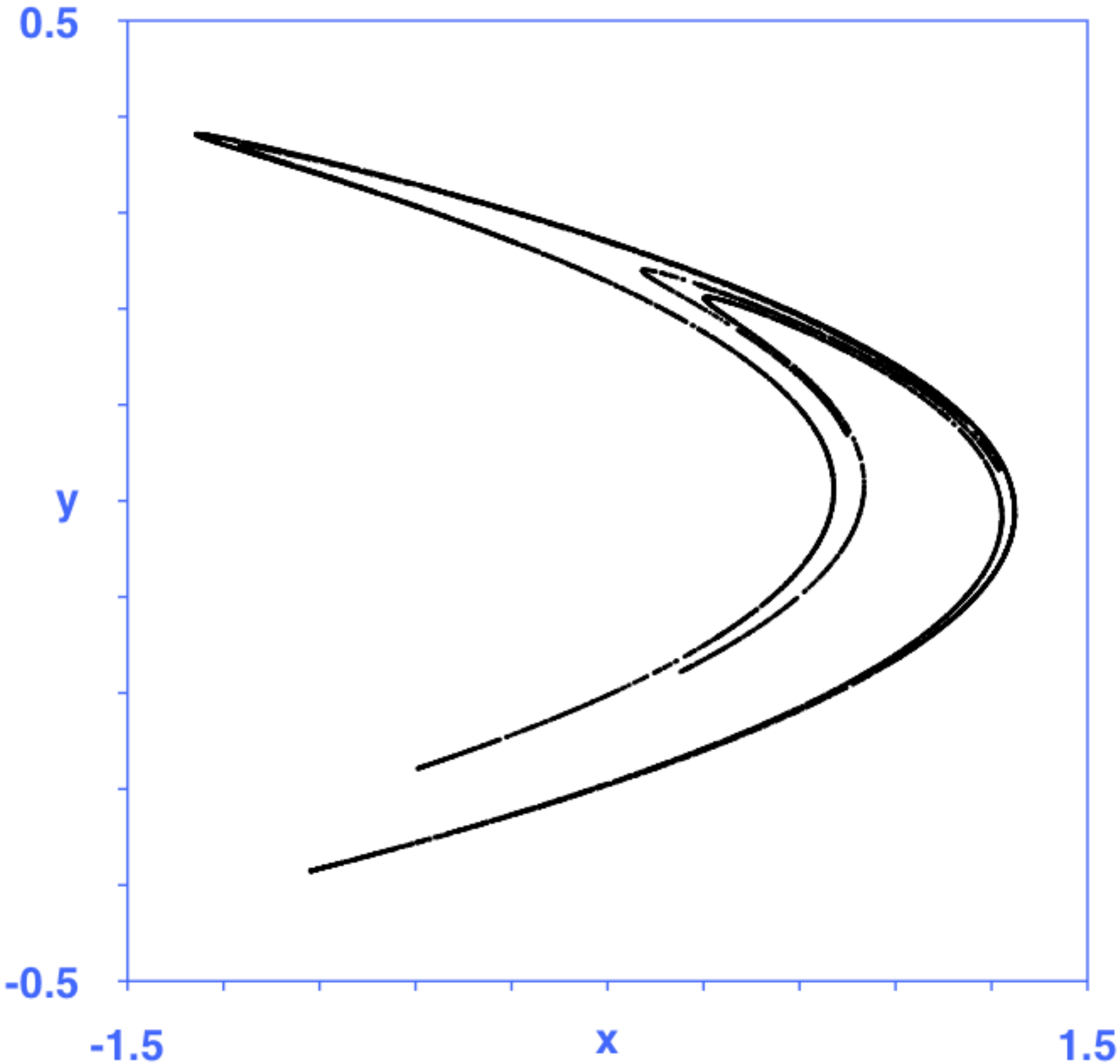
Dissipative (and orientation reversing):

$$\mathrm{Det}(A) = -b$$

Mechanisms of Chaos ...

Hénon Attractor:

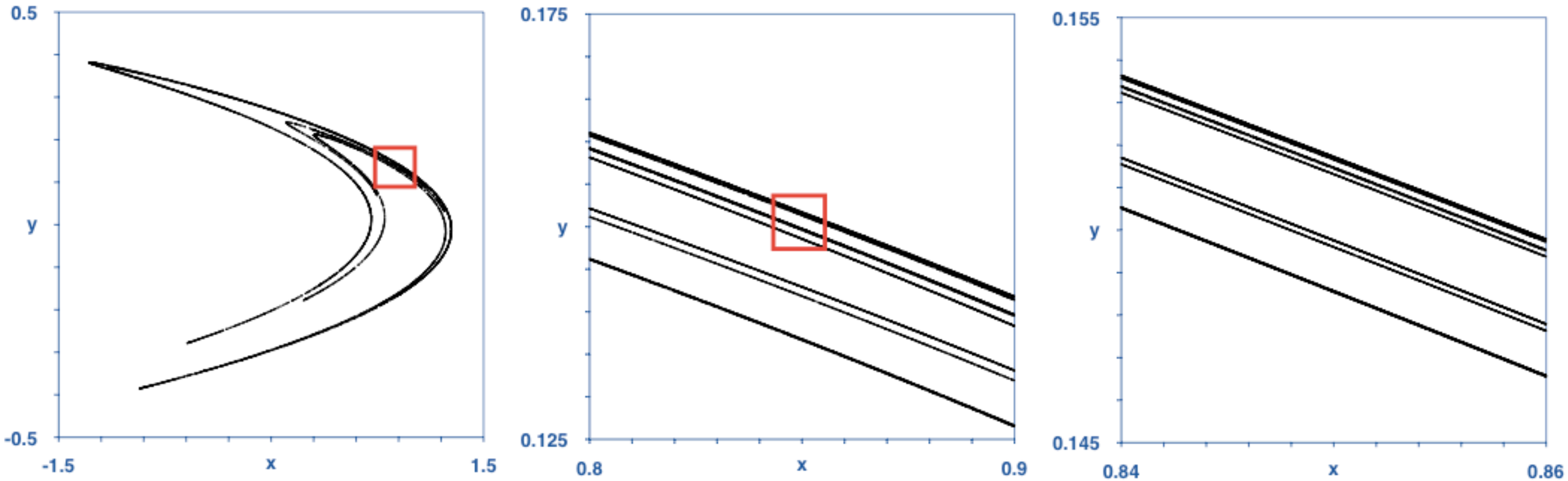
Control parameters: $(a, b) = (1.4, 0.3)$



Mechanisms of Chaos ...

Henon Attractor ...

Self-similar:

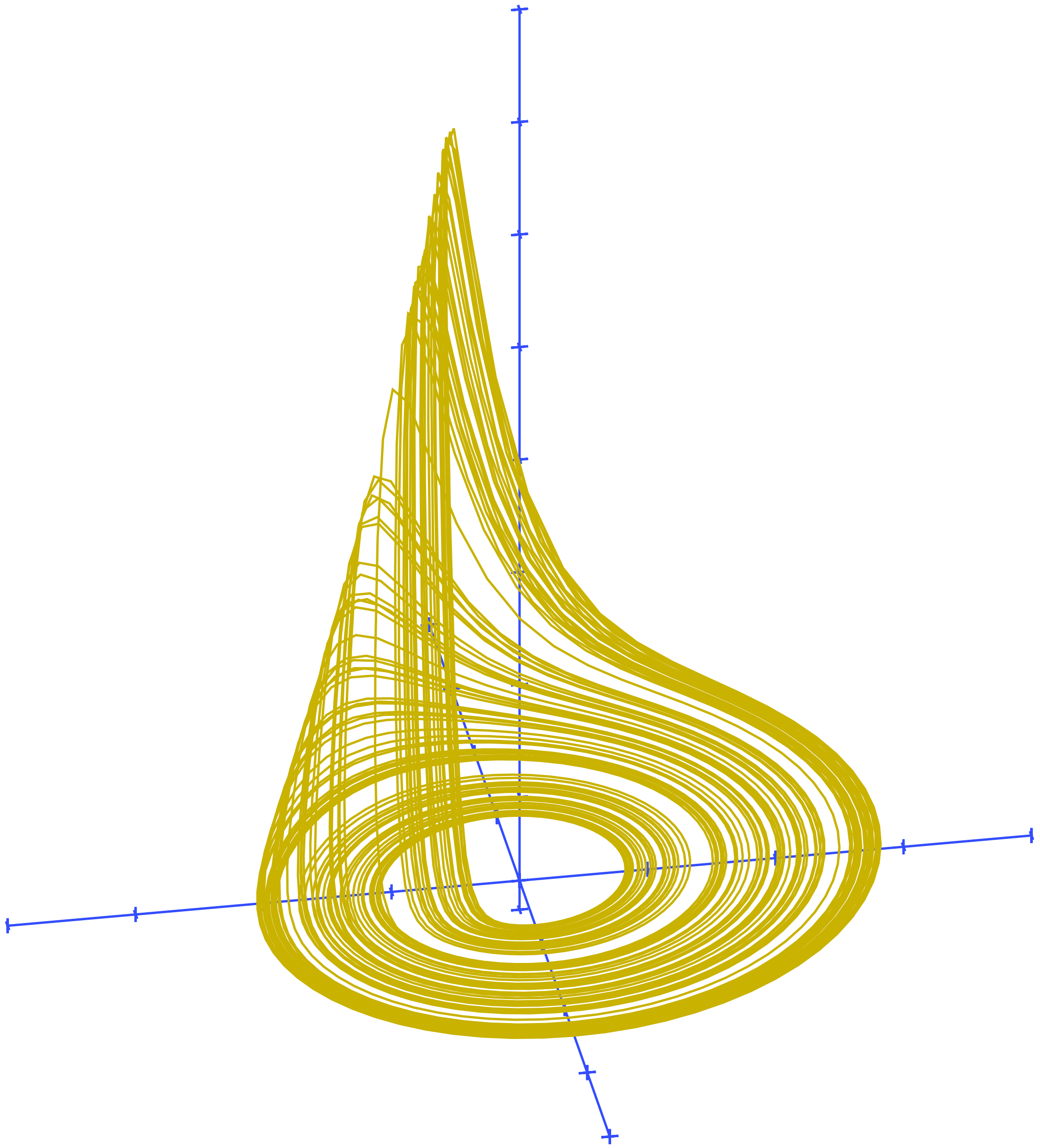


Self-similar attractor = Dissipation + Instability

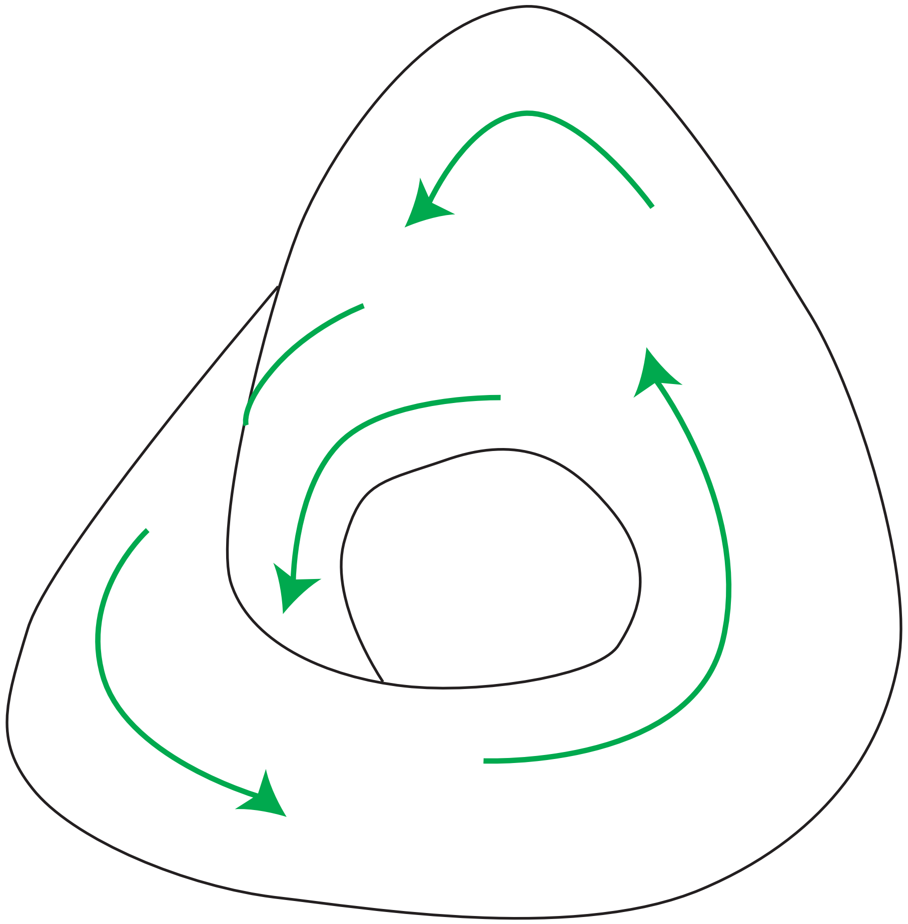
How does the stretch & fold mechanism work in ODEs?

Mechanisms of Chaos ...

Rössler Chaotic Attractor:



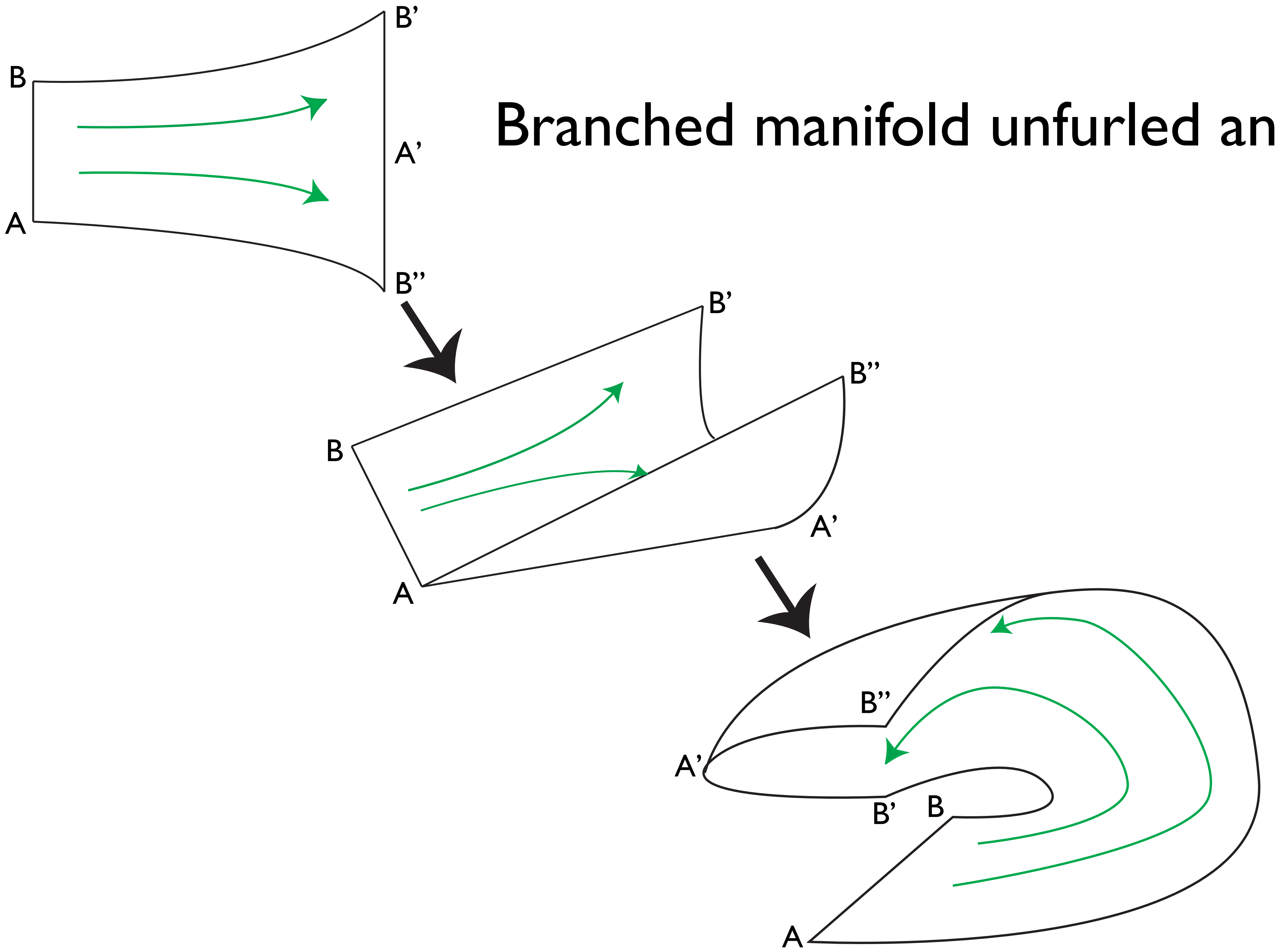
Branched manifold:



Mechanisms of Chaos ...

Rössler Chaotic Attractor ...

Branched manifold unfurled and refurled



Rössler stability + instability:
Dot spreading demo (ds)

Time step = 0.03
Remembered trajectory = 10000
Orient
3000 e
nEns = 100000
IC = (0,-7,0)
radius = .1
1, 1, 1

Mechanisms of Chaos ...

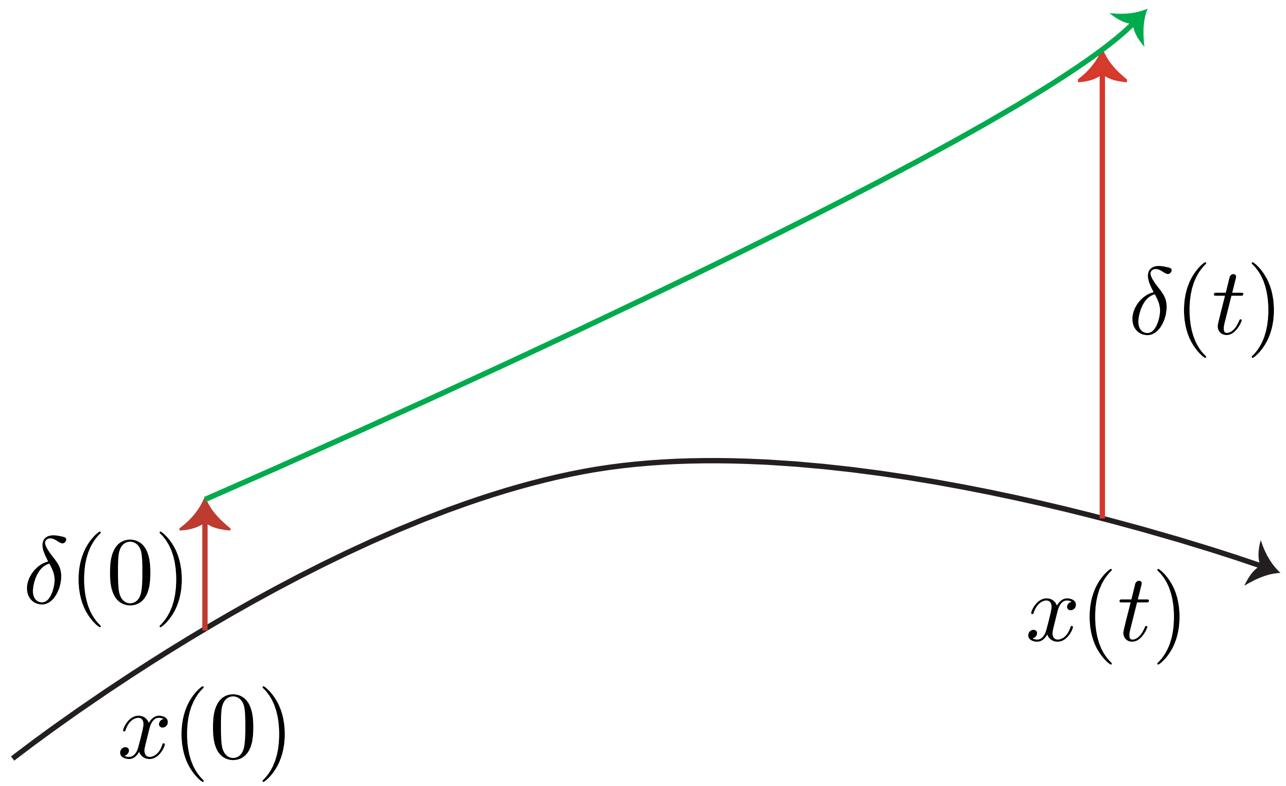
Lorenz stability + instability:
Dot spreading demo (ds)

Time step = .005
Remembered trajectory = 10000
Orient
1000 e
nEns = 60000
IC = (5,5,5)
radius = .05
1, 1, 1

Mechanisms of Chaos ...

Quantifying instability:

Growth-of-error model:



$$||\delta(t)|| \sim ||\delta(0)||\ e^{\lambda t}$$

Or

$$\lambda \sim t^{-1} \ln \frac{||\delta(t)||}{||\delta(0)||}$$

Lyapunov Characteristic Exponent (LCE):

$$\lambda = \lim_{\substack{t \to \infty \\ ||\delta(0)|| \to 0}} \frac{1}{t} \log_2 \frac{||\delta(t)||}{||\delta(0)||}$$

$\delta(t)$ aligns with most unstable direction!

λ: Exponential rate of growth of errors

Units: [bits per second]

Mechanisms of Chaos ...

Measurement Resolution: ϵ

Number of scale factors to locate initial state: $I_0 = -\log_2 \epsilon$

Resolution loss rate (bits per second): λ

Prediction horizon: $t_{\mathrm{unpredict}} \sim \frac{I_0}{\lambda}$

Example:

Loss rate: Factor of 2 each second: $\lambda = 1$

Measurement resolution: $\epsilon = 10^{-3}$

$$I_0 = 10 \text{ bits} \qquad t_{\mathrm{unpredict}} = 10 \text{ seconds}$$

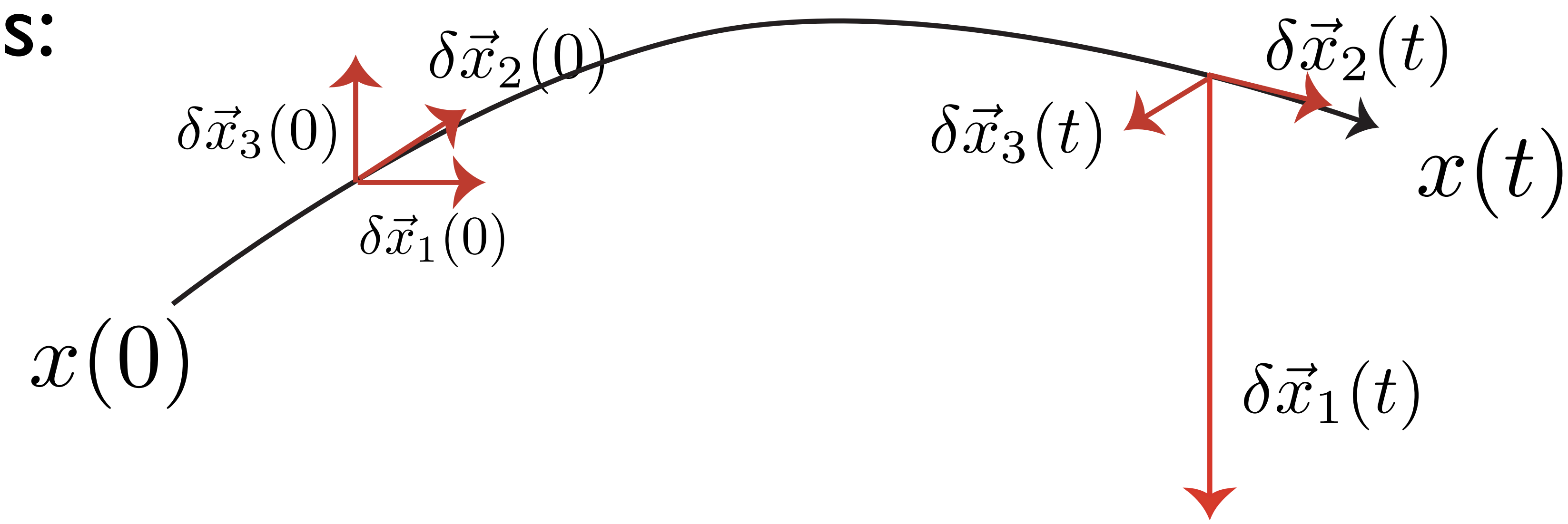
Thousand times higher resolution: $\epsilon = 10^{-6}$

$$I_0 = 20 \text{ bits} \qquad t_{\mathrm{unpredict}} = 20 \text{ seconds}$$

Mechanisms of Chaos ...

Quantifying instability and stability ...

n dimensions:



Lyapunov Characteristic Exponent Spectrum:

$$\chi = \{\lambda_1, \lambda_2, \ldots, \lambda_n\},\ \lambda_i \geq \lambda_{i+1}$$

$$\lambda_i = \lim_{\substack{t \to \infty \\ ||\delta\vec{x}_i|| \to 0}} \frac{1}{t} \log_2 \frac{||\delta\vec{x}_i(t)||}{||\delta\vec{x}_i(0)||}$$

$$\{\delta\vec{x}_1, \delta\vec{x}_2, \ldots, \delta\vec{x}_n\}, \delta\vec{x}_i \cdot \delta\vec{x}_j = 0, i \neq j$$

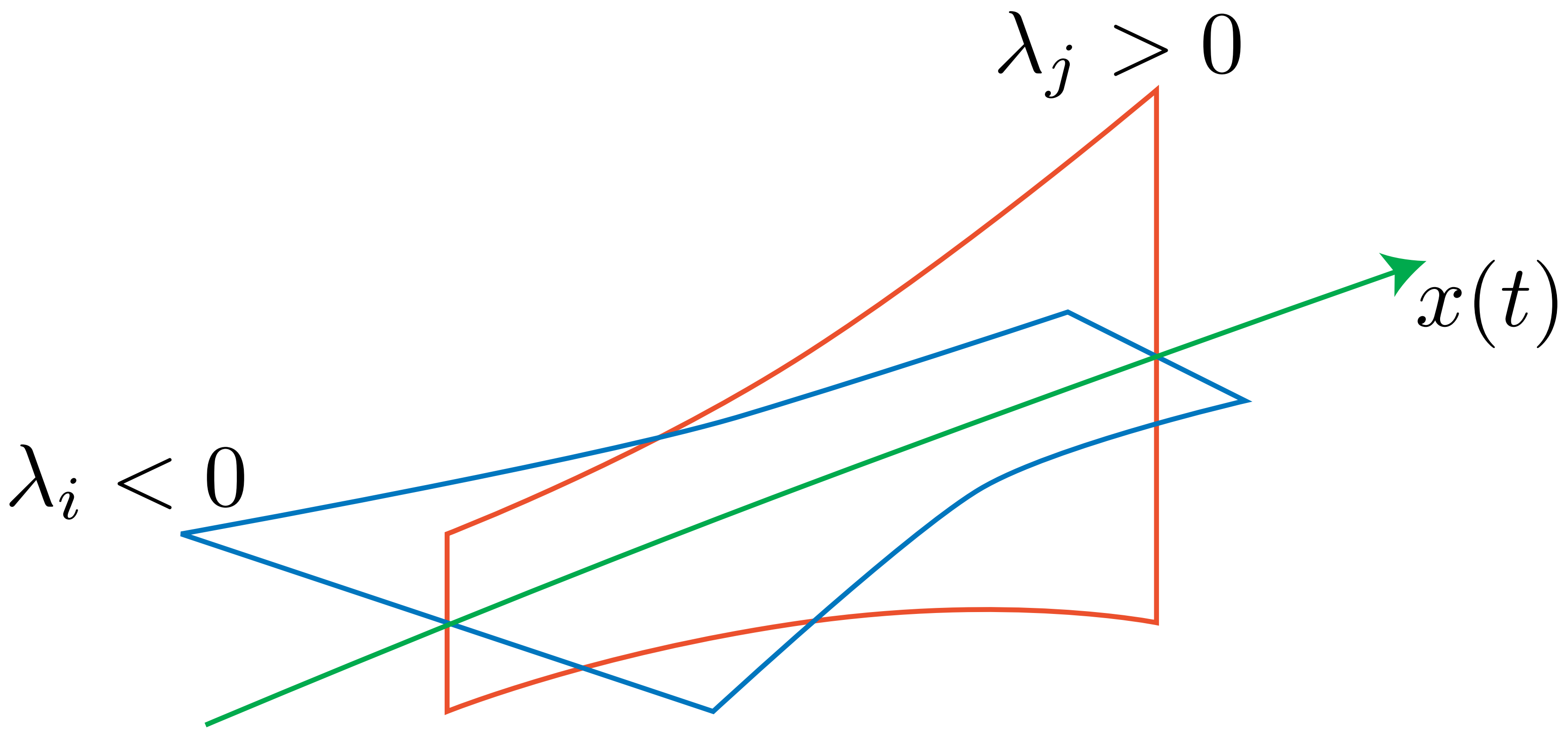
Mechanisms of Chaos ...

Quantifying instability and stability ...

LCE Spectrum and Submanifolds:

$$\lambda_i < 0 \iff \text{stable manifold}$$

$$\lambda_i > 0 \iff \text{unstable manifold}$$



LCE Spectrum: Key to characterizing attractors

Mechanisms of Chaos ...

Dissipation rate:

Divergence of vector field:

$$\nabla \cdot \vec{F}(\vec{x}) = \sum_{i=1}^{n} \left. \frac{\partial \vec{F}}{\partial x_i} \right|_{\vec{x}} = \mathrm{Tr}(A(\vec{x}))$$

$$\mathcal{D} = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \; \nabla \cdot \vec{F}(\vec{x}(t))$$

Theorem: $\mathcal{D} = \sum_{i=1}^{n} \lambda_i$

$$\chi = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \;\; \lambda_i \geq \lambda_{i+1}$$

Mechanisms of Chaos ...

Dimension of attractor: $\chi = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \; \lambda_i \geq \lambda_{i+1}$

$$d = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|}$$

$$j \text{ largest integer such that } \sum_{i=1}^{j} \lambda_i \geq 0$$

(Conjectured to be true in most cases.)

Mechanisms of Chaos ...

Entropy of attractor (jumping ahead a little bit):

$$h_\mu = \sum_{\lambda_i > 0} \lambda_i$$

Rate of information production.

Mechanisms of Chaos ...

LCE Attractor Classification:

An attractor's LCE signature:

$$(\lambda_1, \lambda_2, \ldots, \lambda_n),\ \lambda_i \geq \lambda_{i+1}$$

Constraints:

1. Attracting: $\mathcal{D} < 0$

(A)

$$\Rightarrow \sum_{i=1}^{n} \lambda_i < 0$$

$$\Rightarrow \lambda < 0,\ \text{for at least one } i$$

(B)

$$\Rightarrow |\mathcal{D}| > h_\mu$$

$$\Rightarrow \left| \sum_{\lambda_i < 0} \right| > \sum_{\lambda_i > 0}$$

2. If not fixed point, flow along trajectory neutrally stable:

$$\lambda_i = 0,\ \text{for at least one } i$$

3. If chaotic:

$$\lambda_i > 0,\ \text{for at least one } i$$

Mechanisms of Chaos ...

LCE Spectrum Attractor Classification ...

$$(\mathrm{sgn}(\lambda_1), \mathrm{sgn}(\lambda_2), \ldots, \mathrm{sgn}(\lambda_n))$$

Dimension n	LCE Spectrum	Attractor
1	(-)	Fixed Point
2	(-,-)	Fixed Point
2	(0,-)	Limit Cycle
3	(-,-,-)	Fixed Point
3	(0,-,-)	Limit Cycle
3	(0,0,-)	Torus
3	(+,0,-)	Chaotic
4	(0,0,0,-)	3-Torus
4	(+,0,0,-)	Chaotic 2-Torus
4	(+,+,0,-)	Hyperchaos

Mechanisms of Chaos ...

Definition of chaotic attractor:

(1) Attractor: $\Lambda \subset \mathcal{X}$

(a) Invariant set: $\Lambda = \phi_T(\Lambda)$.

(b) Attracts an open set $U \subset \mathcal{X}$: $\Lambda \subset U$

$$\Lambda = \lim_{T \to \infty} \phi_T(U)$$

(c) Minimal: no proper subset is also (a) & (b).

(2) Aperiodic long-term behavior of a deterministic system with exponential amplification.

(2') Positive maximum LCE: $\lambda_{\max}(\Lambda) > 0$.

(2'') Positive metric entropy: $h_\mu(\Lambda) > 0$.

Mechanisms of Chaos ...

Reading for next lecture:

NDAC, Sections 10.5-10.7.

Thursday Meeting Update

- Invite class to Dept Research
- Overlaps with Thursday class
- 12:10-12:50 PM, 432 Physics
- Then back to 195 Physics