The Big, Big Picture (Bifurcations II)

Reading for this lecture:

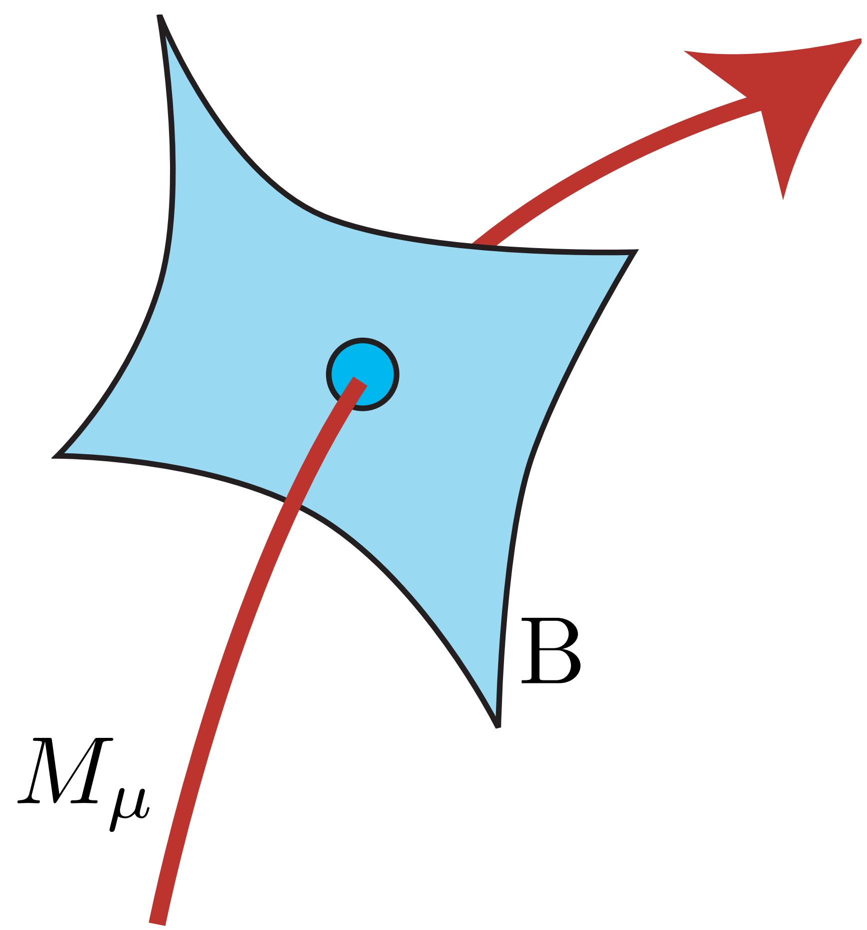
NDAC, Secs. 8.0-8.2, 8.4, and 8.7 and Secs. 10.0-10.4.

The Big, Big Picture (Bifurcations II) ...

Beyond changes in fixed points:

Bifurcation: Qualitative change in behavior as a control parameter is (slowly) varied.

Today: Bifurcations between time-dependent behaviors



The Big, Big Picture (Bifurcations II) ...

Bifurcations of 3D Flows:

Simulation demos of driven van der Pol: (ds)

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = A\sin(\omega t)$$

Why is this a 3D flow?

State $= (\dot{x}(t), x(t), \phi(t))$

Third dimension is phase of driving force:

$$\phi(t) = \omega t \mod 2\pi$$

Topology of state space is periodic in third variable.

The Big, Big Picture (Bifurcations II) ...

Bifurcations of 3D Flows:

Simulation demos of driven van der Pol: (ds)

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = A\sin(\omega t)$$

One way to verify:

Write out equivalent first-order ODEs:

(Goal: RHS is time independent)

$$\begin{aligned}
\dot{x} &= y \\
\dot{y} &= \mu(1 - x^2)y - x + A\sin(\phi) \\
\dot{\phi} &= \omega
\end{aligned}$$

$$(x, y, \phi) \in \mathbb{R}^2 \times S^1$$

The Big, Big Picture (Bifurcations II) ...

Bifurcations of 3D Flows:

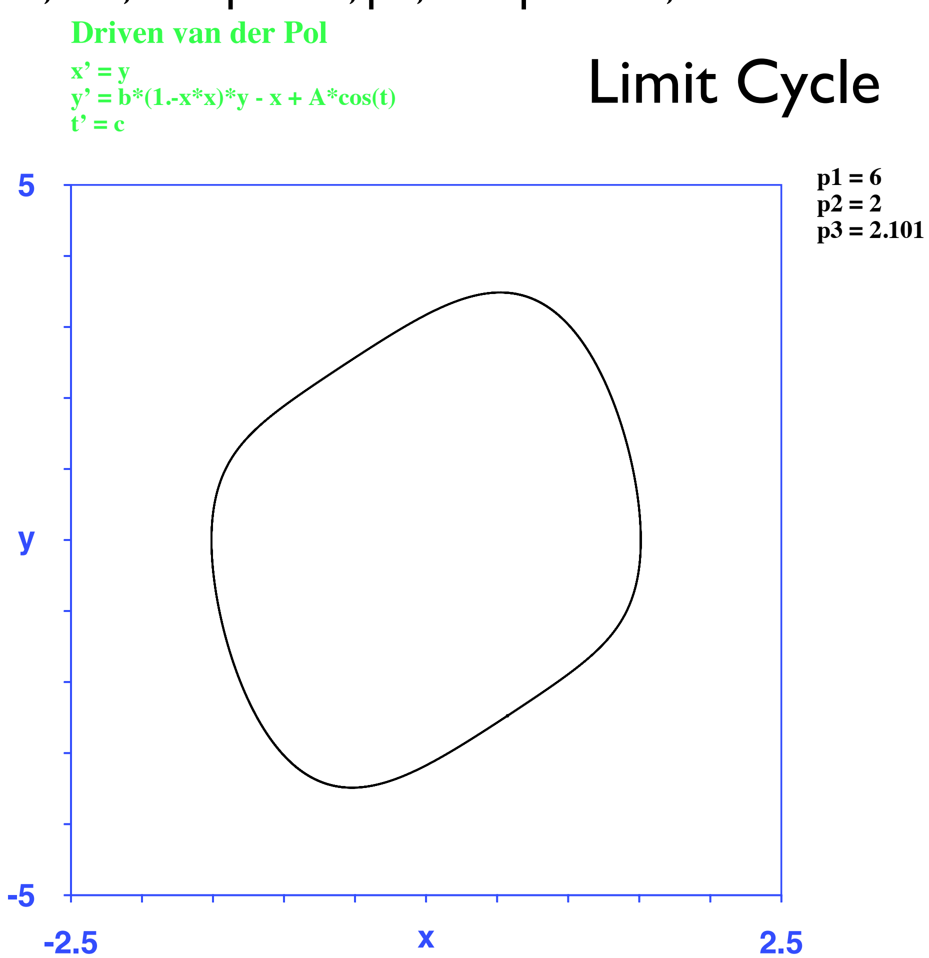
Simulation demos of driven van der Pol:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = A\sin(\omega t)$$

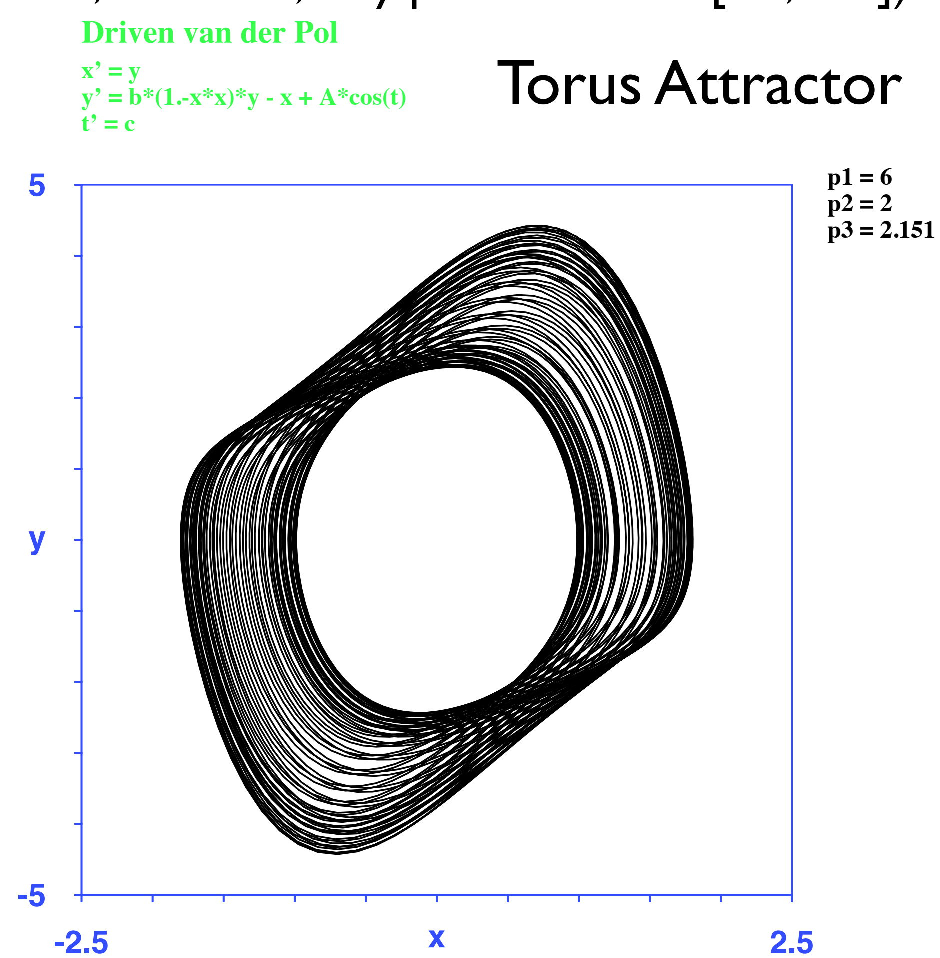
Limit cycle to torus $(A, \mu, \omega) = (6.0, 2.0, [2.1, 2.15])$

(set parms; 2D; nSteps = 1; p2; nSteps = 50; nTrans = 10000; nIts = 3000; color = 0; vary parameter 2 in [2.1,2.15])

Limit Cycle



Torus Attractor



The Big, Big Picture (Bifurcations II) ...

Bifurcations of 3D Flows:

Simulation demos of driven van der Pol: (Shaw's deviant)

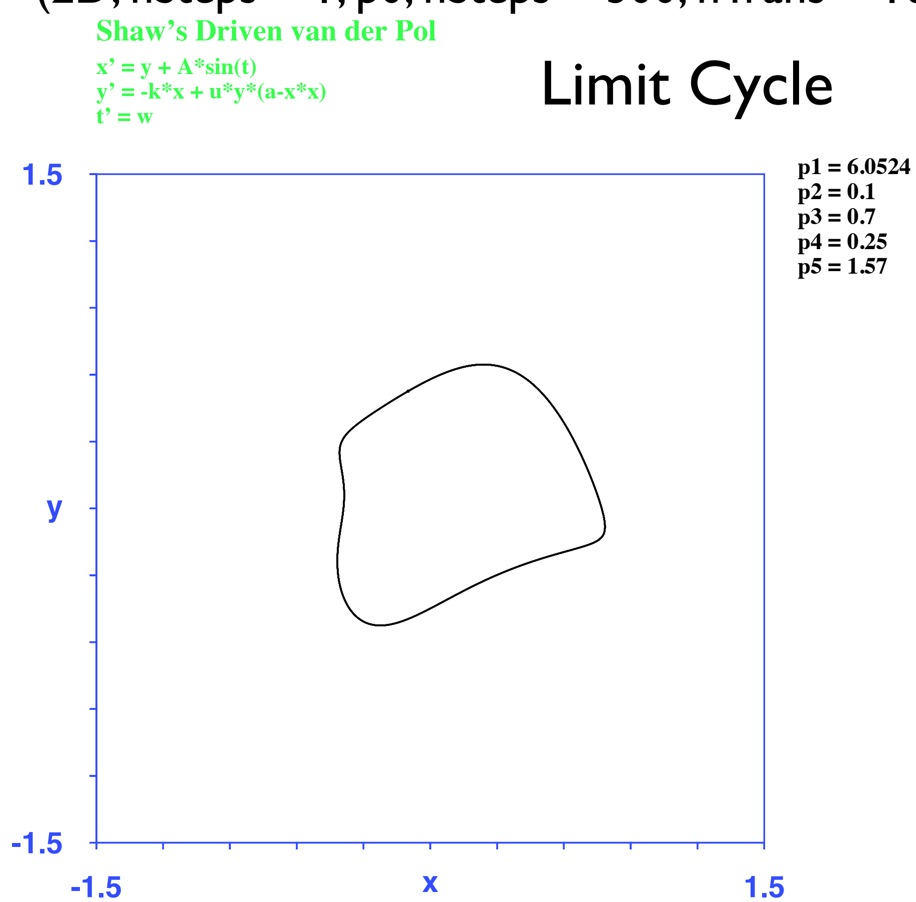
$$
\begin{aligned}
\dot{x} &= y + A\sin(\phi) \\
\dot{y} &= \mu(a - x^2)y - kx \\
\dot{\phi} &= \omega
\end{aligned}
$$

$$(\mu, a, k, A, \omega) \in ([6.0, 8.12], 0.1, 0.7, 0.25, 1.57)$$

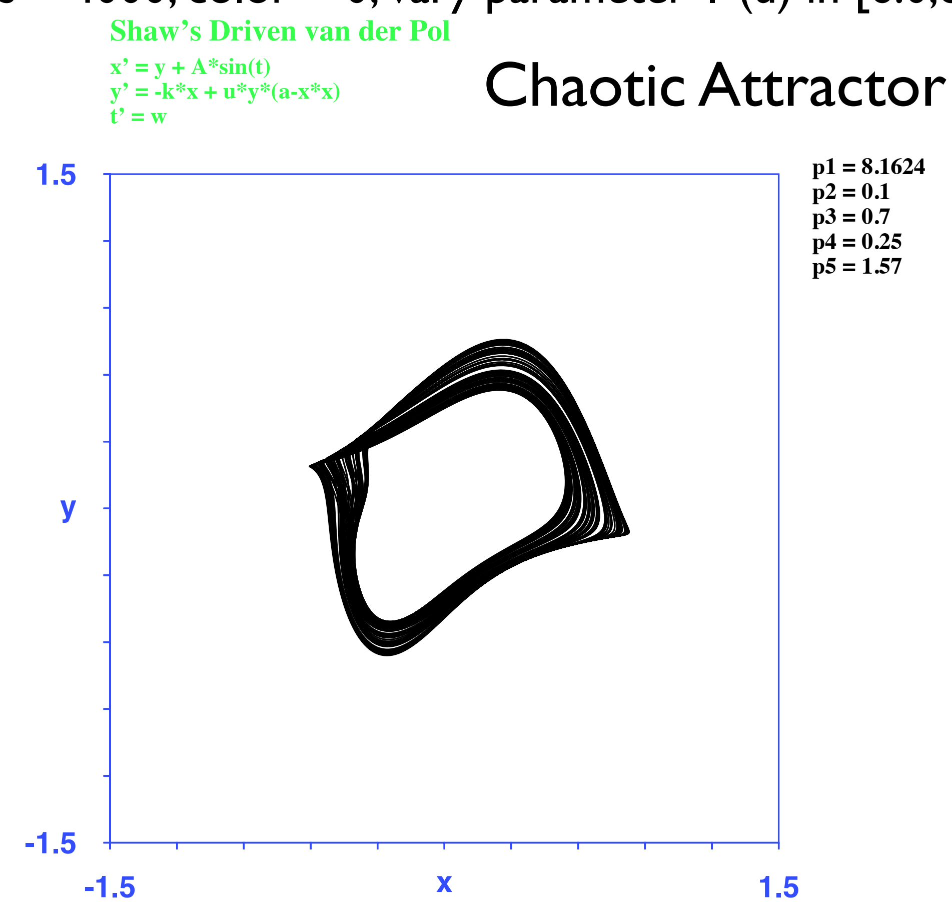
Limit cycle to chaos

(2D; nSteps = 1; p0; nSteps = 300; nTrans = 10000; nIts = 4000; color = 0; vary parameter 1 (u) in [6.0,8.12])

Limit Cycle



Chaotic Attractor



The Big, Big Picture (Bifurcations II) ...

Bifurcations of 3D Flows:

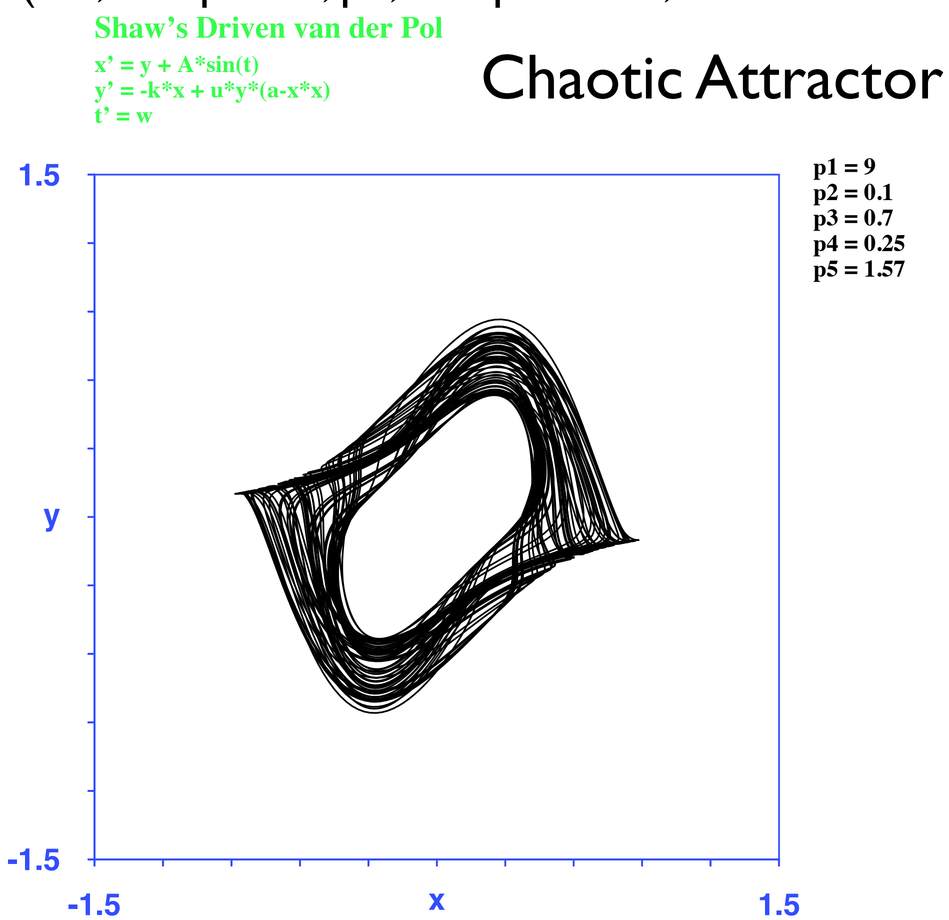
Simulation demos of driven van der Pol: (Shaw's deviant)

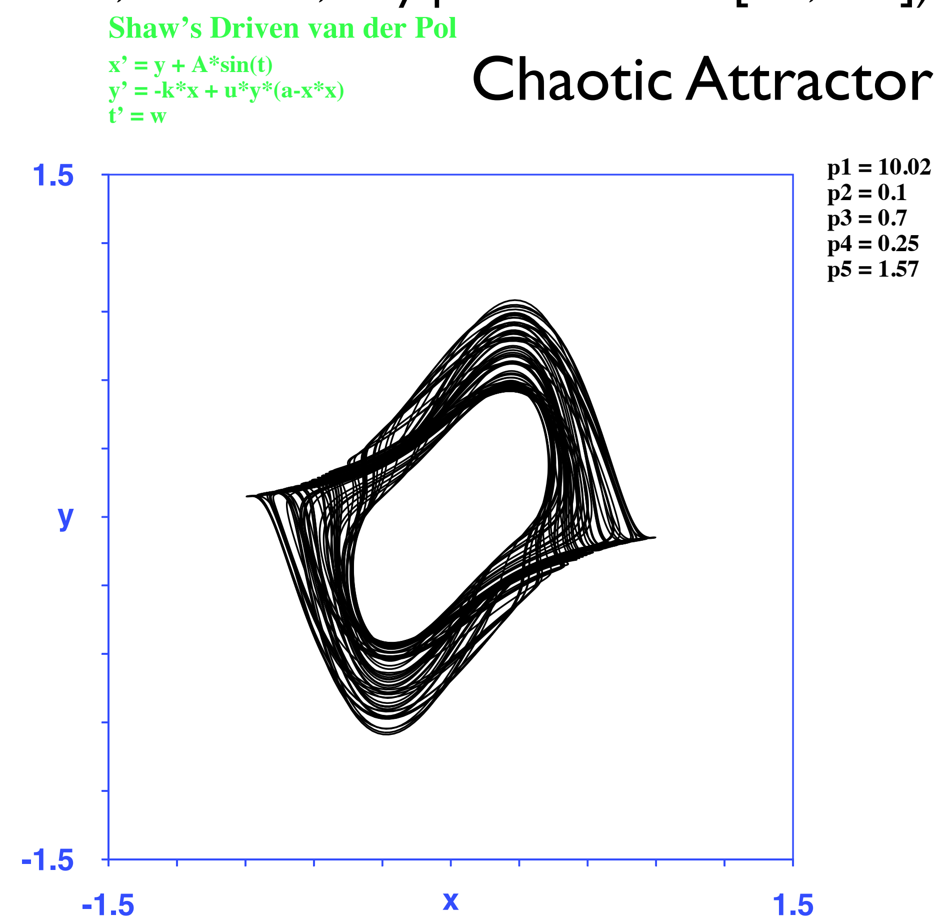
$$\begin{aligned}\dot{x} &= y + A\sin(\phi) \\ \dot{y} &= \mu(a - x^2)y - kx \\ \dot{\phi} &= \omega\end{aligned}$$

$$(\mu, a, k, A, \omega) \in ([9.0, 10.0], 0.1, 0.7, 0.25, 1.57)$$

Chaos to chaos

(2D; nSteps = 1; p0; nSteps = 100; nTrans = 10000; nIts = 3000; color = 0; vary parameter 0 in [9.0,10.0])





The Big, Big Picture (Bifurcations II) ...

Bifurcations of 3D Flows:

Simulation demos of driven van der Pol: (Shaw's deviant)

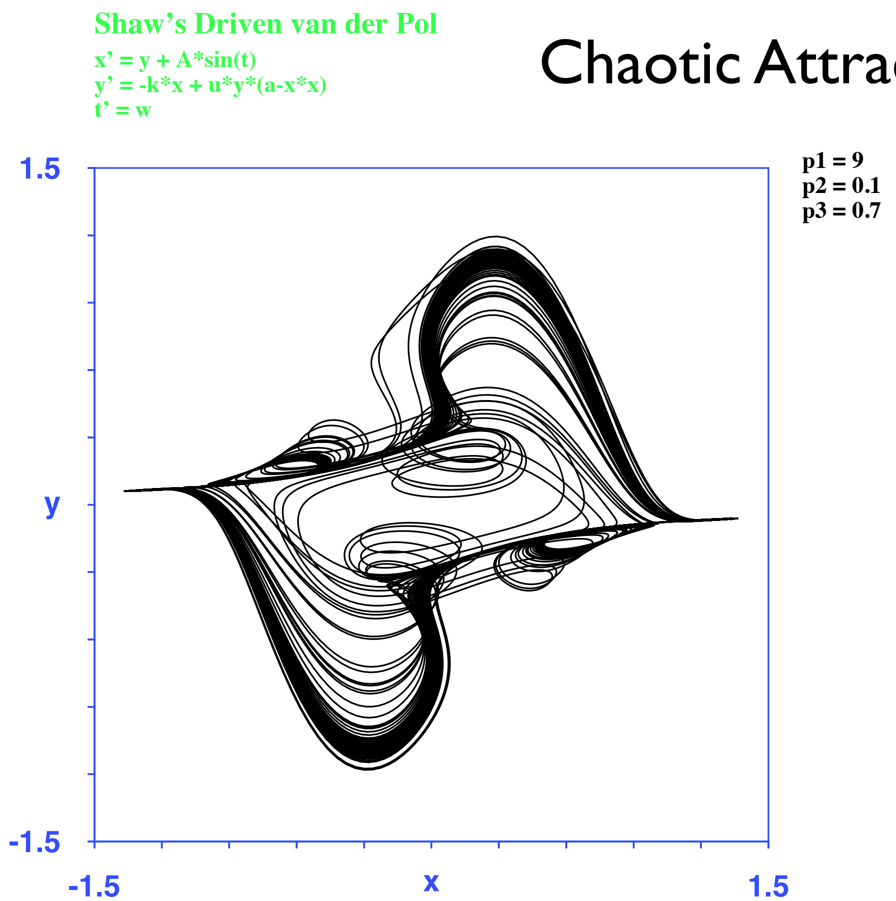
$$\begin{aligned} \dot{x} &= y + A\sin(\phi) \\ \dot{y} &= \mu(a - x^2)y - kx \\ \dot{\phi} &= \omega \end{aligned}$$

$$(\mu, a, l, A, \omega) = (9.0, 0.1, 0.7, [0.45, 0.73], 2.0)$$

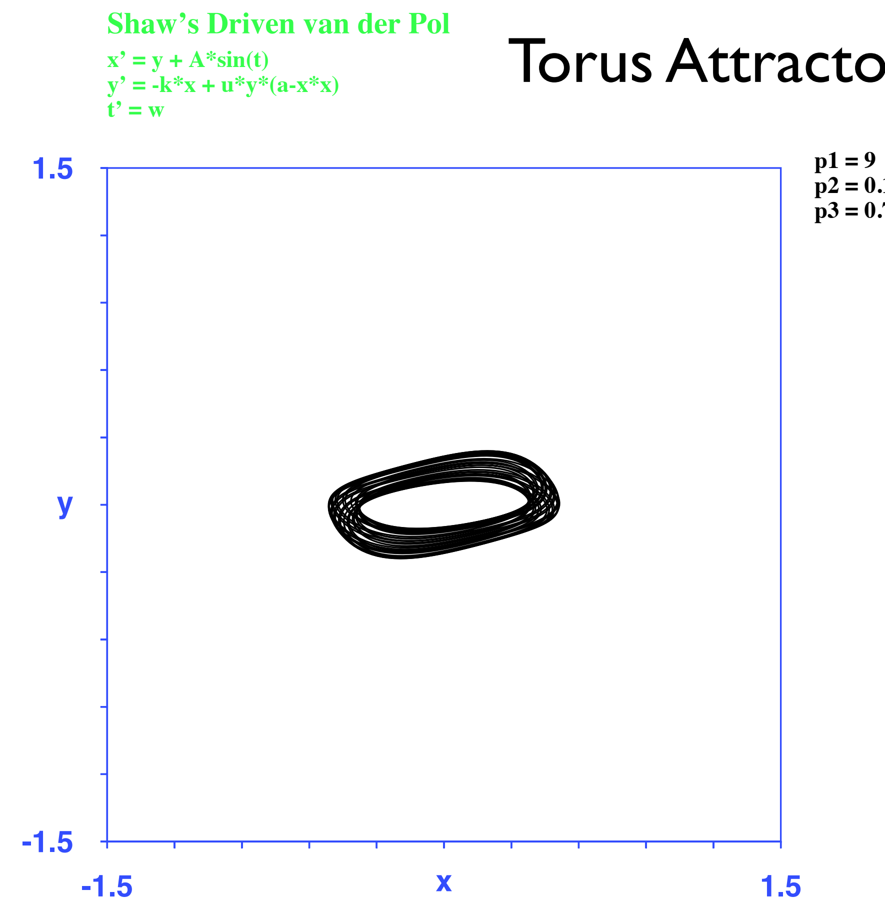
Chaos to torus

(2D; nSteps=1; p3; nSteps = 200; nTrans = 20000; nIts = 4000; color = 0; vary parameter 3 in [0.45,0.73])

Chaotic Attractor



Torus Attractor



The Big, Big Picture (Bifurcations II) ...

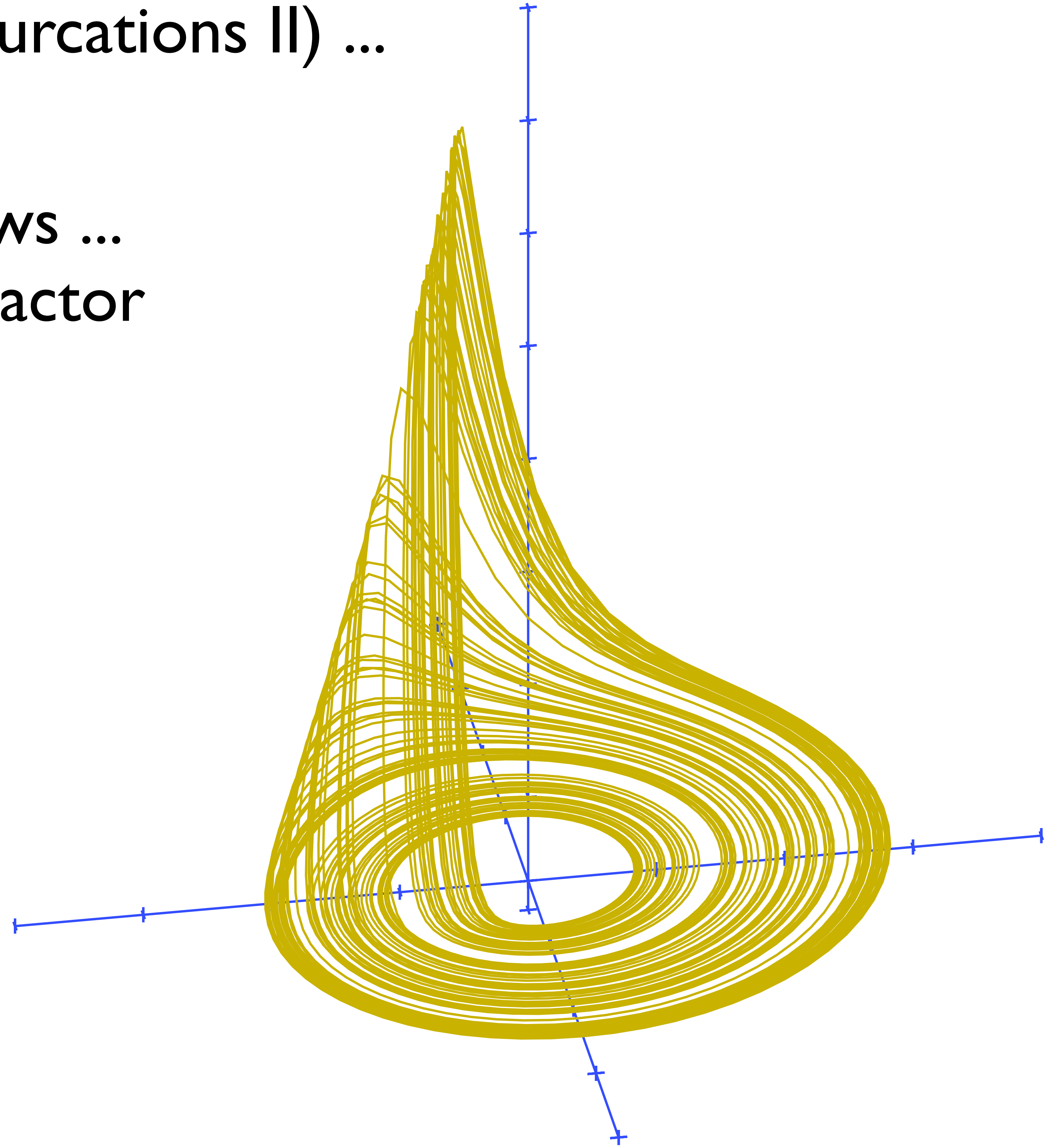
Bifurcations of 3D Flows:
Rössler equations

$$
\begin{aligned}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
\end{aligned}
$$

Parameters: $a, b, c > 0$

The Big, Big Picture (Bifurcations II) ...

Bifurcations of 3D Flows ...
Rössler chaotic attractor



Parameters: $(a, b, c) = (0.2, 0.2, 5.7)$

The Big, Big Picture (Bifurcations II) ...

Bifurcations of 3D Flows ...

Simulation demos of Rössler: (ds)

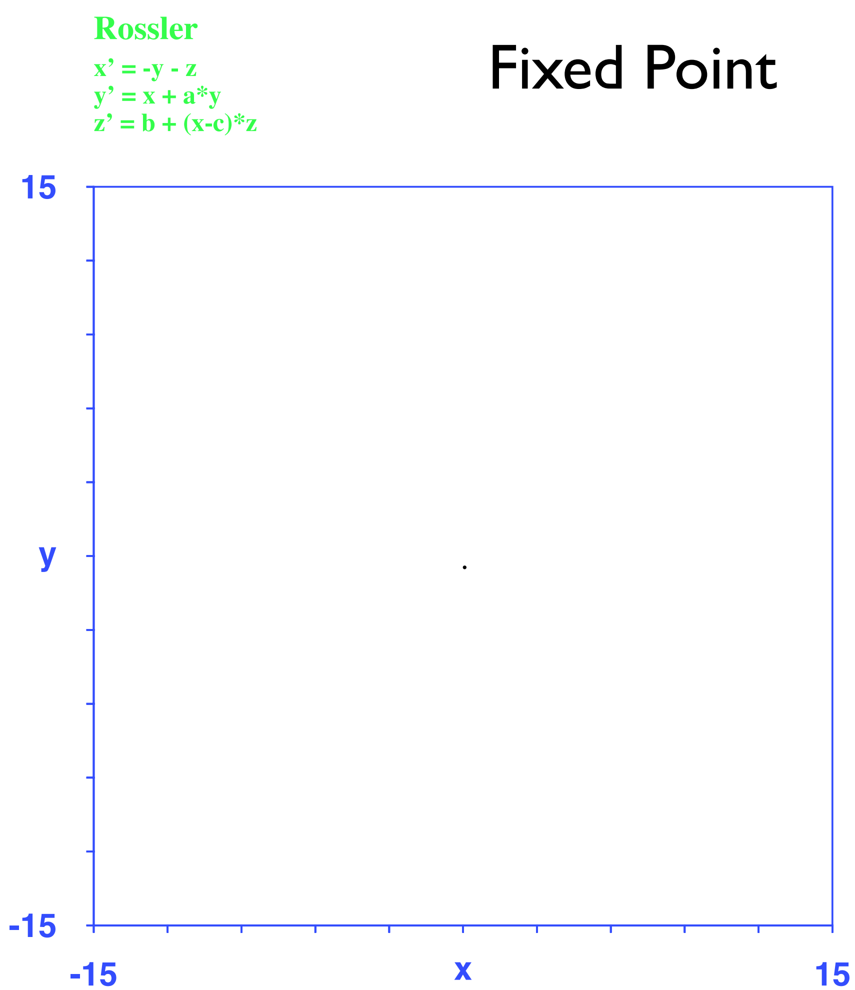
Hopf bifurcation:

Fixed point to limit cycle:

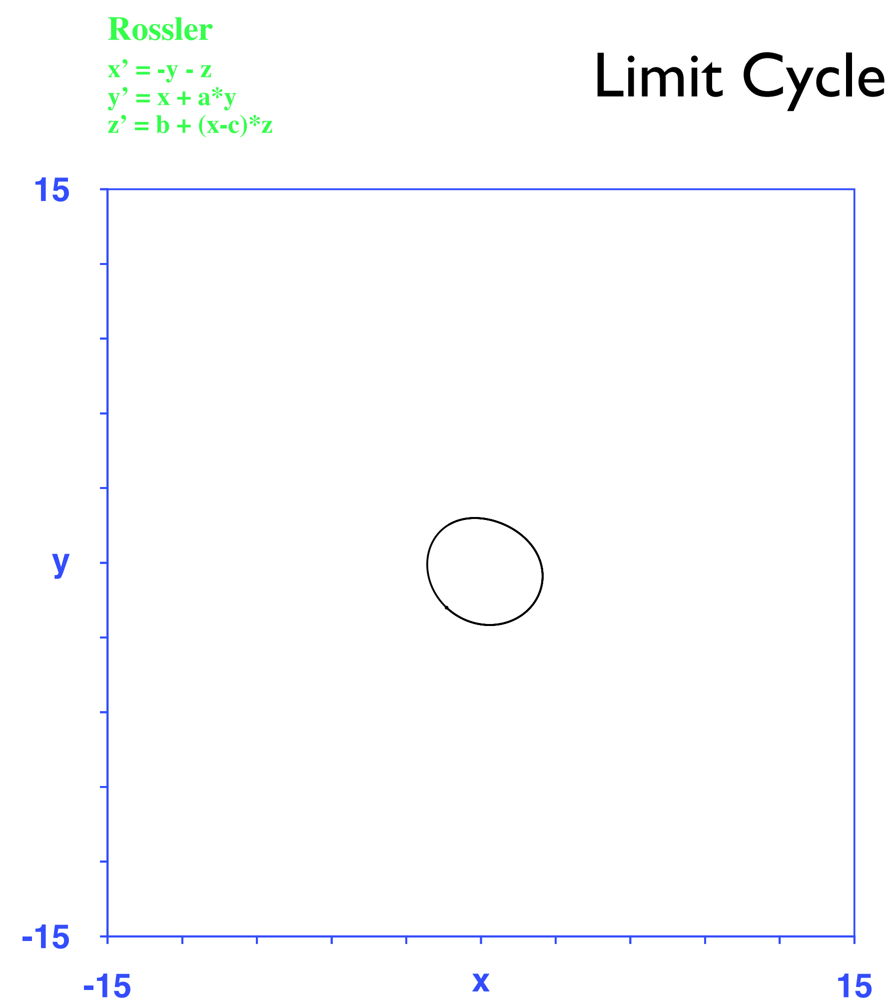
$$(a, b, c) \in ([0.1, 0.2], 0.2, 0.6)$$

(2D; nSteps = 1; p0; nSteps = 1000; nTrans = 10000; nIts = 1000; color = 0; vary parameter 0 in [0.1,0.2])

Fixed Point



Limit Cycle



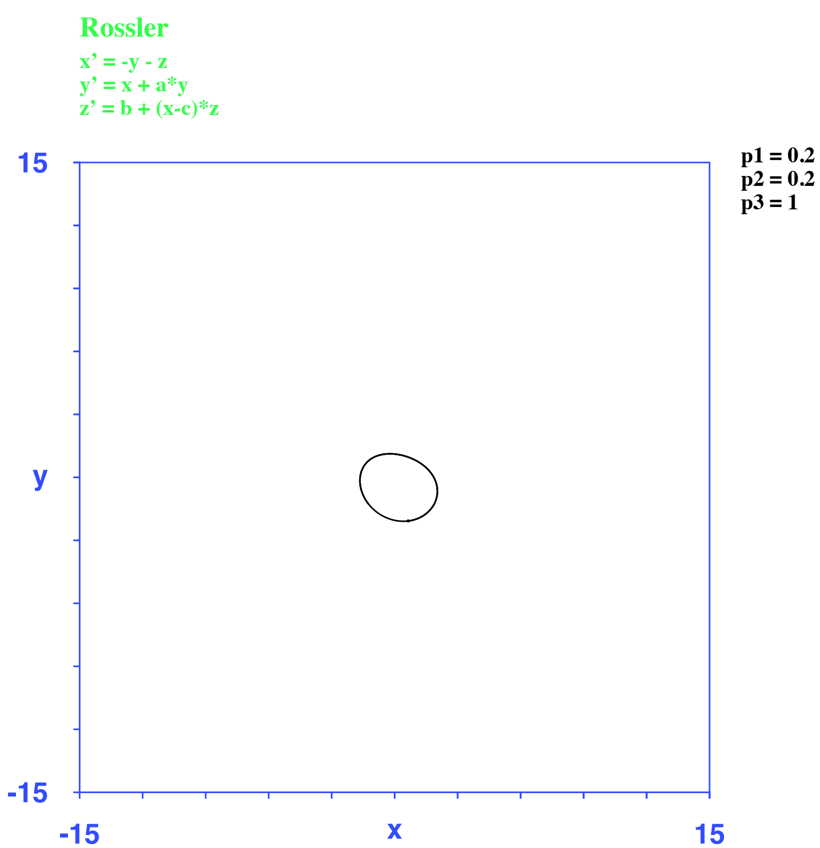
Bifurcations of 3D Flows ...

Simulation demos of Rössler: (ds)

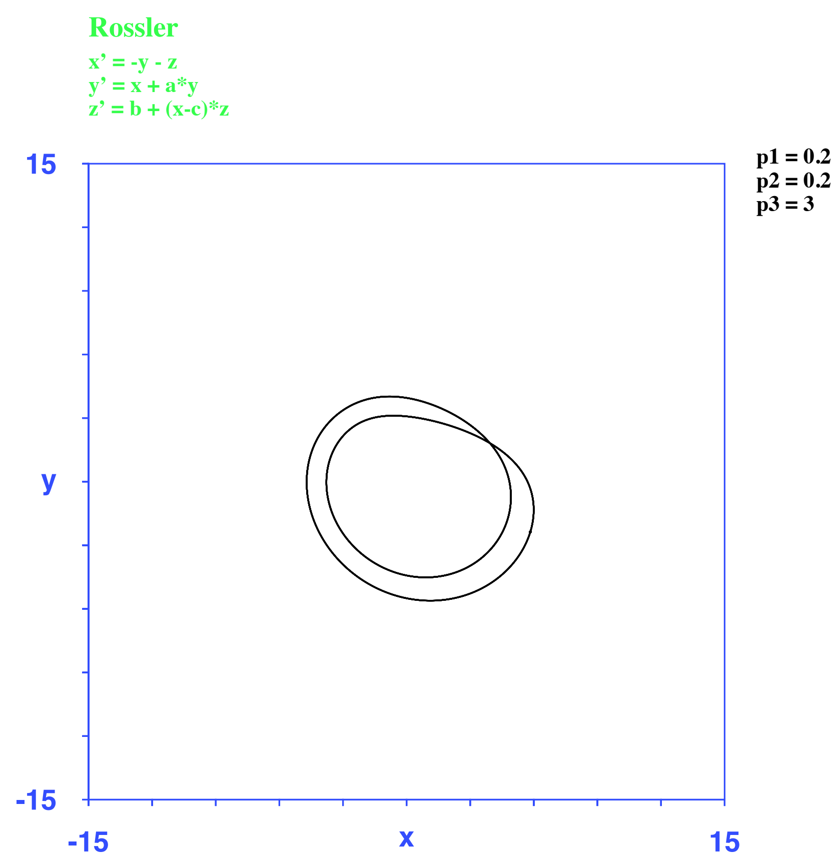
Period-doubling route:

$$(a, b, c) \in (0.2, 0.2, [1.0, 4.0])$$

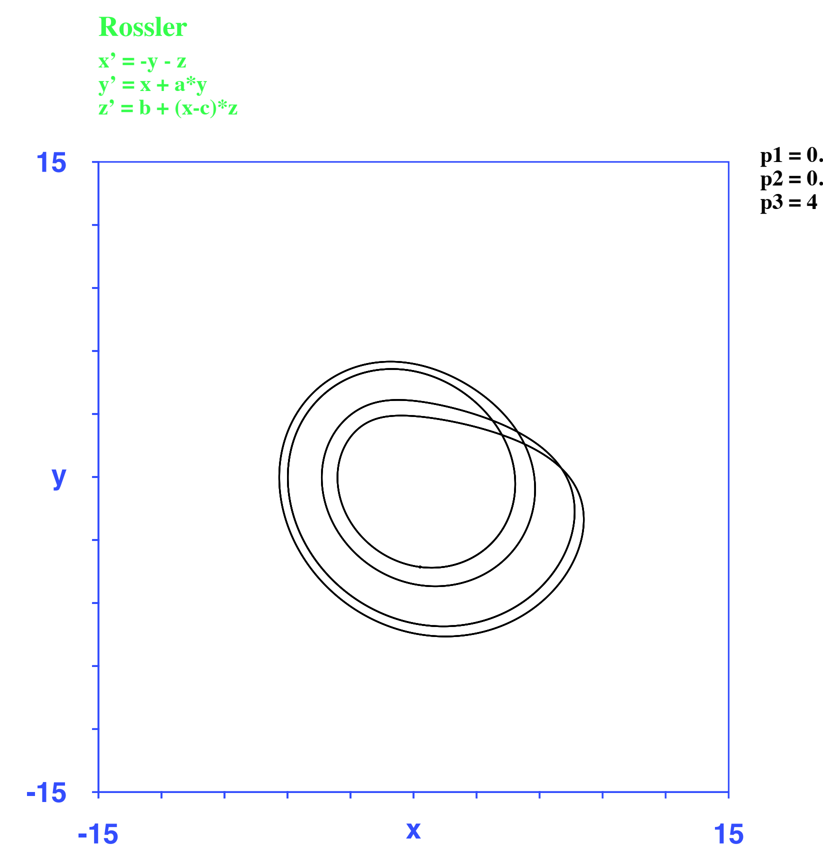
(2D; nSteps = 1; p2; nSteps = 2000; nTrans = 20000; nIts = 2000; color = 0; c is parameter 2)



Period 1



Period 2



Period 4

The Big, Big Picture (Bifurcations II) ...

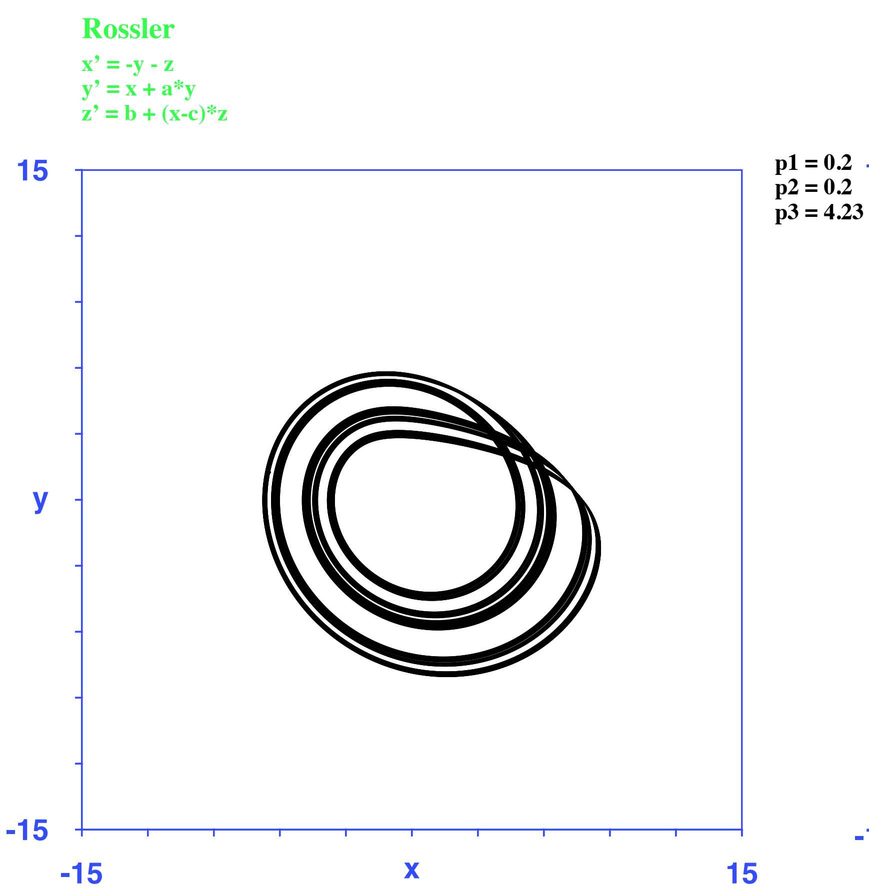
Bifurcations of 3D Flows ...

Simulation demos of Rössler: (ds)

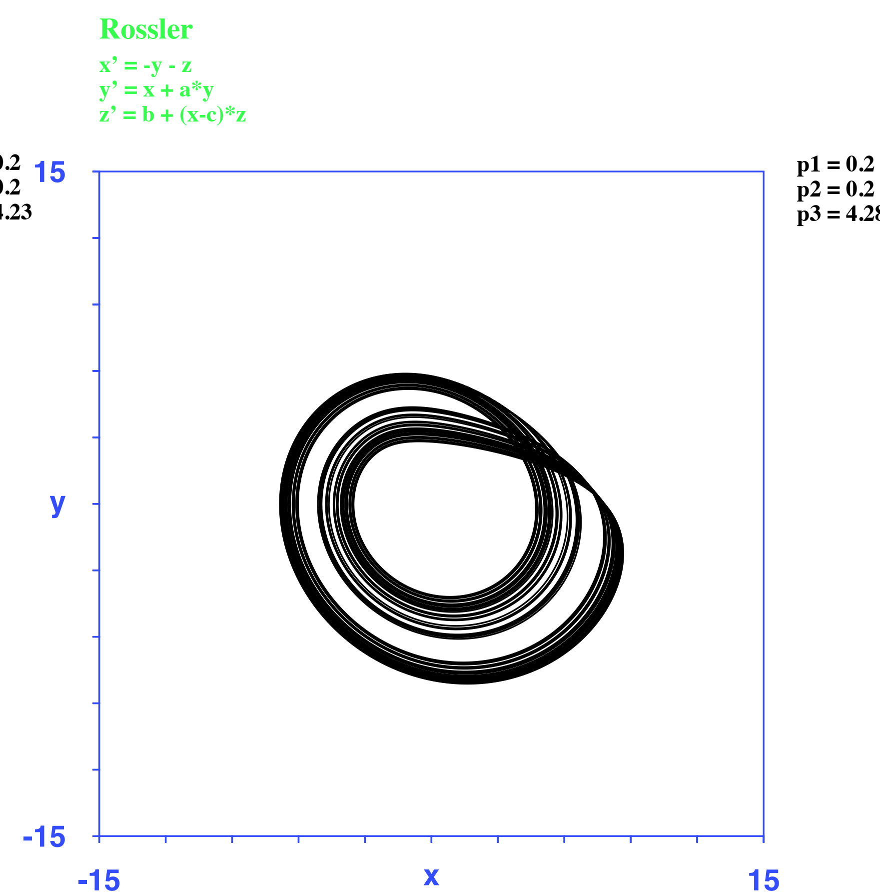
Period-doubling route:

$$(a, b, c) \in (0.2, 0.2, [4.0, 6.0])$$

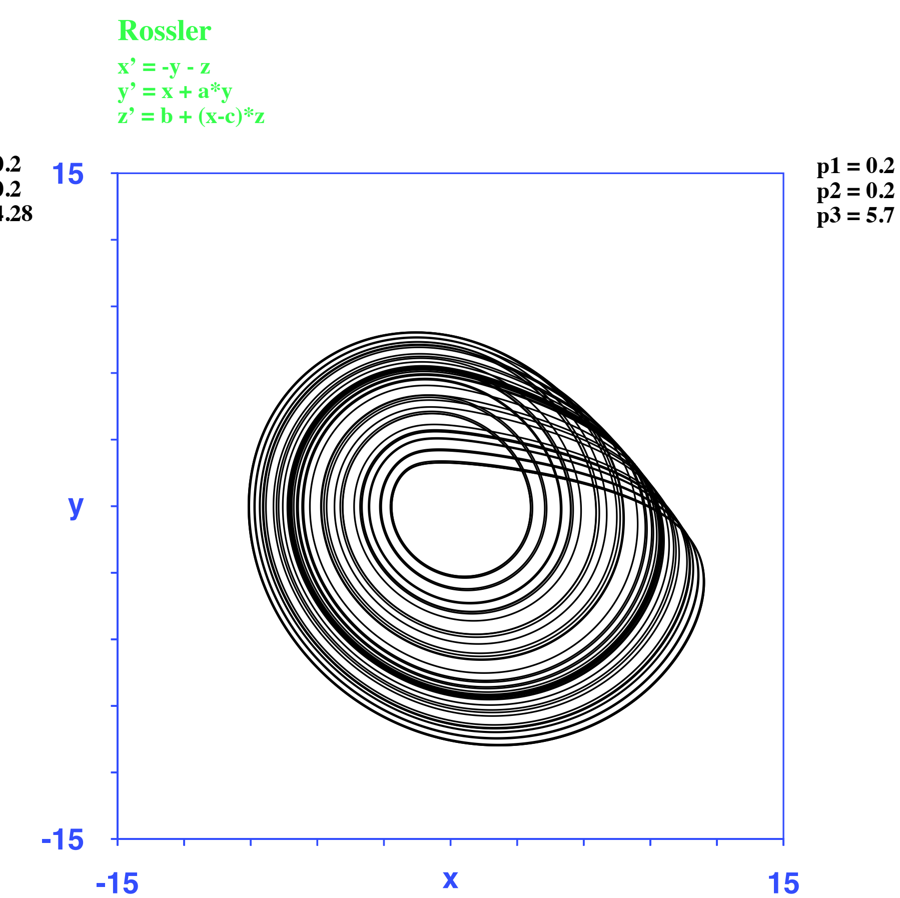
(2D; nSteps = 1; p2; nSteps = 2000; nTrans = 20000; nIts = 2000; color = 0; c is parameter 2)



4 Bands



2 Bands



1 Band

The Big, Big Picture (Bifurcations II) ...

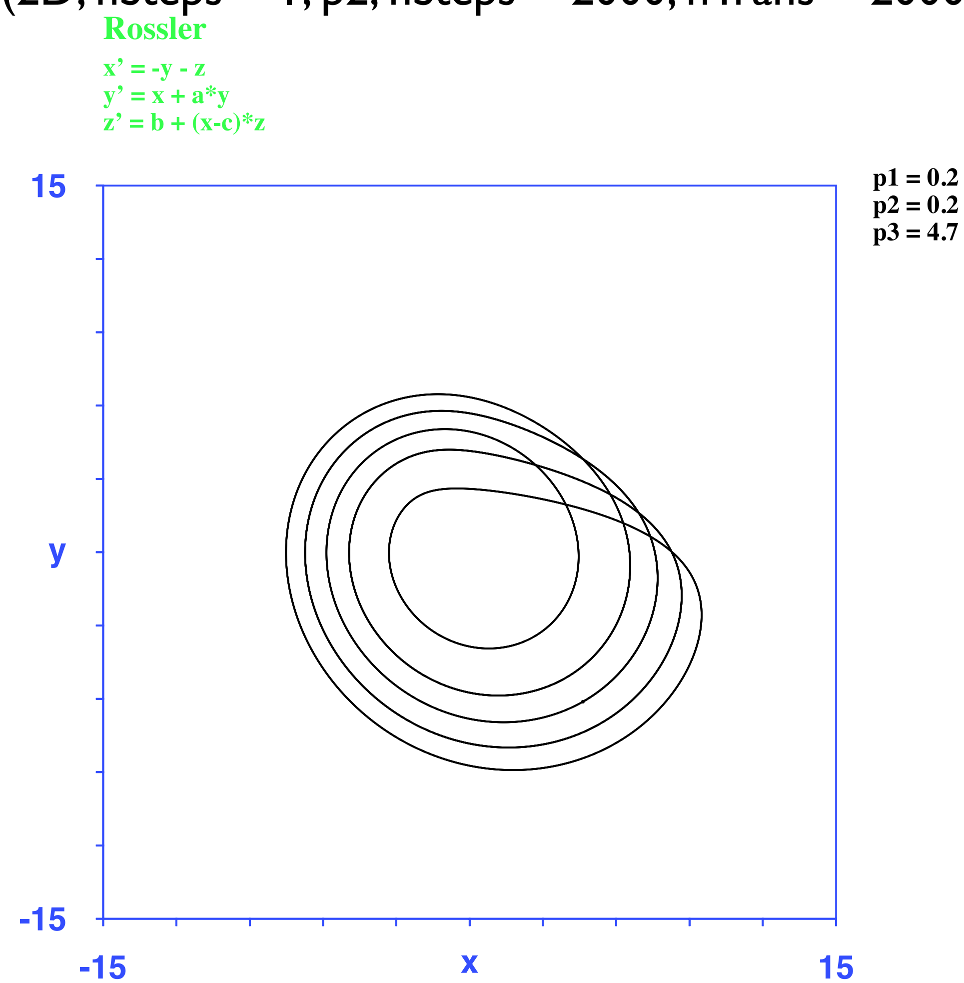
Bifurcations of 3D Flows ...

Simulation demos of Rössler: (ds)

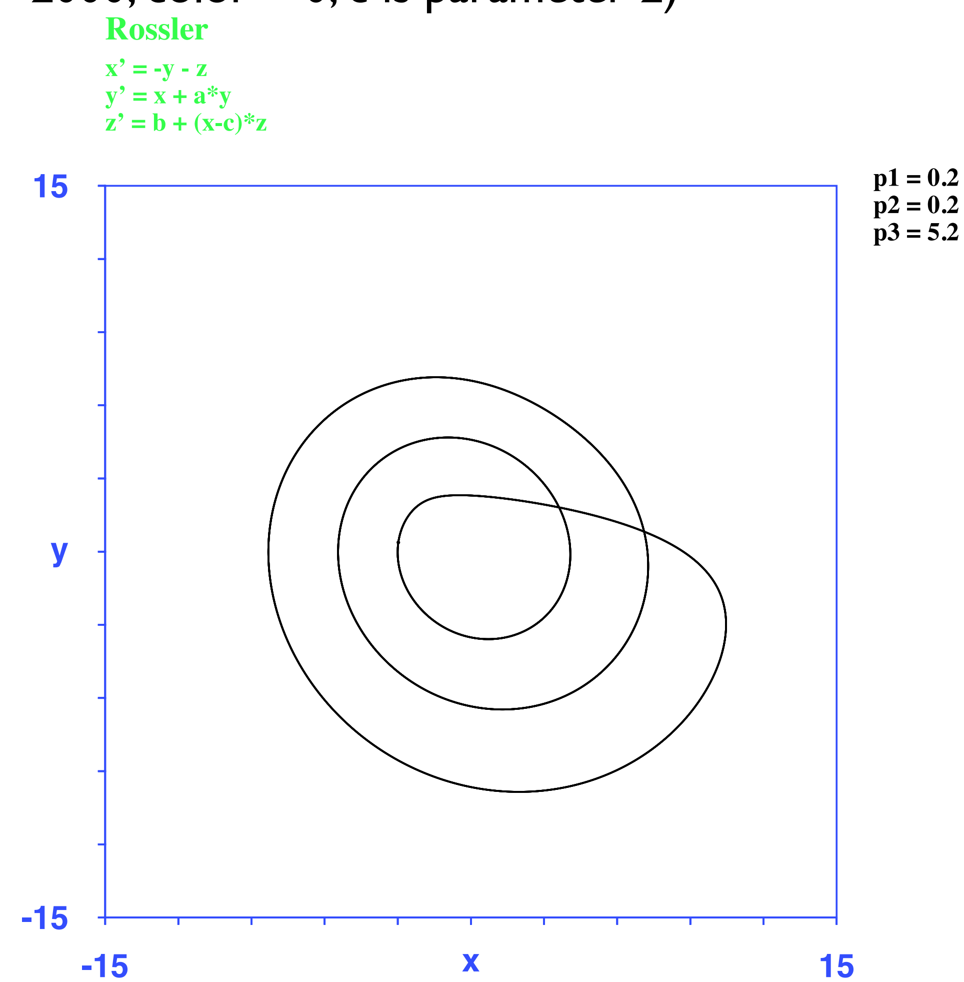
Period-doubling route:

$$(a, b, c) \in (0.2, 0.2, [4.7, 5.2])$$

(2D; nSteps = 1; p2; nSteps = 2000; nTrans = 20000; nIts = 2000; color = 0; c is parameter 2)



Period 5



Period 3

The Big, Big Picture (Bifurcations II) ...

Bifurcations of 3D Flows ...

Period-doubling route order of bifurcations:

$$\text{P-1} \rightarrow \text{P-2} \rightarrow \text{P-4} \rightarrow \text{P-8} \rightarrow \ldots \rightarrow \text{P-}2^n \rightarrow \text{P-}\infty$$

Then what? Chaotic bands:

$$\text{B-}\infty \rightarrow \text{B-}2^n \rightarrow \ldots \rightarrow \text{B-8} \rightarrow \text{B-4} \rightarrow \text{B-2} \rightarrow \text{B-1}$$

Period-5 Period-3

Sarkovski ordering:

Universal for all period-doubling systems

Theorem : Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous with a periodic point of principal period *k*. If *k* > *l* in the ordering

$$3 > 5 > 7 > \ldots > 3 \times 2^n > 5 \times 2^n > \ldots > 2^n > 2^{n-1} > \ldots > 4 > 2$$

then *f* also has a periodic point of period *l*.

The Big, Big Picture (Bifurcations II) ...

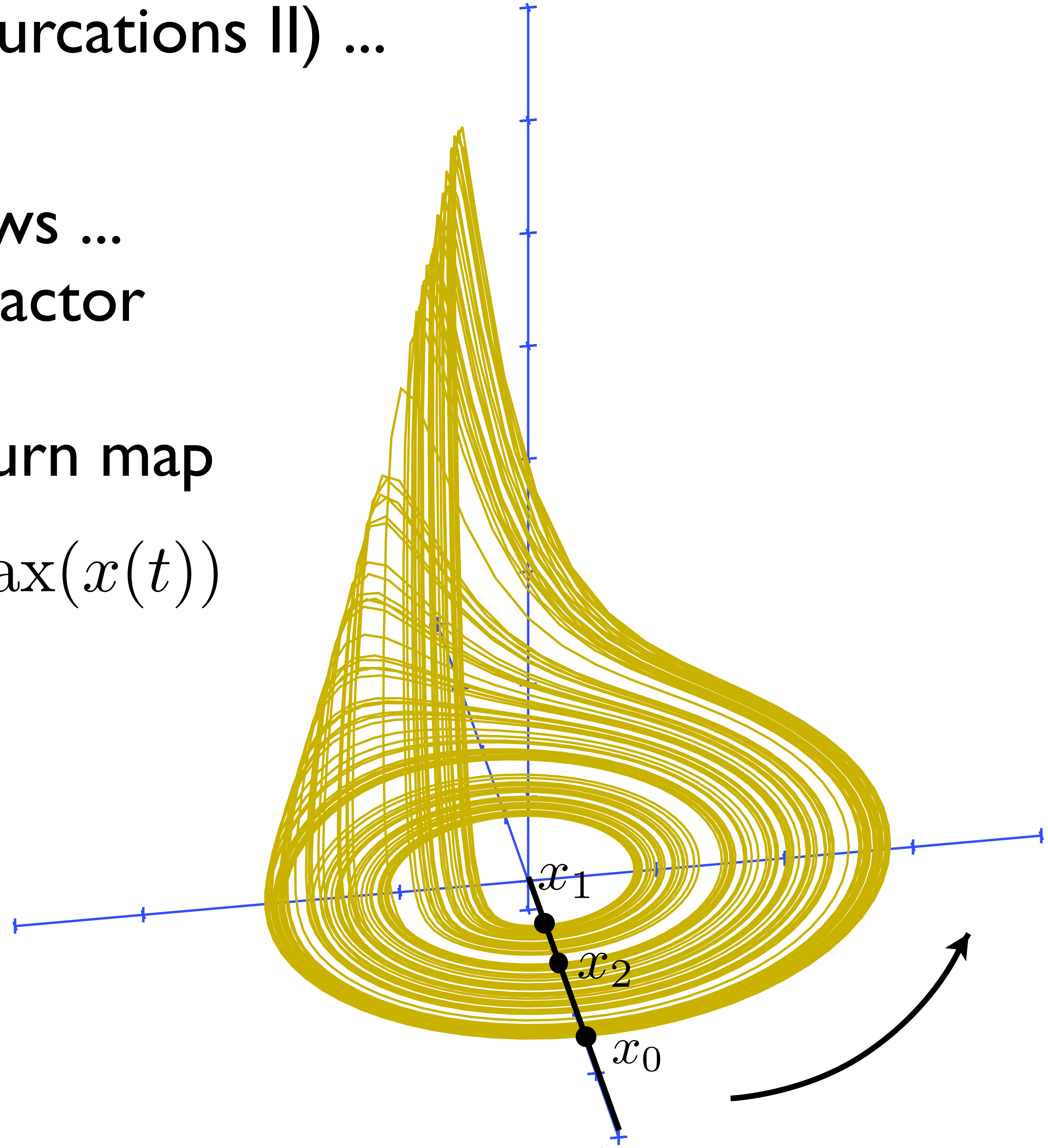
Bifurcations of 3D Flows ...

Rössler chaotic attractor

reduction to

x(t)-maxima return map

$$x_n \equiv n^{\text{th}} \ \max(x(t))$$

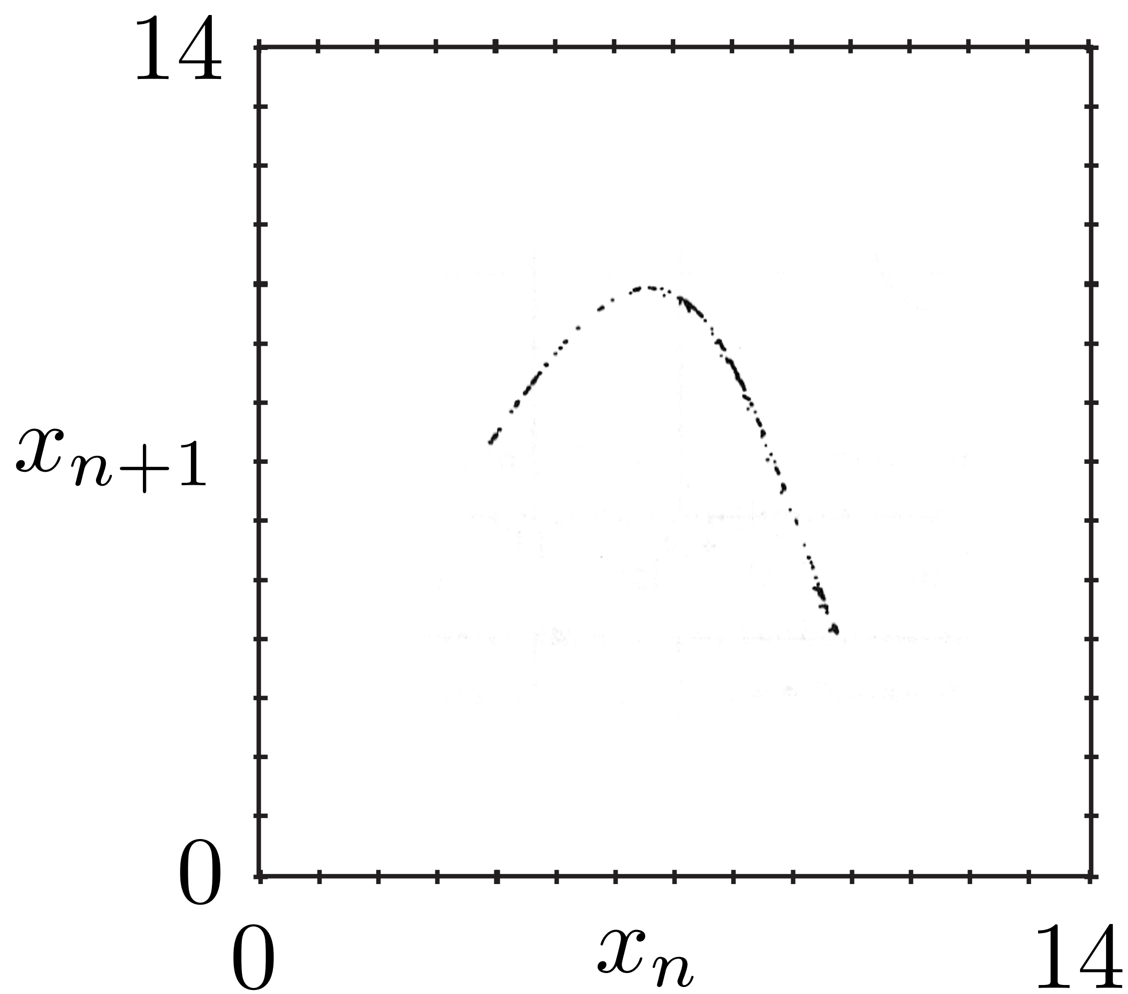


Parameters: $(a, b, c) = (0.2, 0.2, 5.7)$

The Big, Big Picture (Bifurcations II) ...

Bifurcations of 3D Flows ...

Rössler maximum-x return map: $x_{n+1} = f(x_n)$



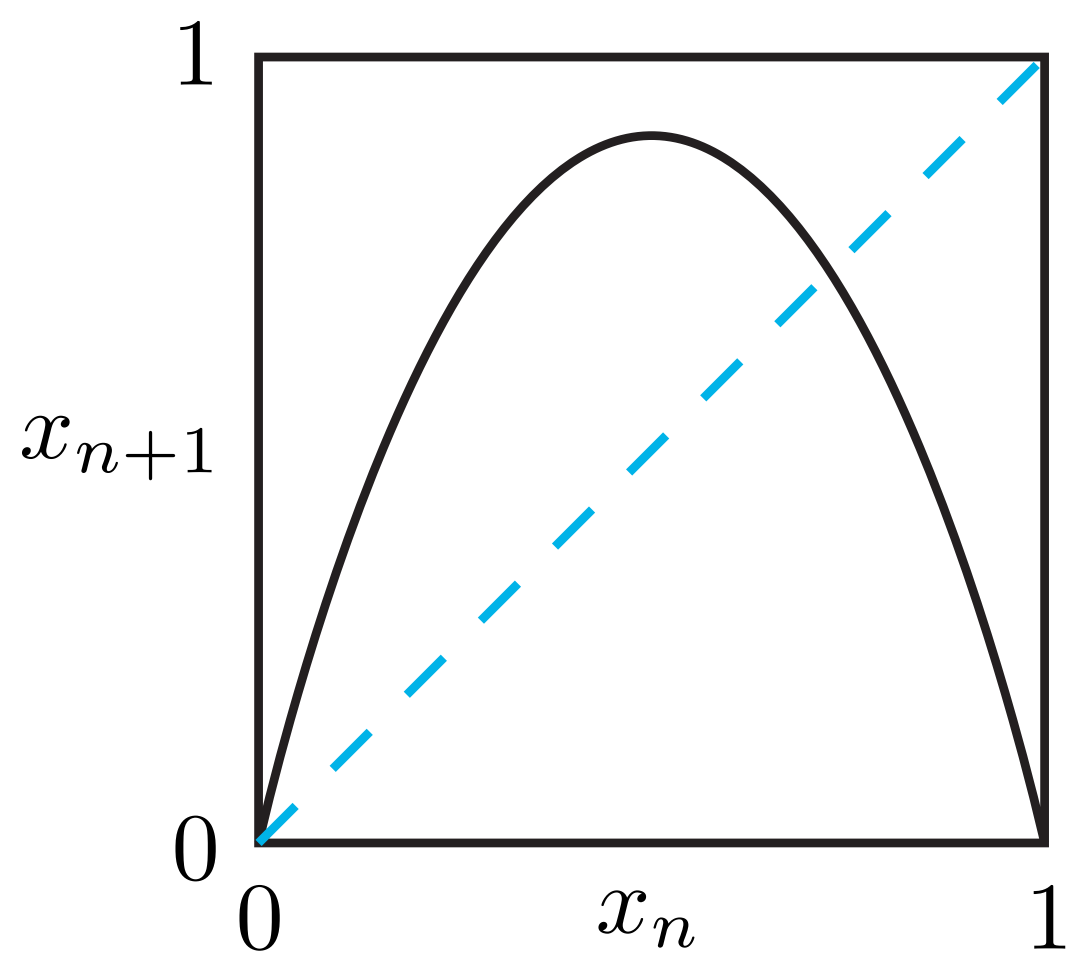
The Big, Big Picture (Bifurcations II) ...

Bifurcations of 3D Flows ...

When normalized to $x_n \in [0, 1]$

get the Logistic Map:

$$x_{n+1} = r x_n (1 - x_n)$$

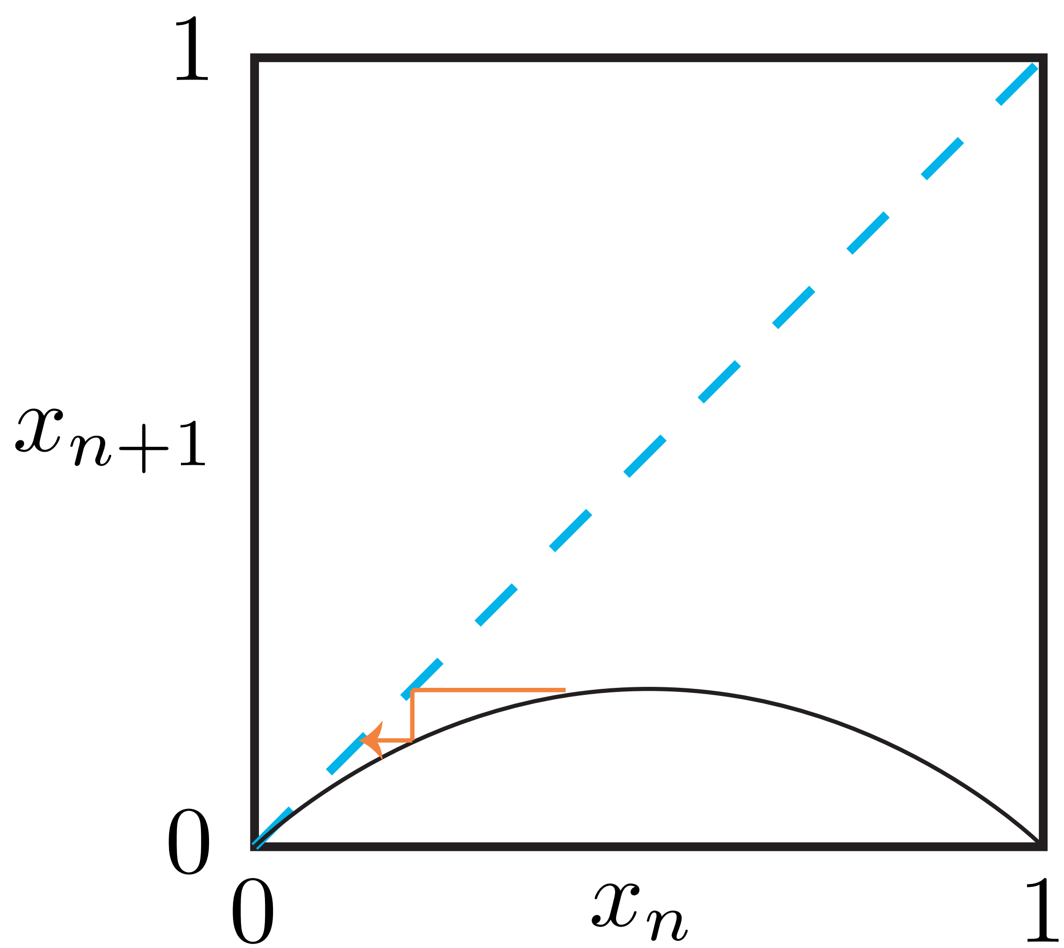


Parameter (height): $r \in [0, 4]$

The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps:

Logistic map: $x_{n+1} = r x_n (1 - x_n)$
State space: $x_n \in [0, 1]$
Parameter (height): $r \in [0, 4]$



Maximum: $x = \frac{1}{2}$

Fixed Points: $x^* = f(x^*)$

Fixed Point: $x^* = 0, \ \forall r$

Stable:

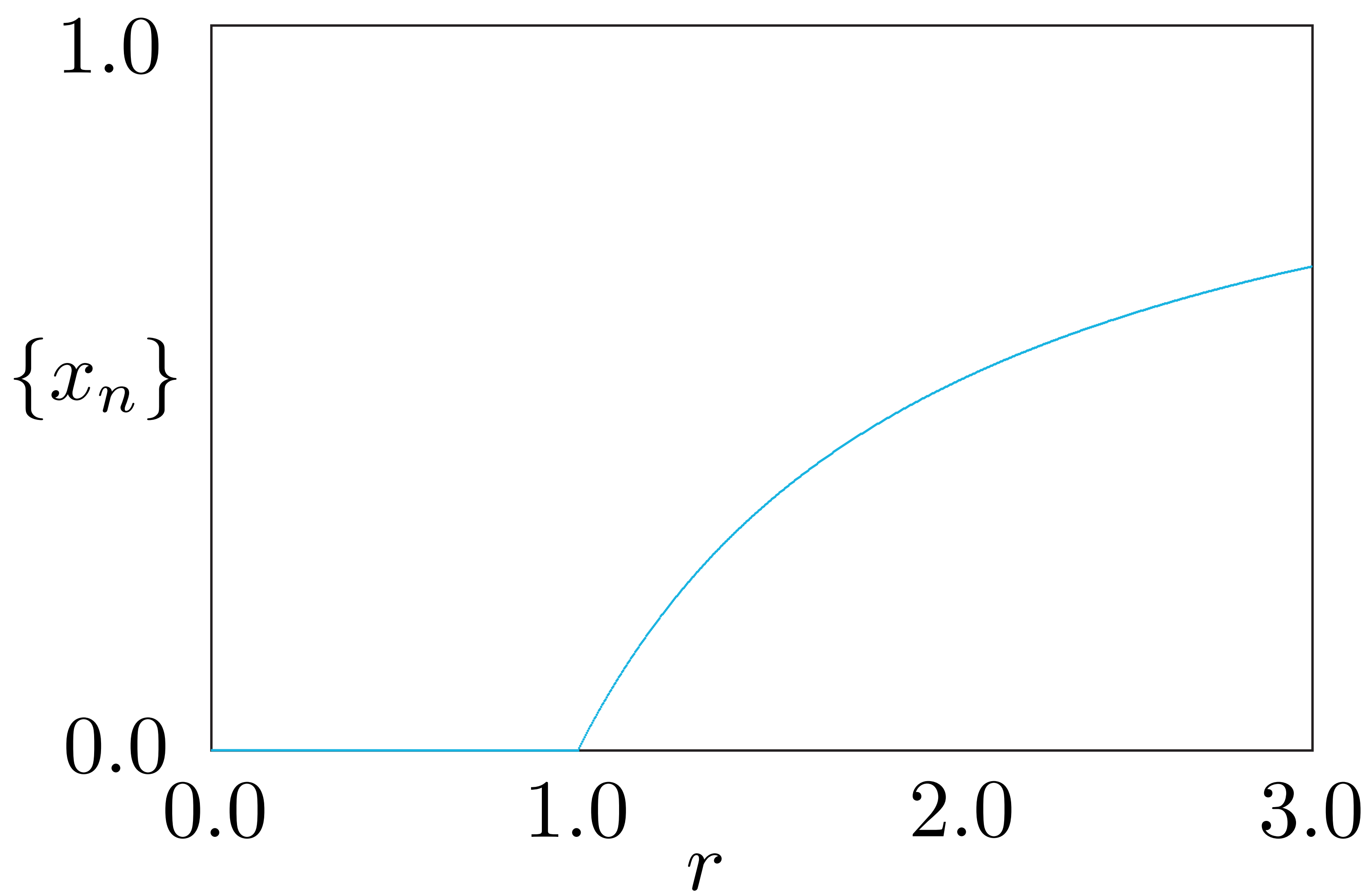
$$\left| \left. \frac{df}{dx} \right|_{x^*} \right| < 1, \ r \in [0, 1)$$

The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

But that fixed point goes unstable



The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

The other fixed point: $x^* = 1 - r^{-1}$

At what parameter value?

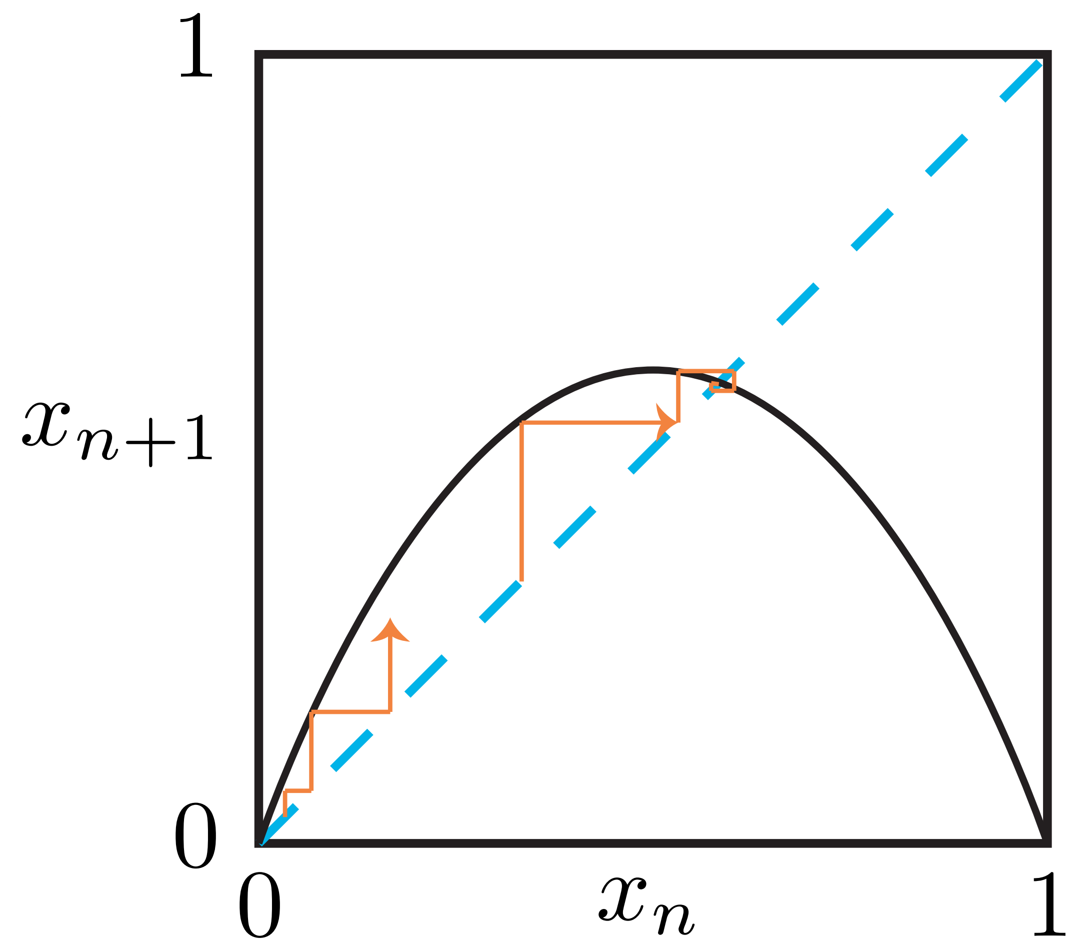
Where the other loses stability

$$|f'(x^*)| = 1$$

$$f'(x) = r - 2rx$$

$$x^* = 0 \;\Rightarrow\; f'(x^*) = r$$

x^* is unstable when $r \geq 1$

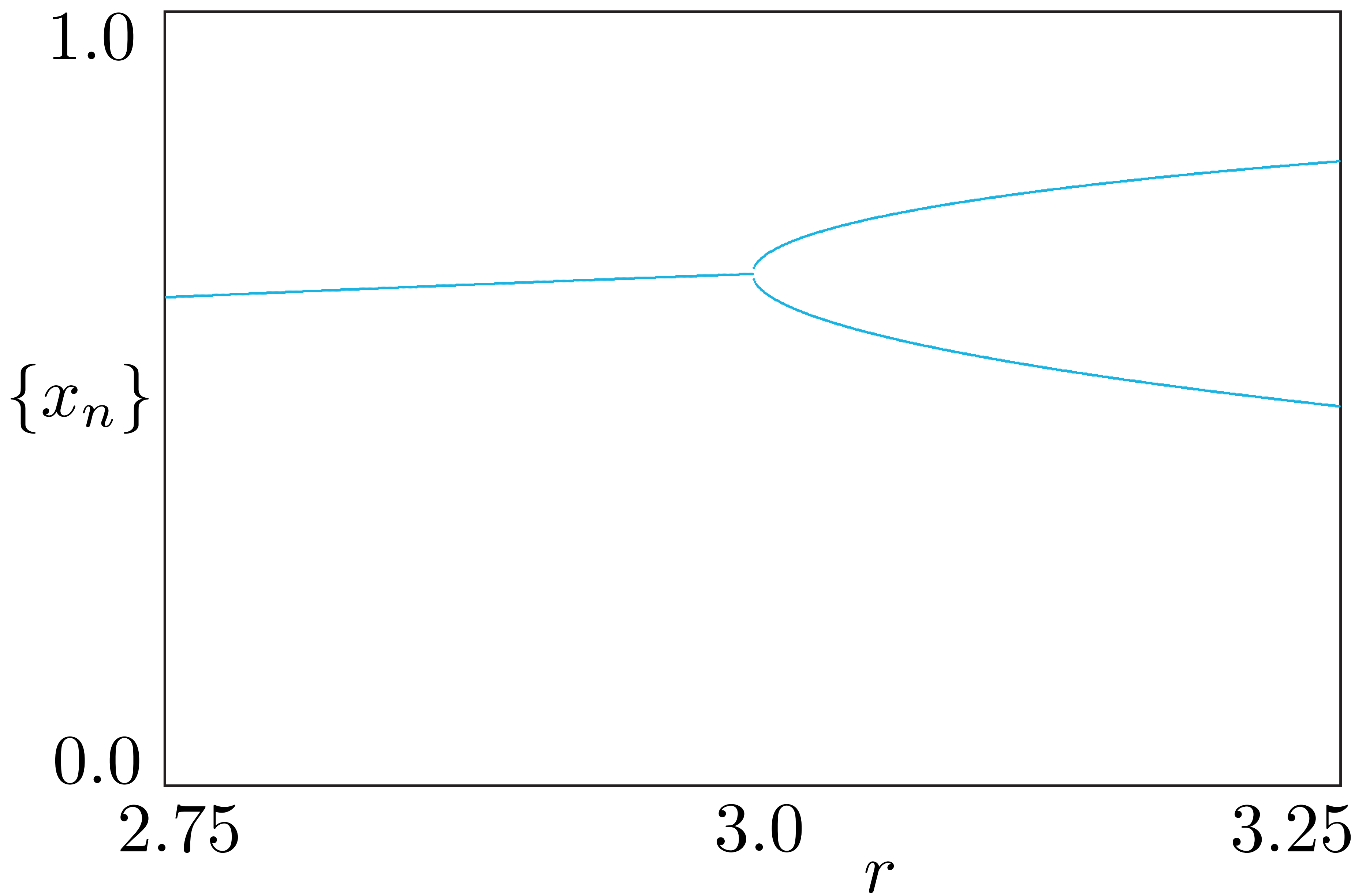


The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

Fixed point to period-2 limit cycle



Did period-1 fixed point disappear?

The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

At what bifurcation parameter value is P-2 orbit stable?

P-2 orbit: $\{x_1^*, x_2^*\}$

$$x_1^* = f(x_2^*) = f \circ f(x_1^*)$$

Fixed points: $x_1^* = f^2(x_1^*)$ and $x_2^* = f^2(x_2^*)$

Calculate: $x^* = rf(x^*)(1 - f(x^*))$

$$x^* = r^2 x^*(1 - x^*)(1 - rx^*(1 - x^*))$$

Find parameter such that this quartic equation has solutions!

The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

Simpler: When does P-1 go unstable?

Nontrivial P-1: $x^* = 1 - r^{-1}$

Slope: $f'(x) = r(1 - 2x)$

Slope at fixed point: $f'(x^*) = 2 - r$

Marginally stable: $|f'(x^*)| = 1$

$$|2 - r| = 1$$

First, P-1 to P-1 Bifurcation: $r = 1$

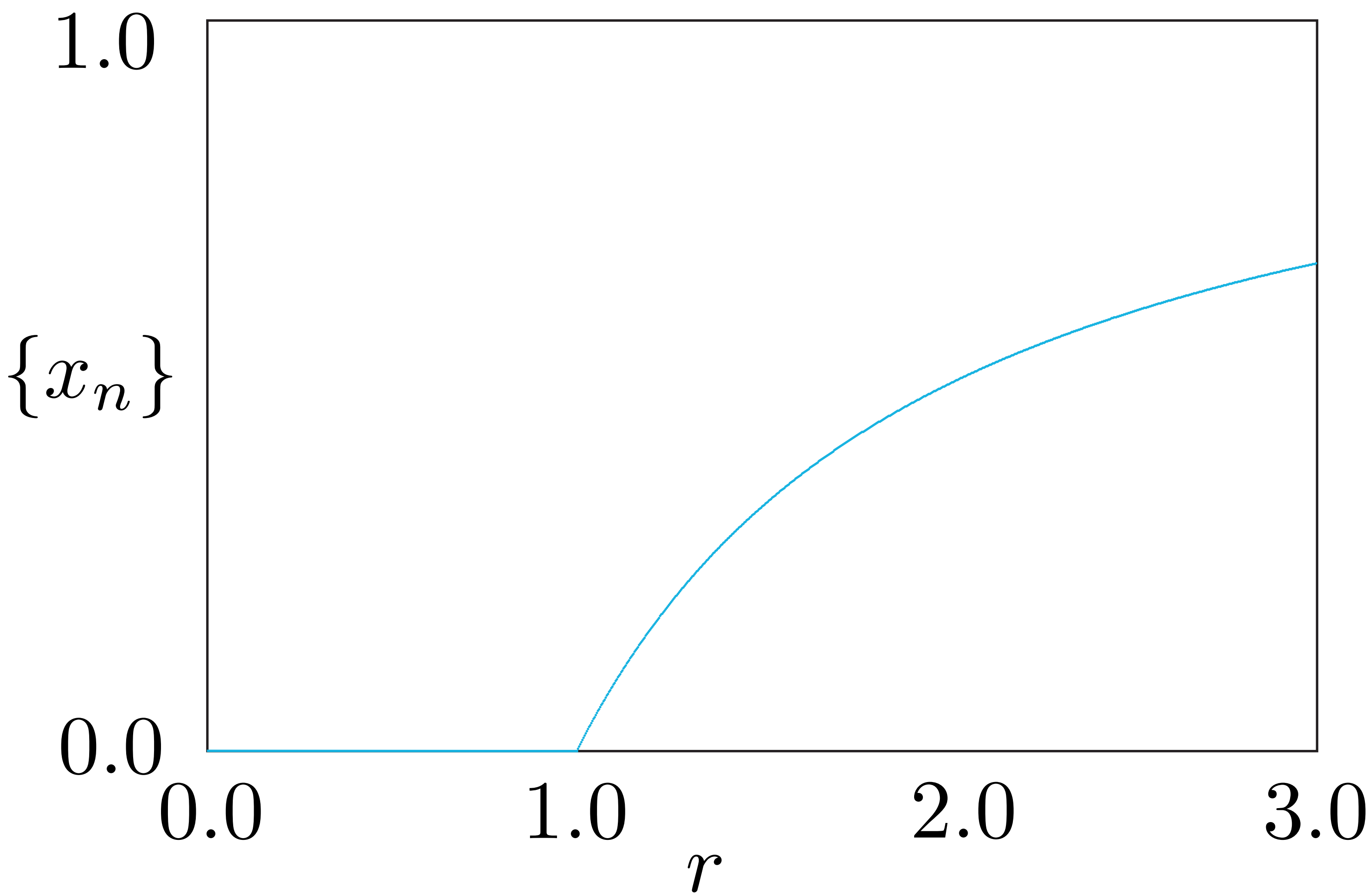
P-1 to P-2 Bifurcation: $r = 3$

The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

P-1 fixed point goes unstable: r = 1

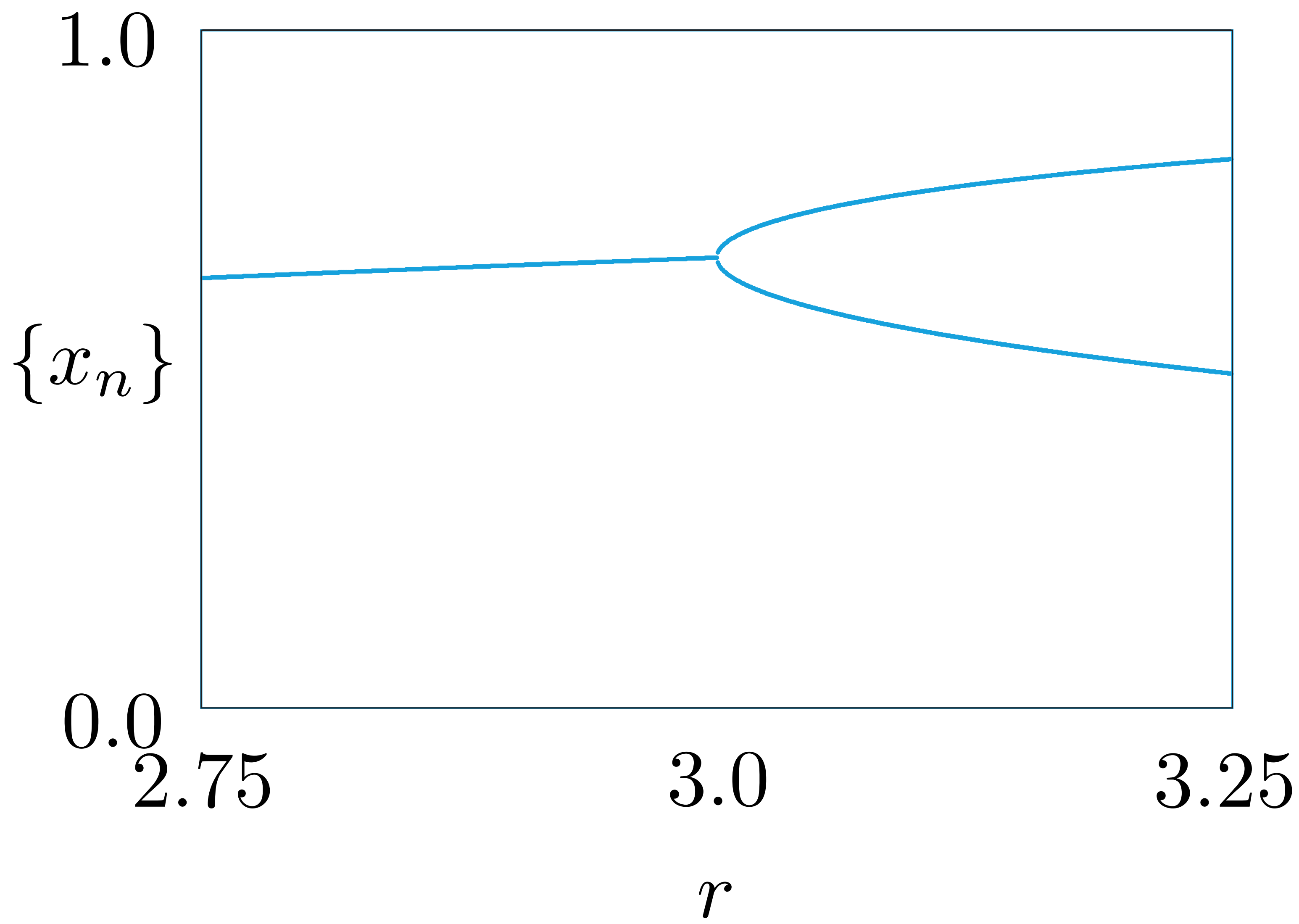


The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

P-2 limit cycle goes unstable: r = 3

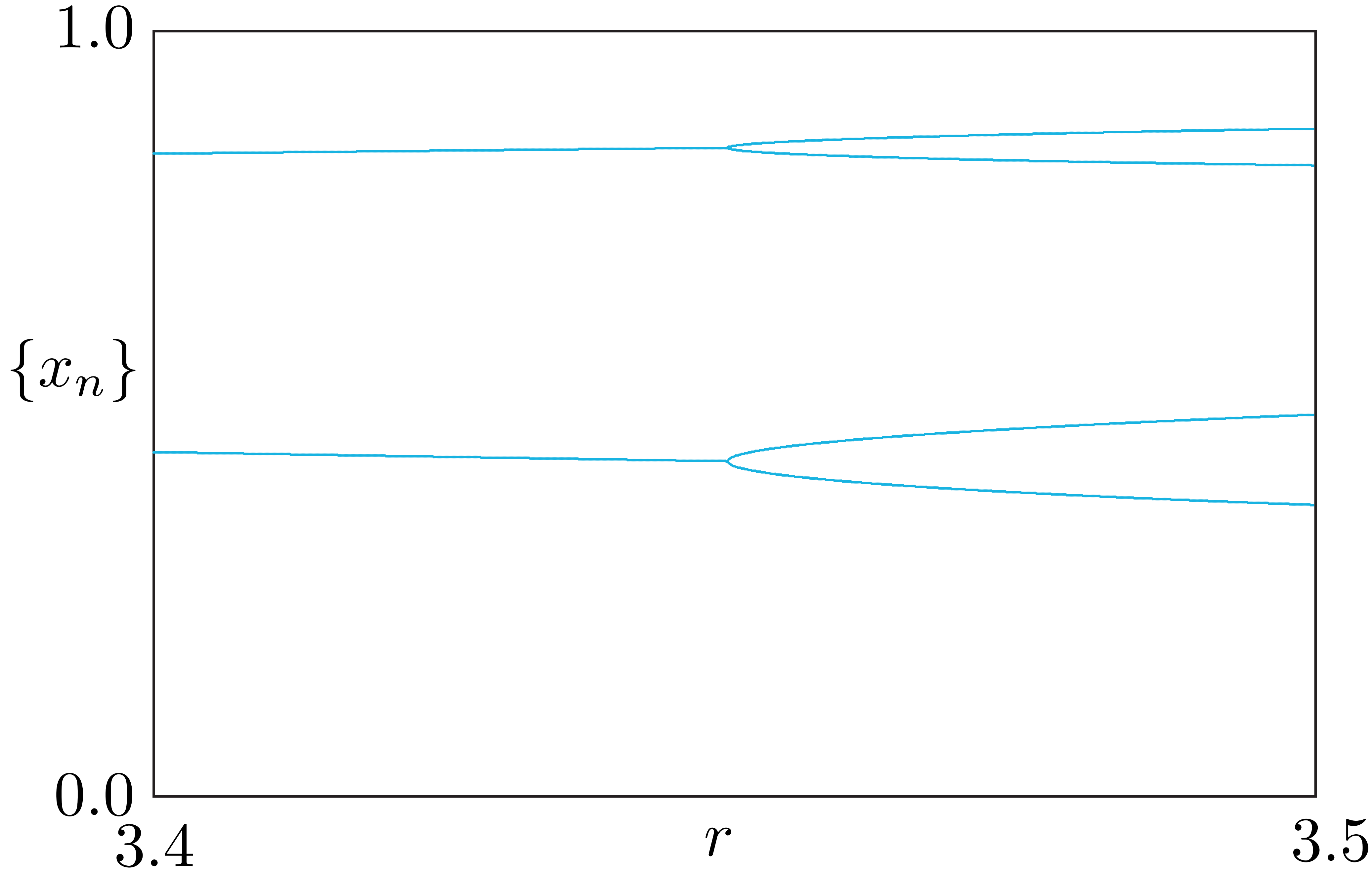


The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

Limit cycle to limit cycle: Period-2 to Period-4



The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

What parameter value P-2 to P-4 bifurcation?

Way too messy ... solve numerically:

Period-p limit cycle: $x_1 \to x_2 \to \cdots x_p \to x_1$

Criteria:

Fixed point of p-iterate: $x_i = f^p(x_i), \ i = 1, \ldots, p$

Onset of instability: $\left| \frac{d}{dx} f^p(x) \right| = 1$

Stability along the orbit: $|f'(x_1) f'(x_2) \cdots f'(x_p)| = 1$

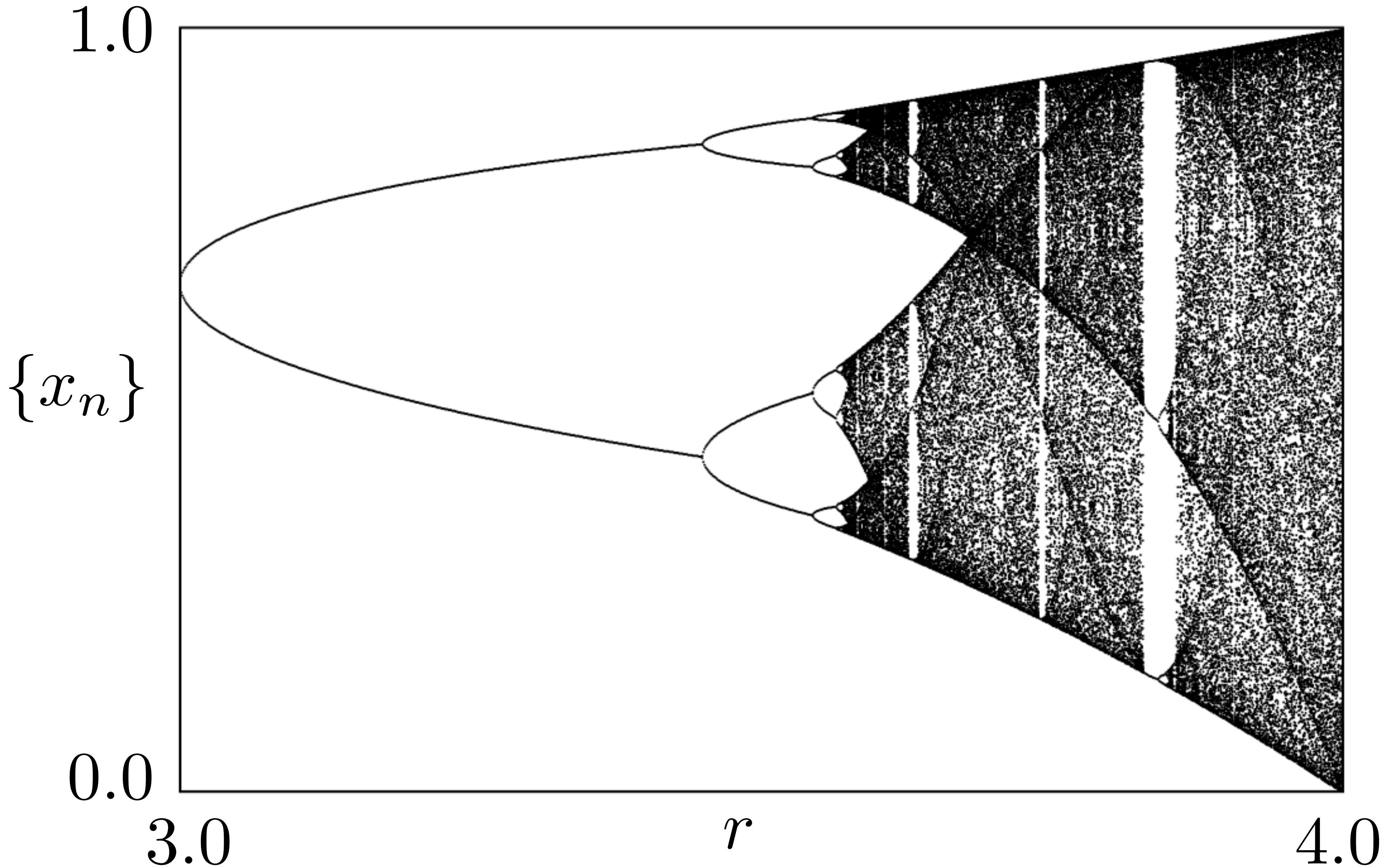
Numerically: Search in r to match this

The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

Route to chaos via period-doubling cascade

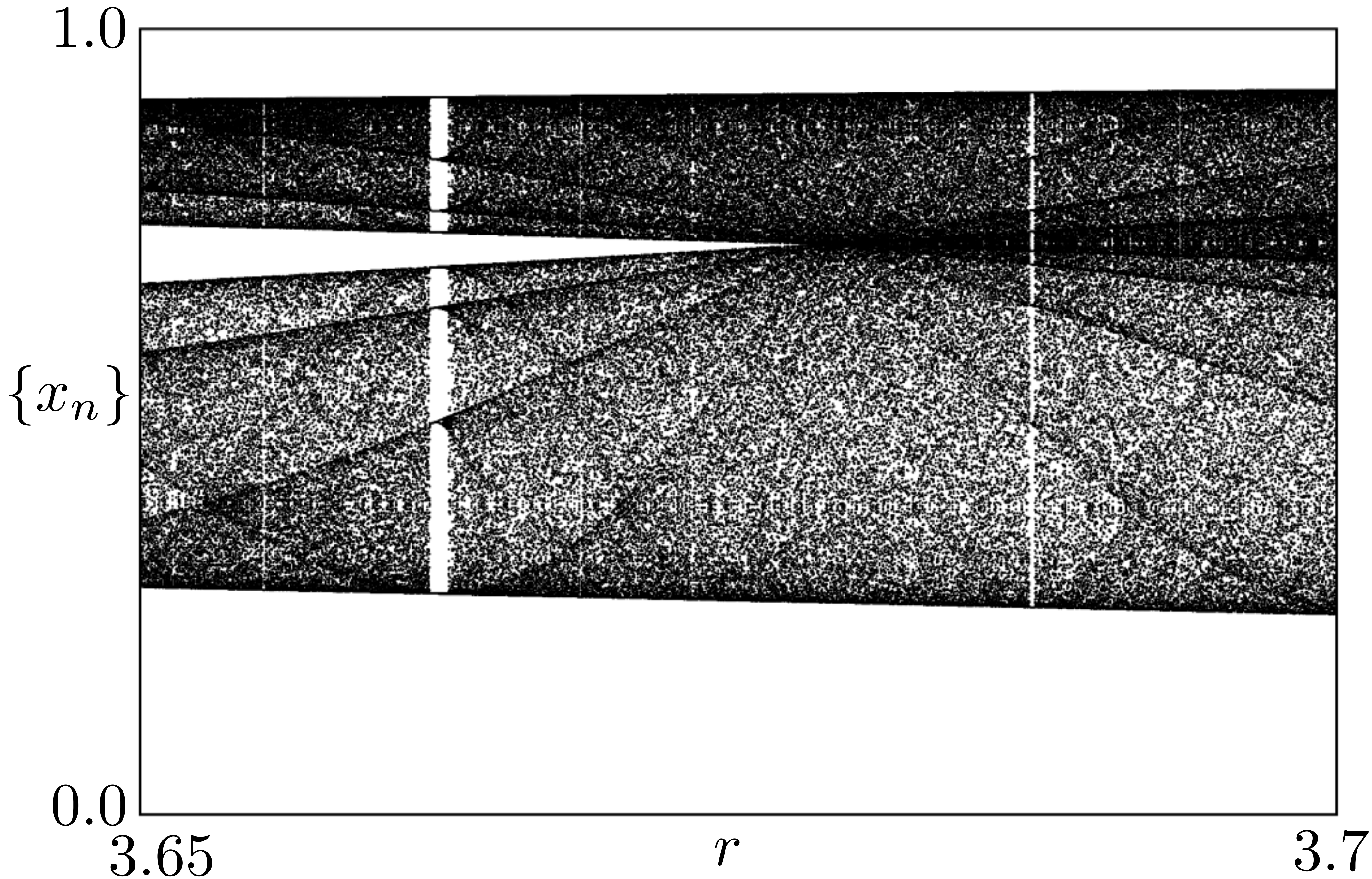


The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

Band-merging (mirror of period-doubling)



The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

What parameter values for band-merging?

Veils: Iterates $f^n(x_c)$ of map maximum $x_c = 1/2$

Upper bound on attractor: $f(x_c)$

Lower bound on attractor: $f^2(x_c)$

Two bands merge to one band: $f^k(x_c)$ becomes P-1

Specifically: $f^3(x_c) = f^4(x_c)$

Solve numerically: $r_{2B \rightarrow 1B} = 3.678\ldots$

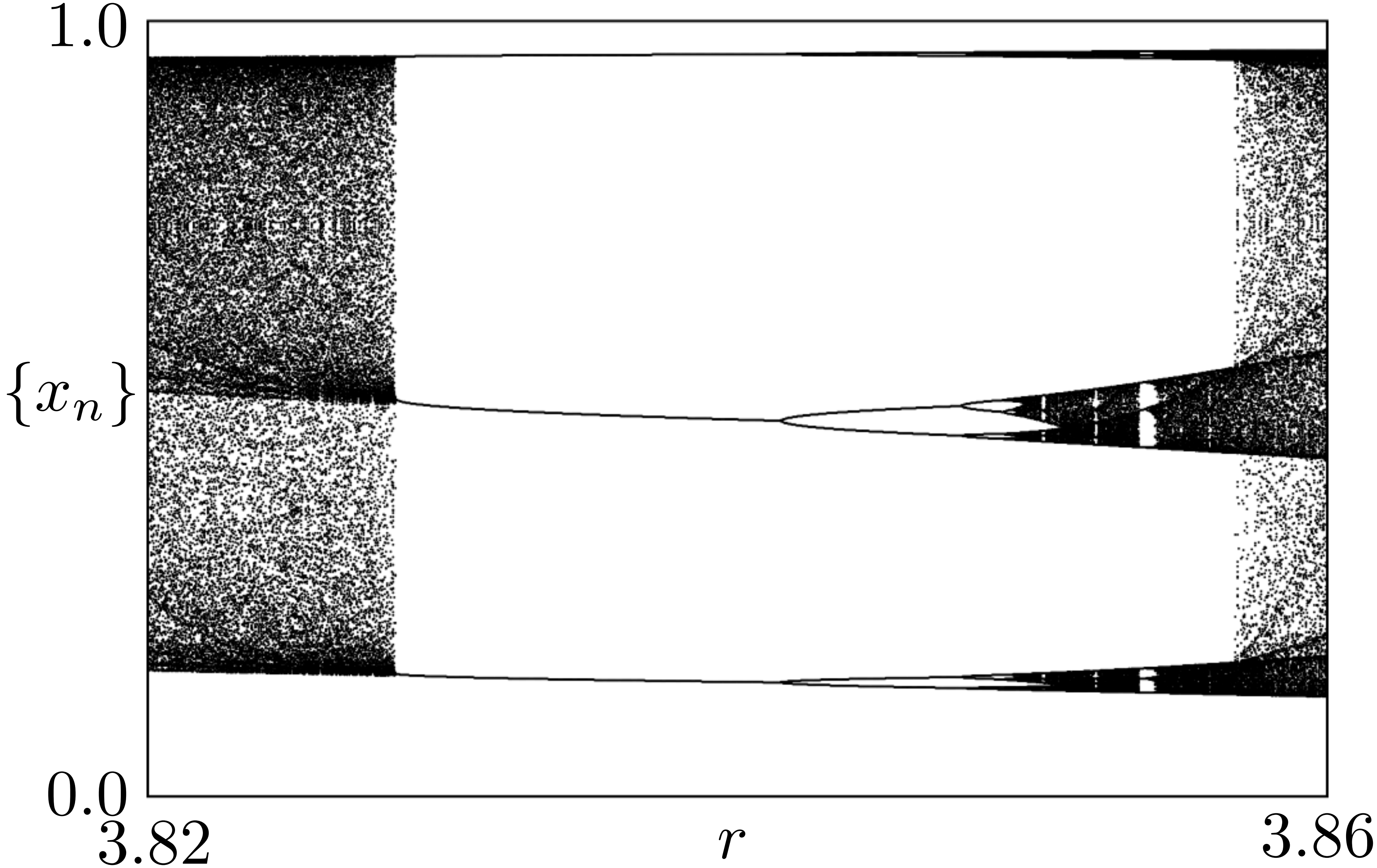
Generally: 2^n bands merge to 2^{n-1} bands: $f^k(x_c)$ is period 2^{n-1}

The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

Periodic windows



Entire period-doubling cascade inside window: $P = 3 \times 2^n$

The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

Periodic windows ...

How to locate:

Superstable periodic orbits: $x_i = x_c = \frac{1}{2}$

Why "superstable"? $f'(x_c) = 0$

Period-3: $f^3(x_c) = x_c$

Solve numerically: $r_{P-3} = 3.83\ldots$

The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

Simulation demos:

Animation as a function of parameter (ds)

(2D; nSteps = 10; p0; r in [0.95,4]; nSteps = 8000; nTrans = 4000; nIts = 1000; colors = 0)

Bifurcation diagrams (bifn1d)

(see usage)

The Big, Big Picture (Bifurcations II) ...

Next:

- Chaotic mechanisms
- Quantify the degree of chaos and unpredictability

Today:

- A preview: Sounds of chaos
  - Rössler and Lorenz chaotic attractors

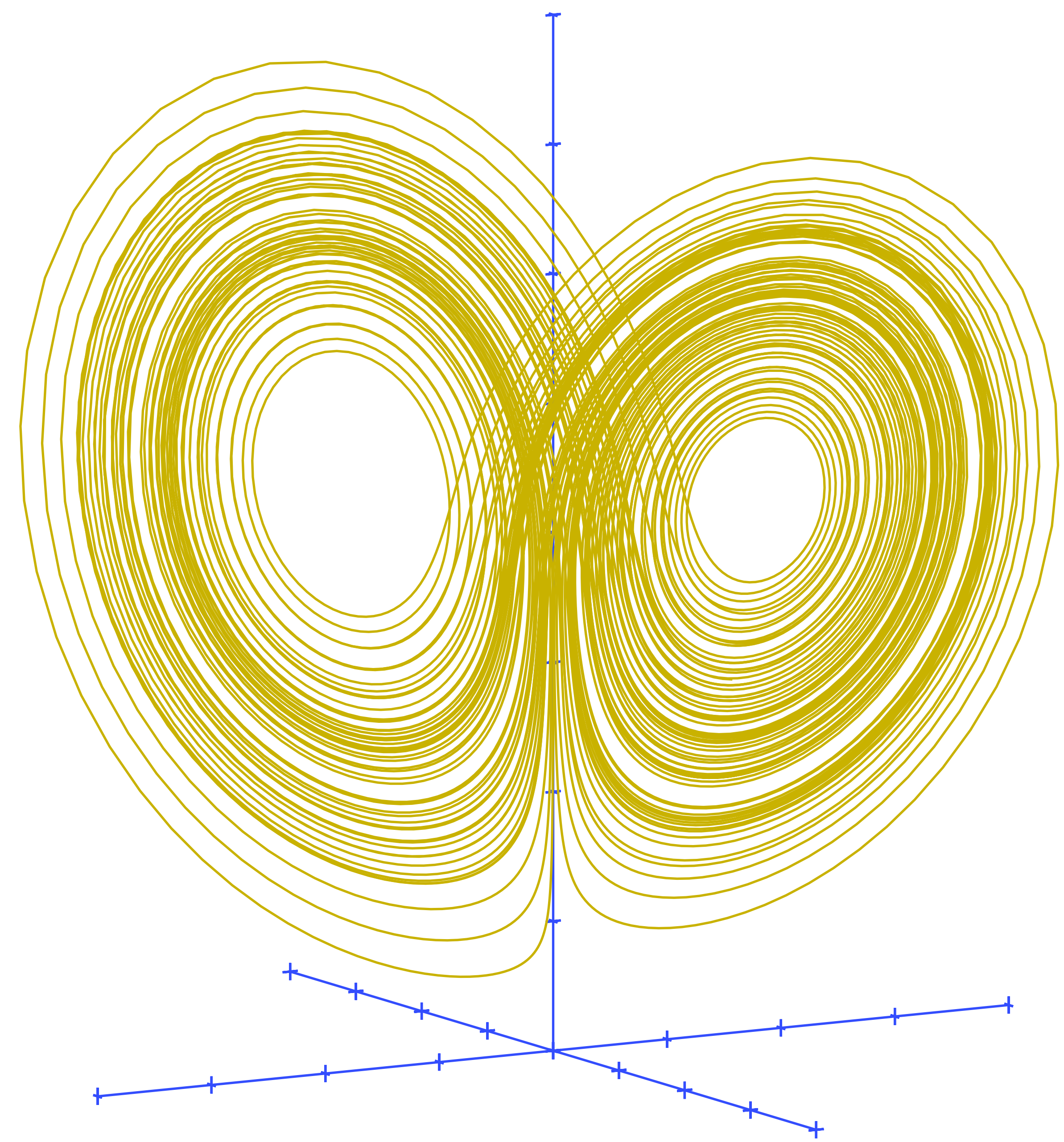
~/Programming/Audio/SoC

The Big, Big Picture (Bifurcations II) ...

Lorenz chaotic attractor ...

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\end{aligned}$$

$$(\sigma, r, b) = (10,28,8/3)$$

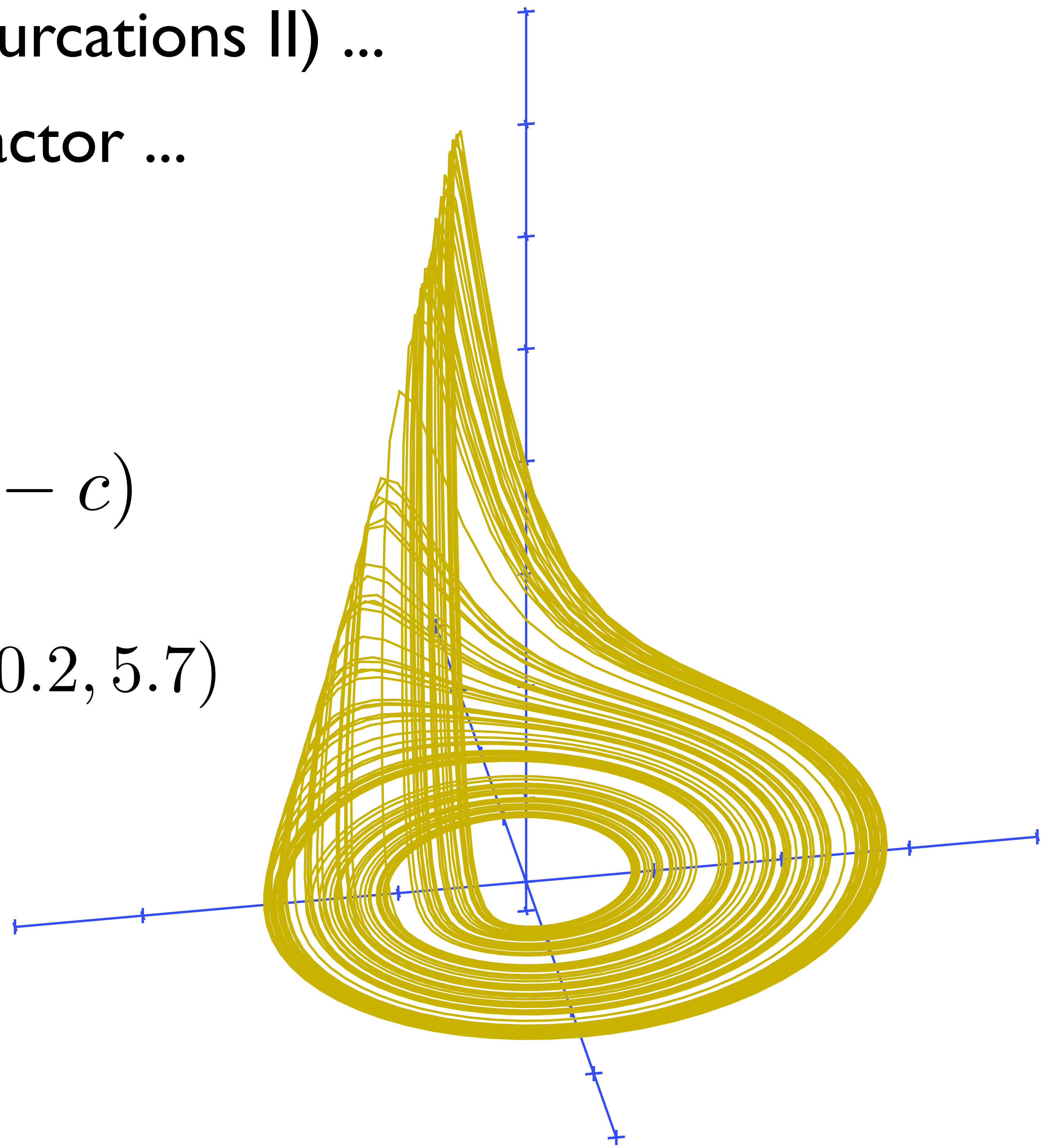


The Big, Big Picture (Bifurcations II) ...

Roessler chaotic attractor ...

$$
\begin{aligned}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
\end{aligned}
$$

$$(a, b, c) = (0.2, 0.2, 5.7)$$



The Big, Big Picture (Bifurcations II) ...

Reading for next lecture:

NDAC, Sec. 12.0-12.3, 9.3, and 10.5.