STRUCTURE IN QUANTUM REPRESENTATIONS OF PROCESSES

John Mahoney, Cina Aghamohammadi , James Crutchfield

UC DAVIS
UNIVERSITY OF CALIFORNIA

John Templeton Foundation

WHAT’S THE BIG IDEA?

- What is “structure”? - illustrate for discrete processes.
- Does the same process in a quantum “substrate” have different structure?
- Connection to the “cryptic order”
- Advantages / tradeoffs

STATIONARY STOCHASTIC PROCESSES

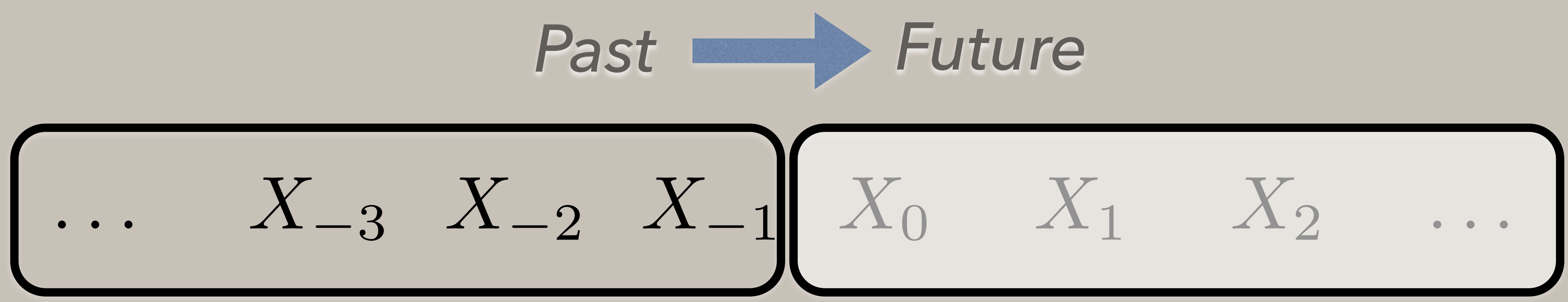
$$\ldots \quad X_{-2} \quad X_{-1} \quad X_0 \quad X_1 \quad X_2 \quad \ldots$$

Symbols from discrete alphabet $x \in \mathcal{A}$

Stationary

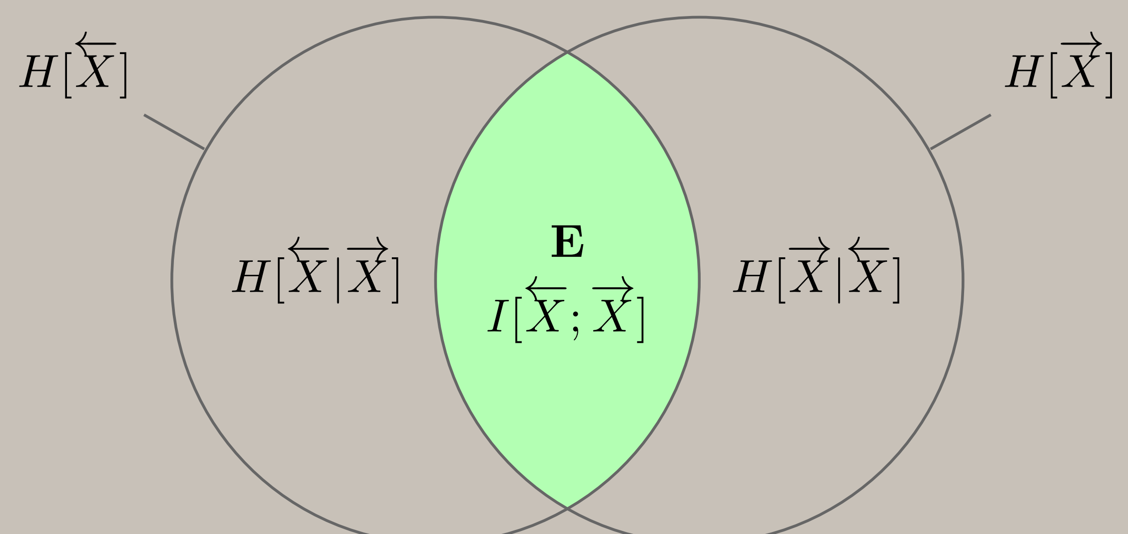
$$Pr(X_t, ..X_{t+L-1}) = Pr(X_0, ..X_{L-1})$$

PREDICTING THE FUTURE



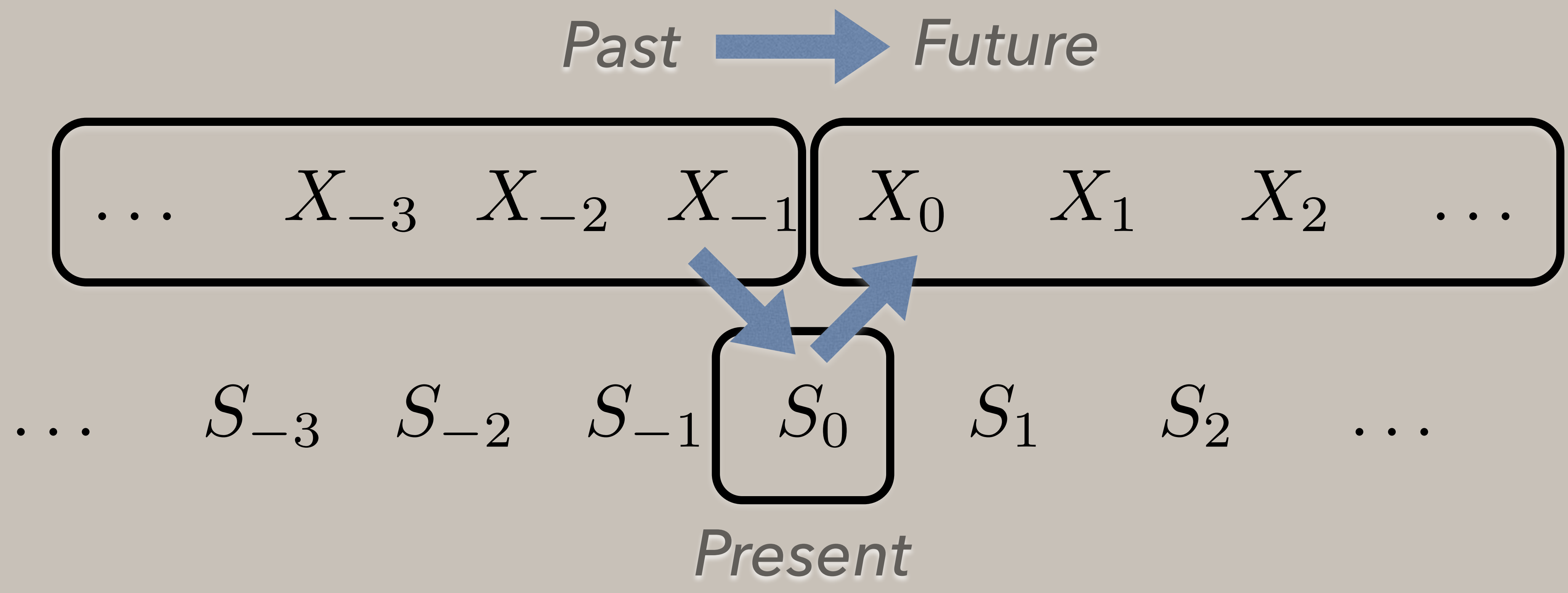
Excess entropy = predictive potential

$$\mathbf{E} = I[\ldots, X_{-2}, X_{-1}; X_0, X_1, \ldots]$$

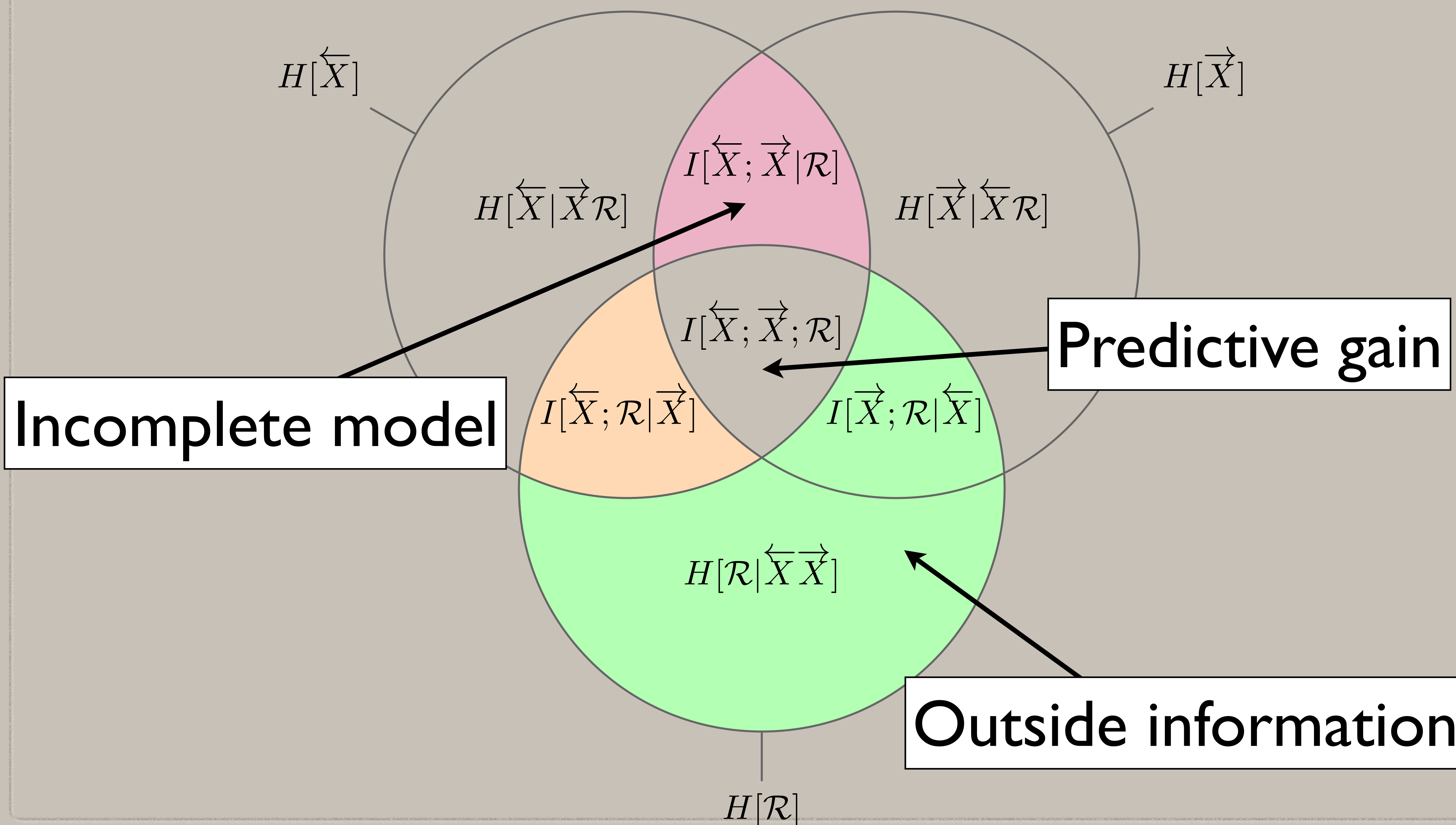


But is this a good measure of "structure"?

BUILDING MODELS



PREDICTIVE MODELS



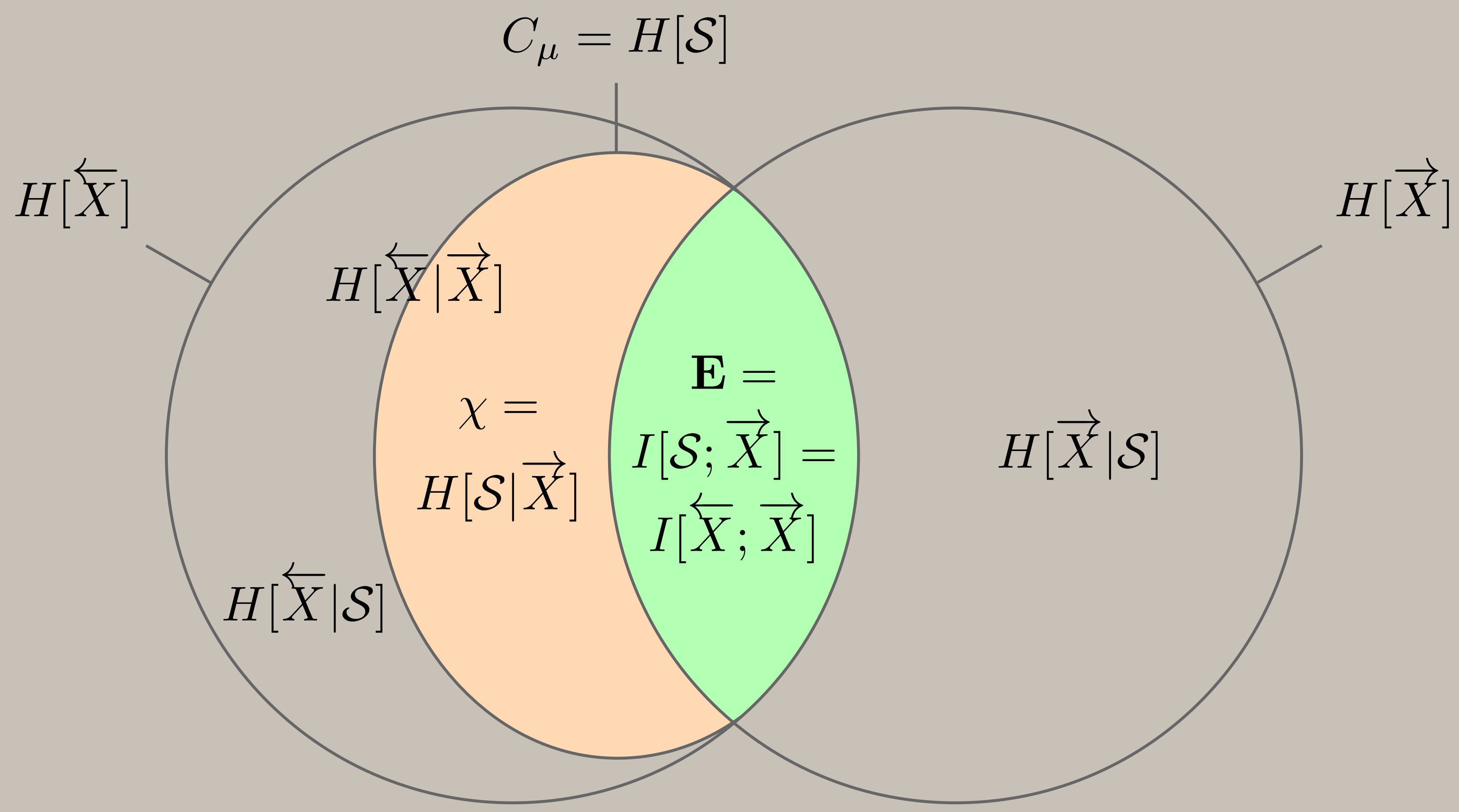
CAUSAL STATES

Causal states are equivalence classes of histories

$$\overleftarrow{x} \sim \overleftarrow{x}' \equiv Pr(\overrightarrow{X}|\overleftarrow{x}) = Pr(\overrightarrow{X}|\overleftarrow{x}')$$

"Distinguish only between pasts that distinguish themselves."

ϵ-MACHINE I-DIAGRAM



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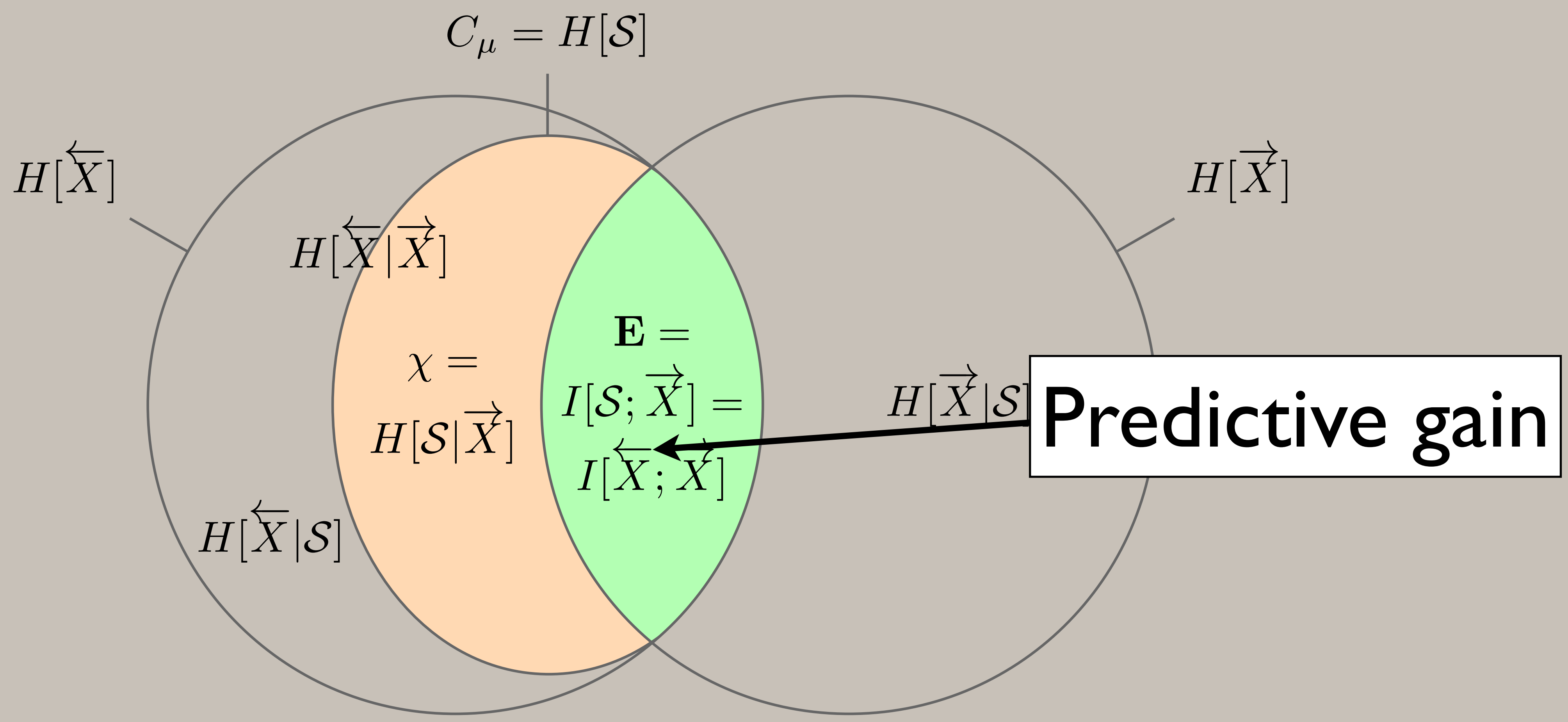

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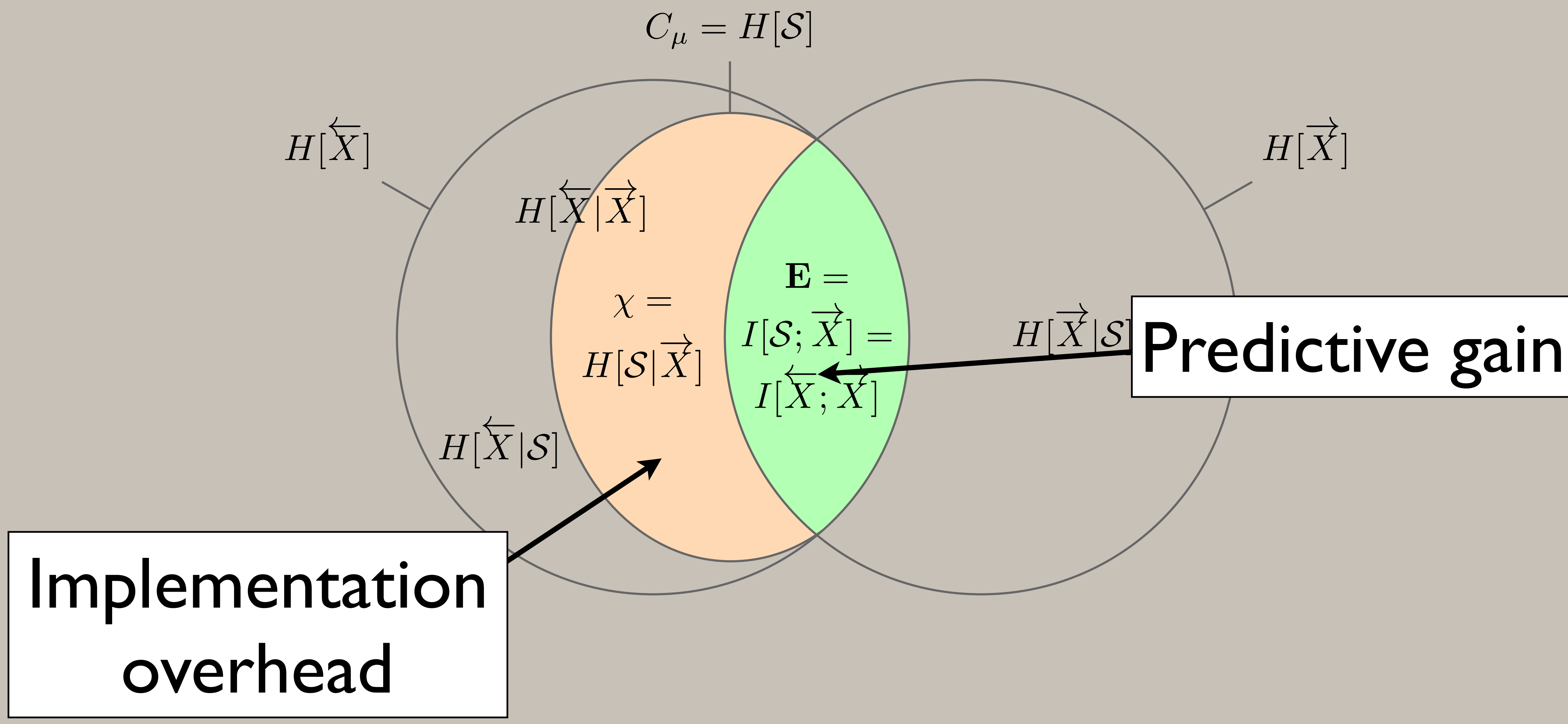

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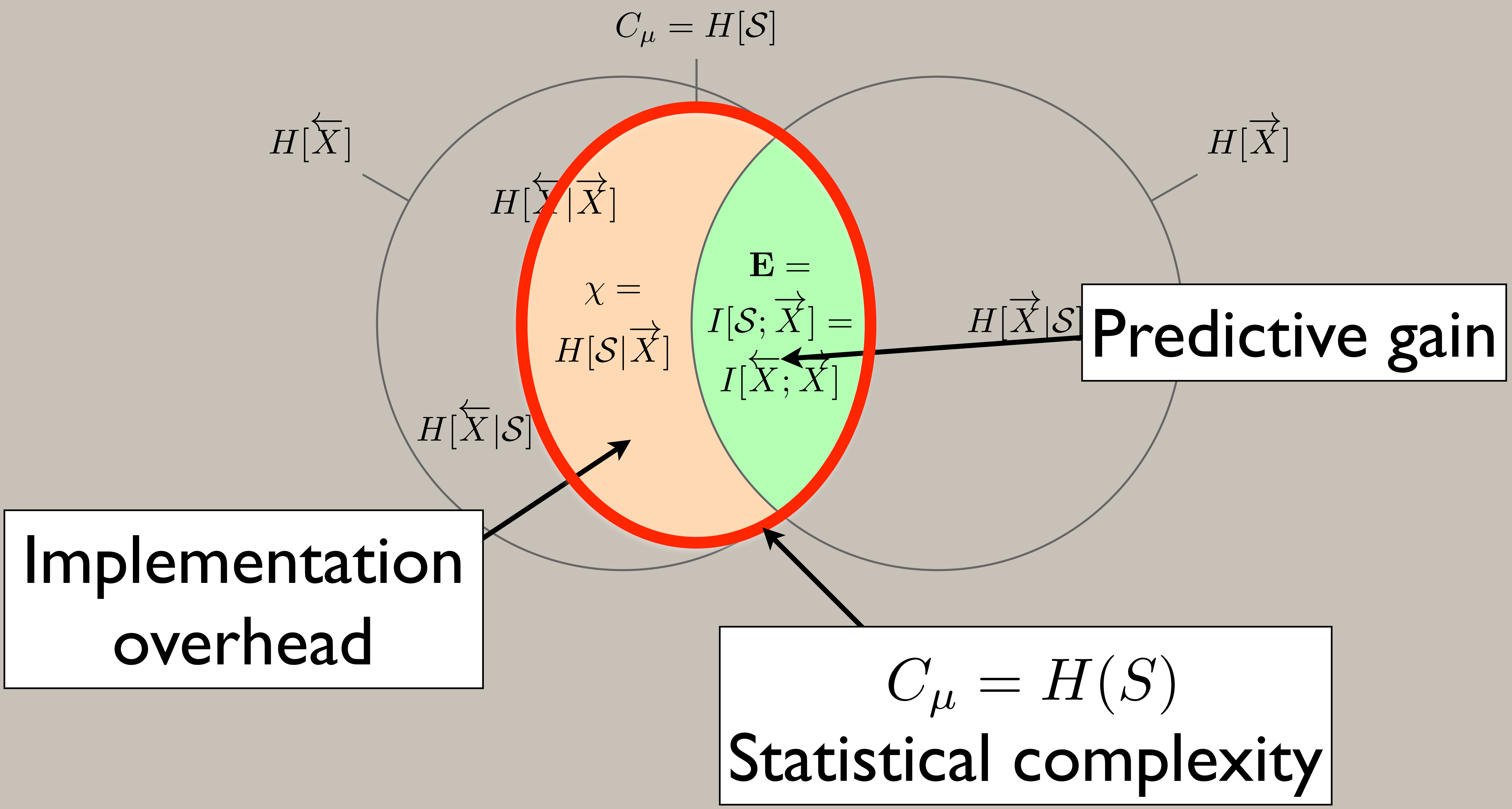


THE EPSILON-MACHINE

- Equivalence relation defines causal state
- Unifilar
- Leads to natural computation of entropy rate, etc
- Canonical representation

ϵ-MACHINE: EXAMPLES

Biased Coin:

0101010001110011100000011011110101

Period 2:

101010101010101010101010101010101

Golden Mean:

110101011011010101010110111110111

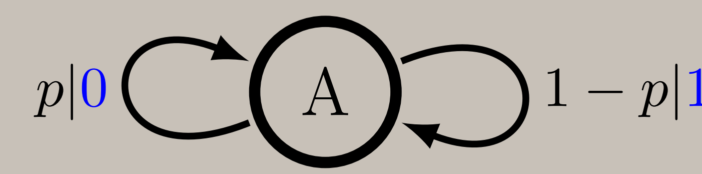
Even Process:

110011001111111111001111011111111

ϵ-MACHINE: EXAMPLES

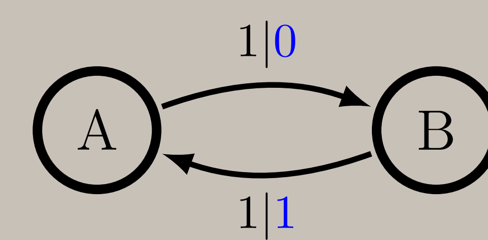
Biased Coin:

010101000111001110000011011110101



Period 2:

101010101010101010101010101010101



Golden Mean:

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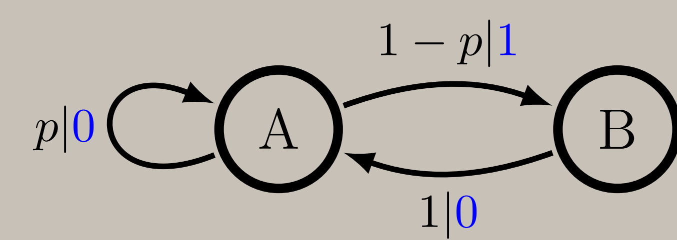

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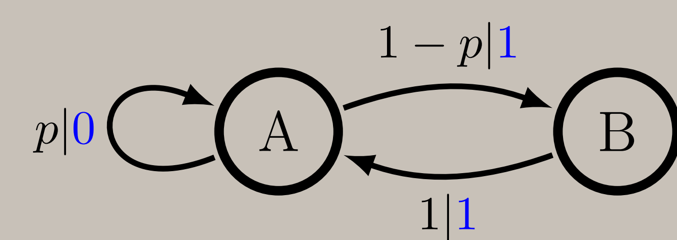

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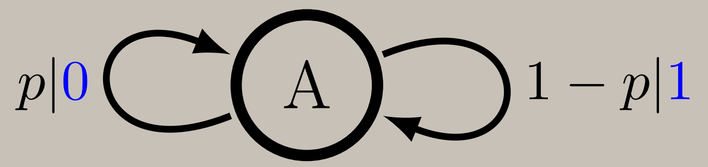
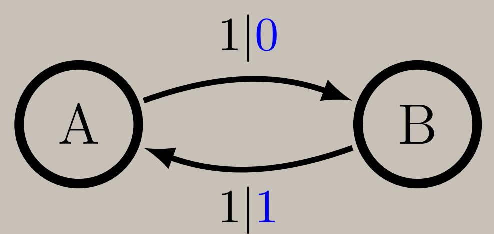


010101000111001110000011011110101

$$E = C_\mu = 0, R = 0$$

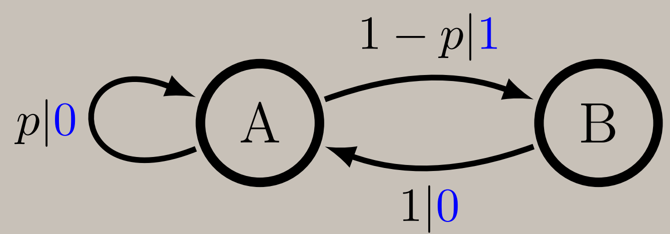
Period 2:



101010101010101010101010101010101

$$E = C_\mu = 1, R = 1$$

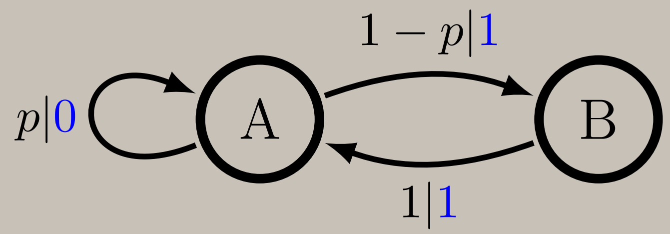
Golden Mean:



110101011011010101010110111110111

$$E = 0.252 < C_\mu = 0.918, R = 1$$

Even Process:



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$$E = C_\mu = 0.918, R = \infty$$

ϵ-MACHINE: EXAMPLES

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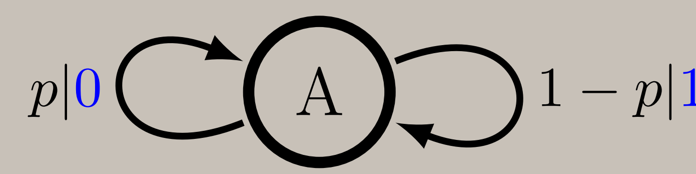
0101010001110011100000110111110101



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Period 2:

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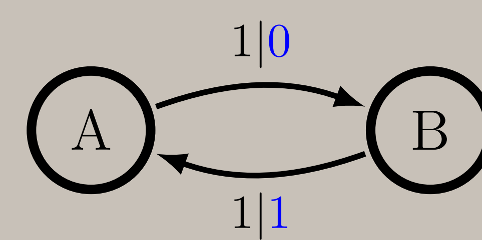

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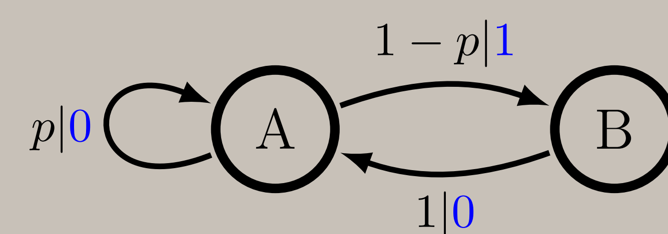

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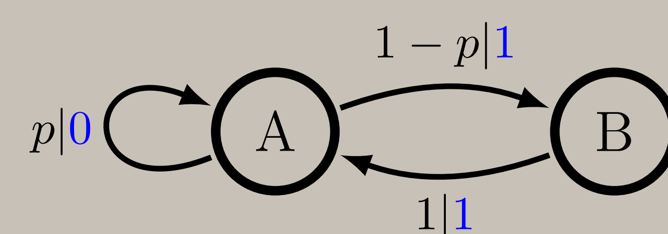

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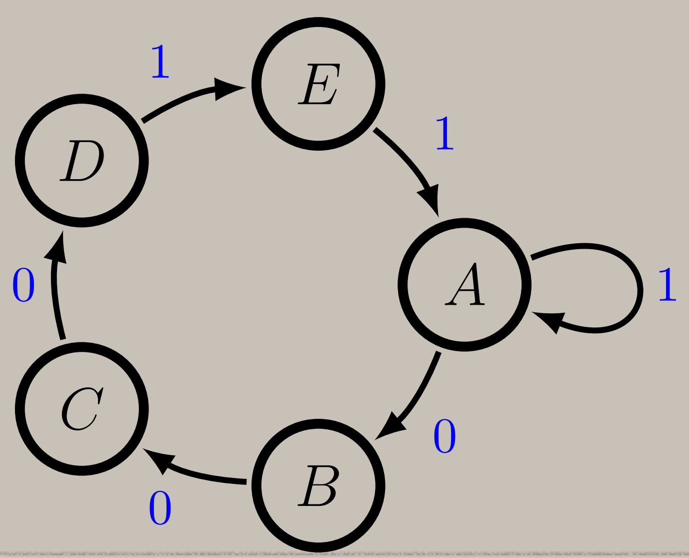
MARKOV ORDER



$$Pr(\overrightarrow{X}_0|\overleftarrow{X}_0) = Pr(\overrightarrow{X}_0|X_{-R}, \ldots, X_{-1})$$

$$R = min\{L \in \mathbb{Z}^+ : H[S_R|X_0^R] = 0\}$$

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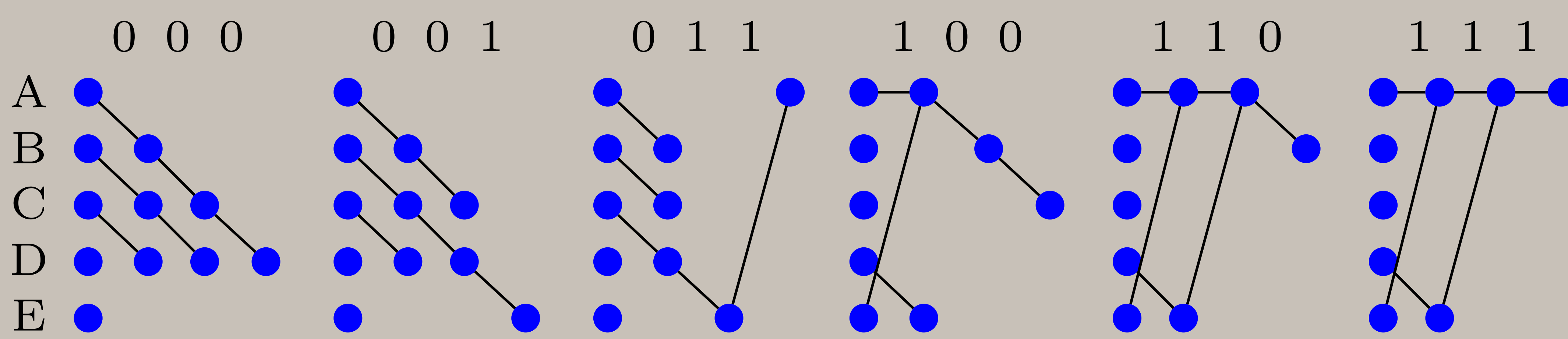

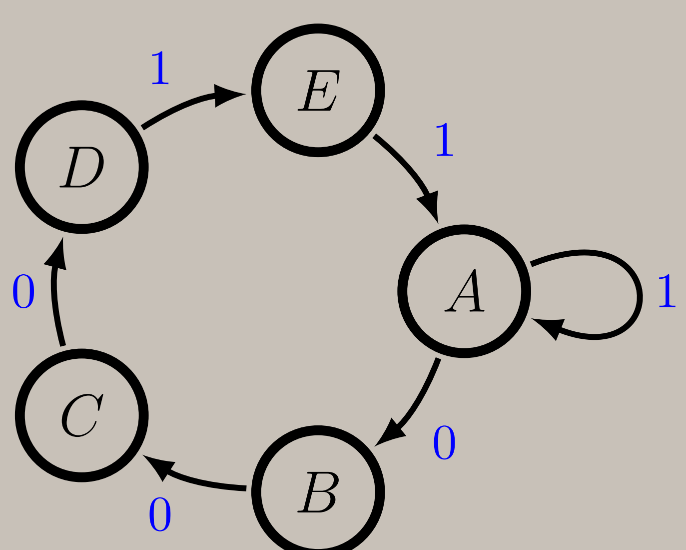

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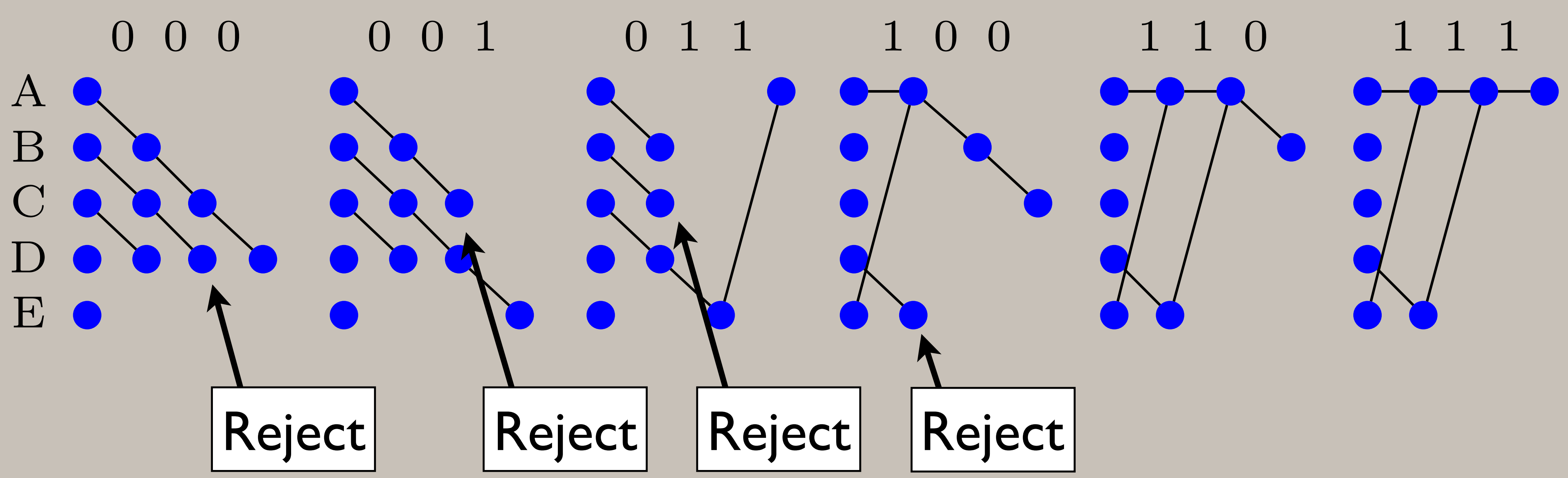

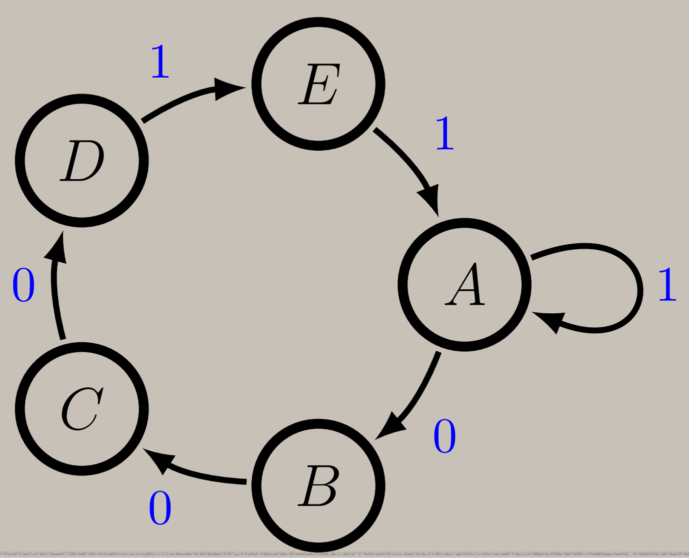

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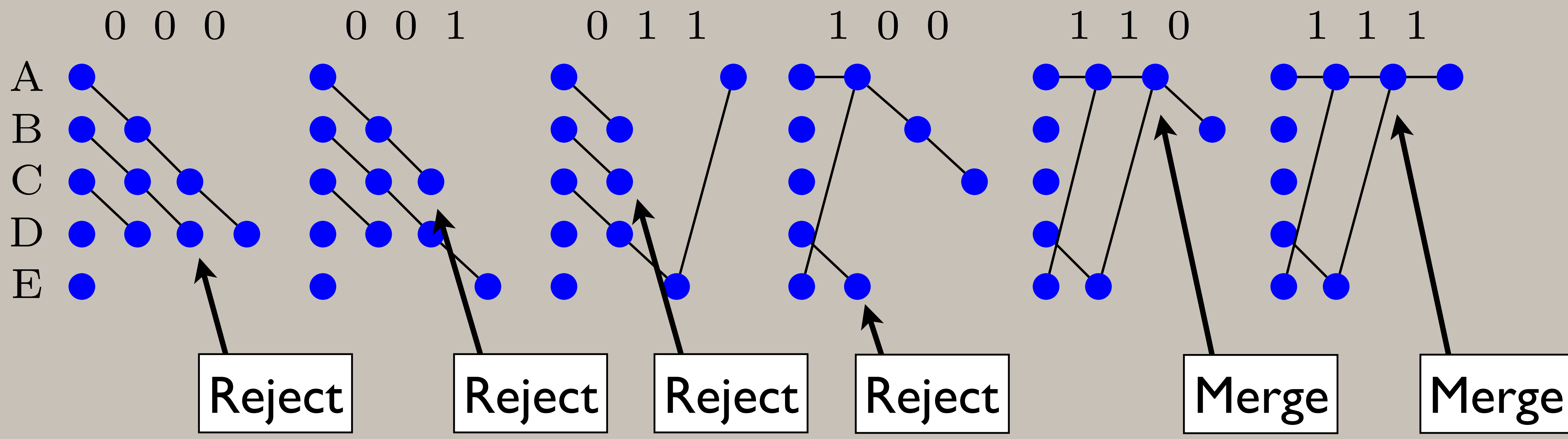

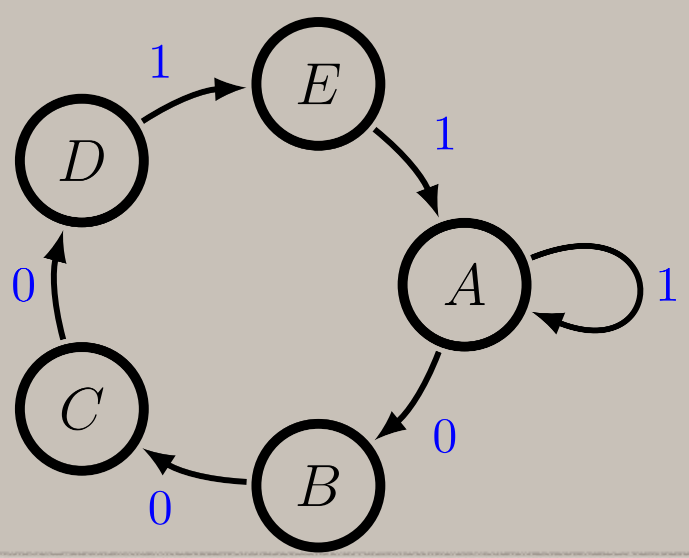

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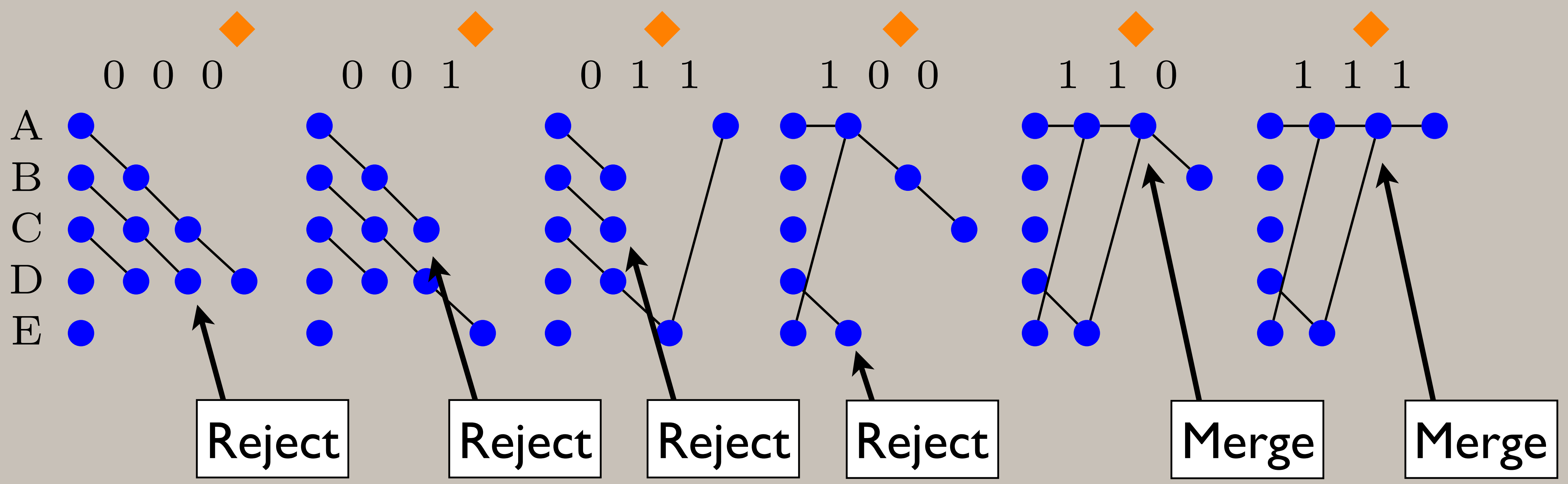

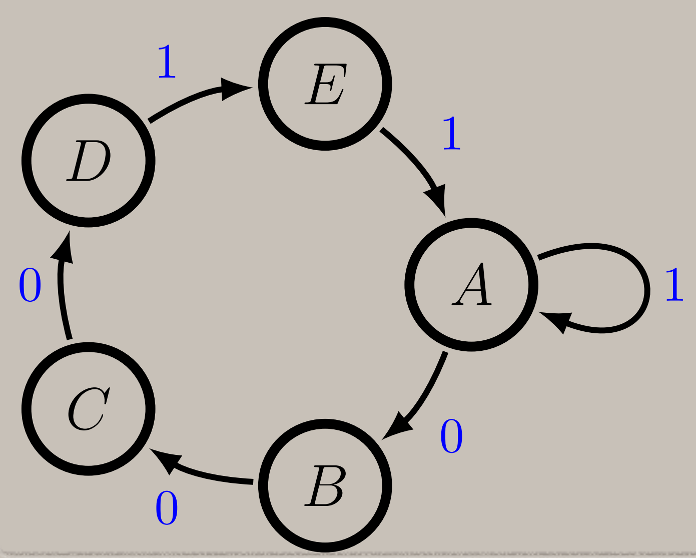

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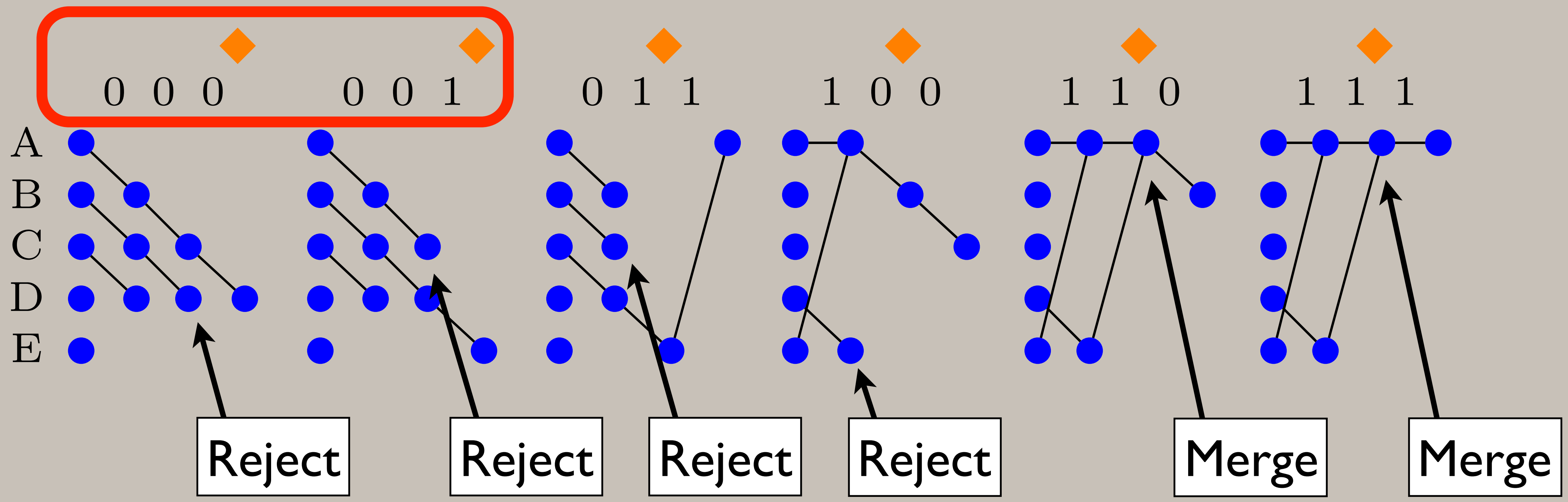

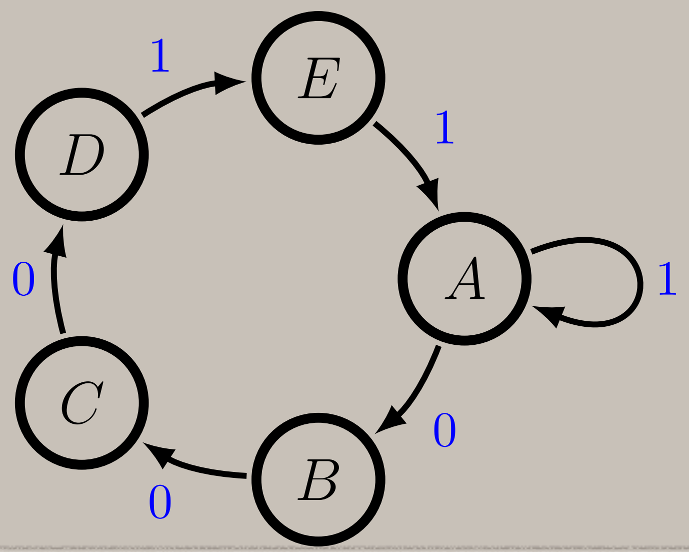

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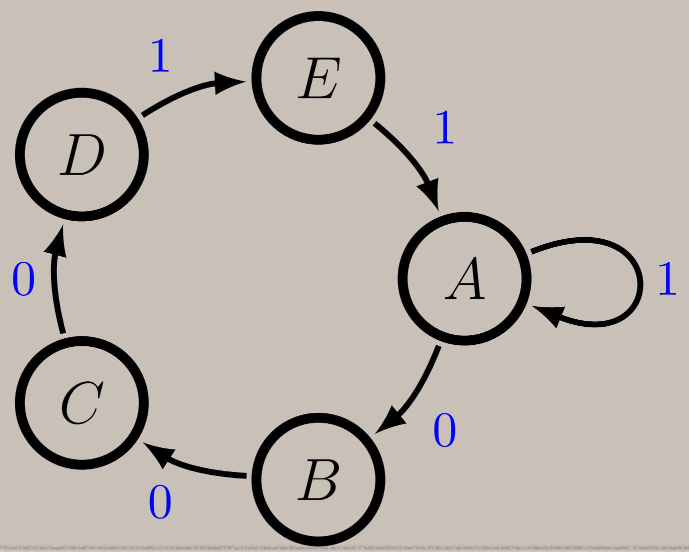
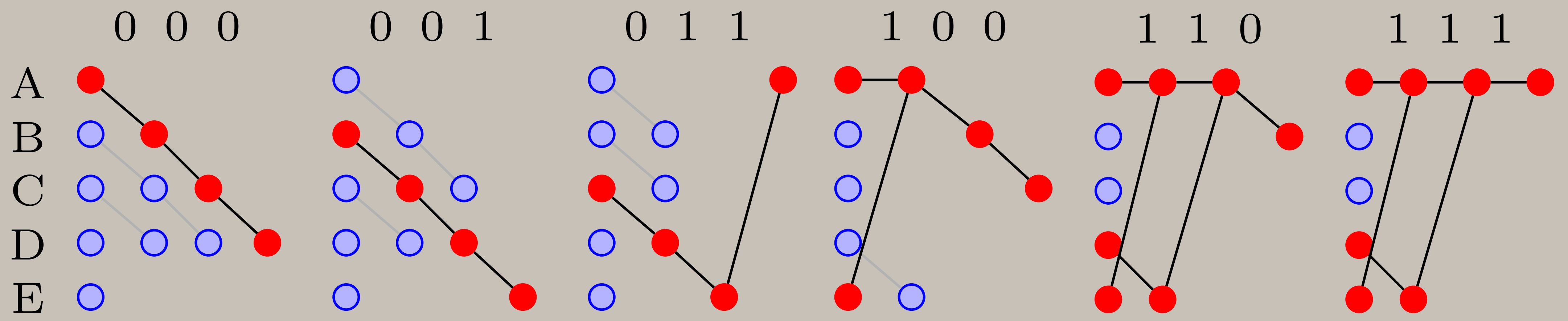
CRYPTIC ORDER

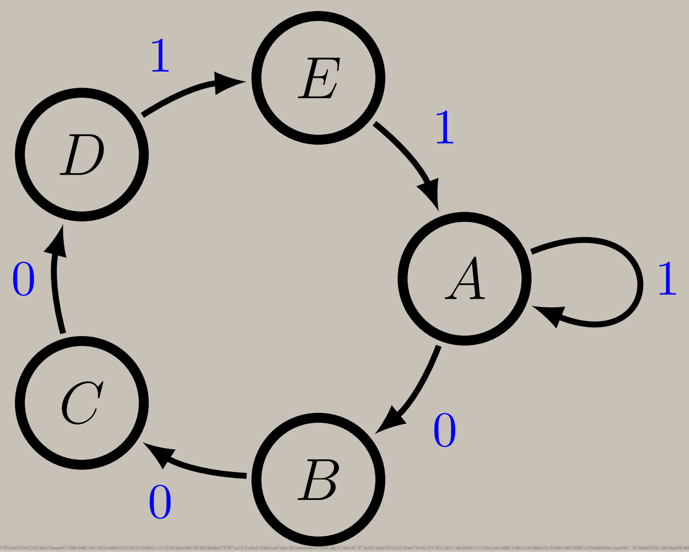


$$k = min\{L \in \mathbb{Z}^+ : H[S_L|\overrightarrow{X}_0] = 0\}$$

Conditioning on future ensures a complete path.

CRYPTIC ORDER





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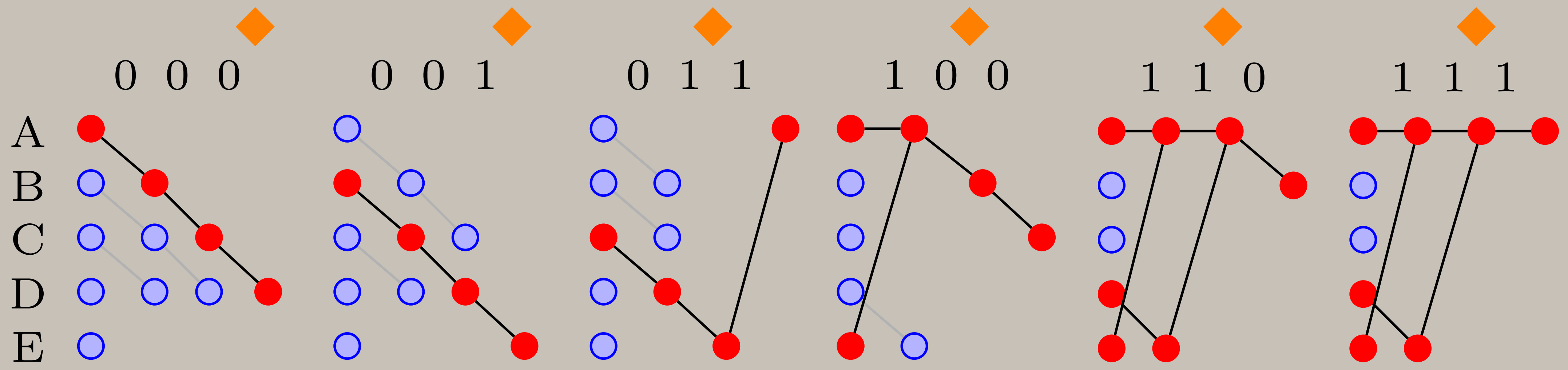

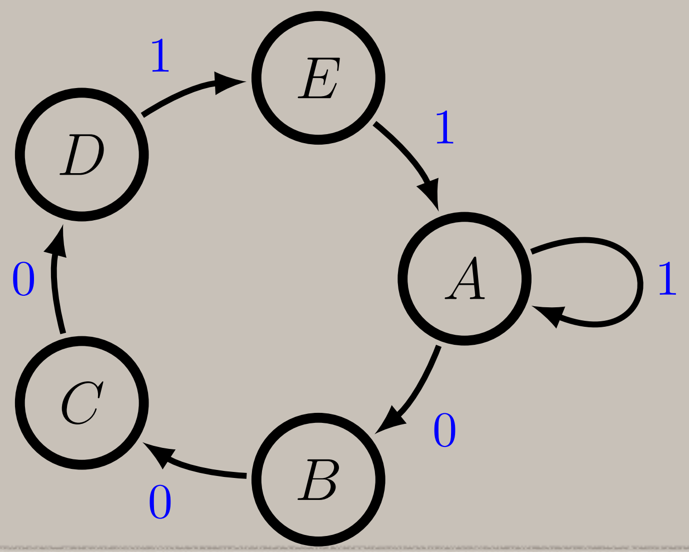

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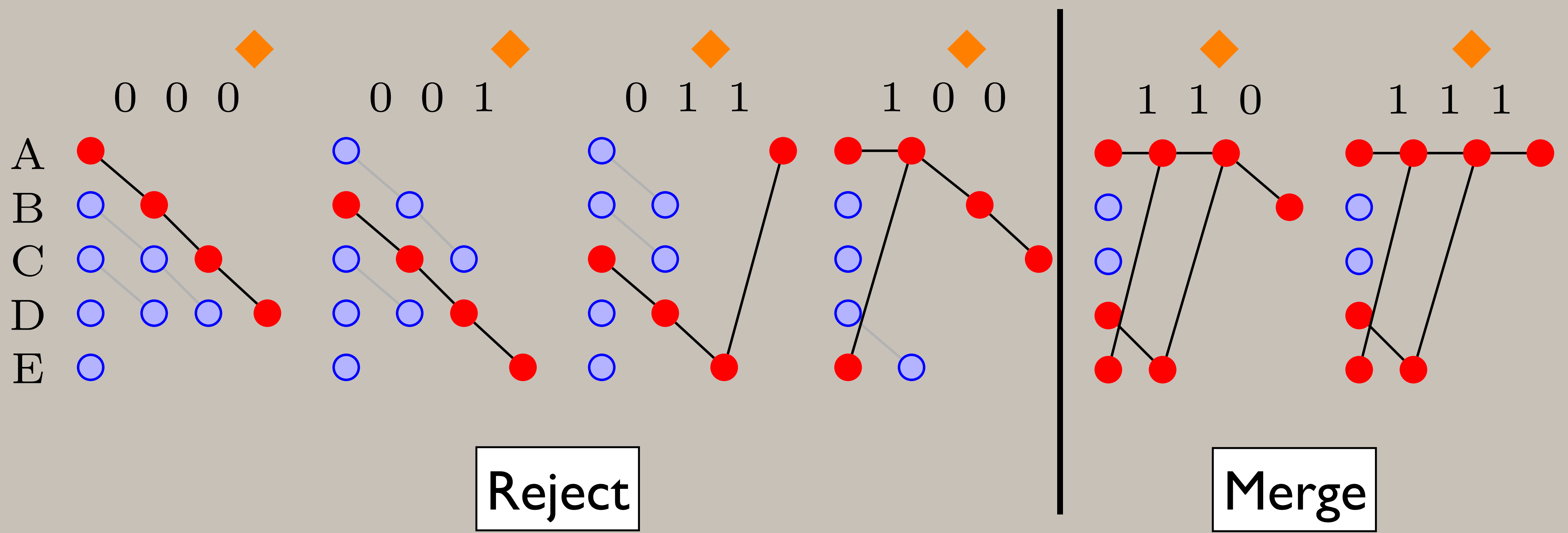

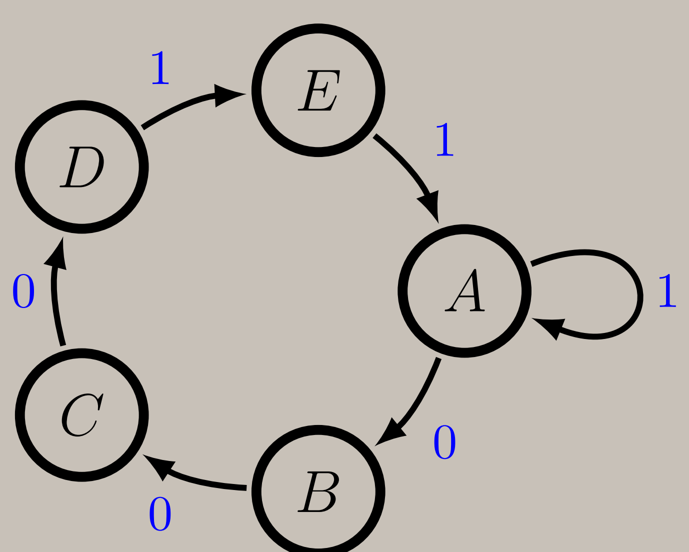

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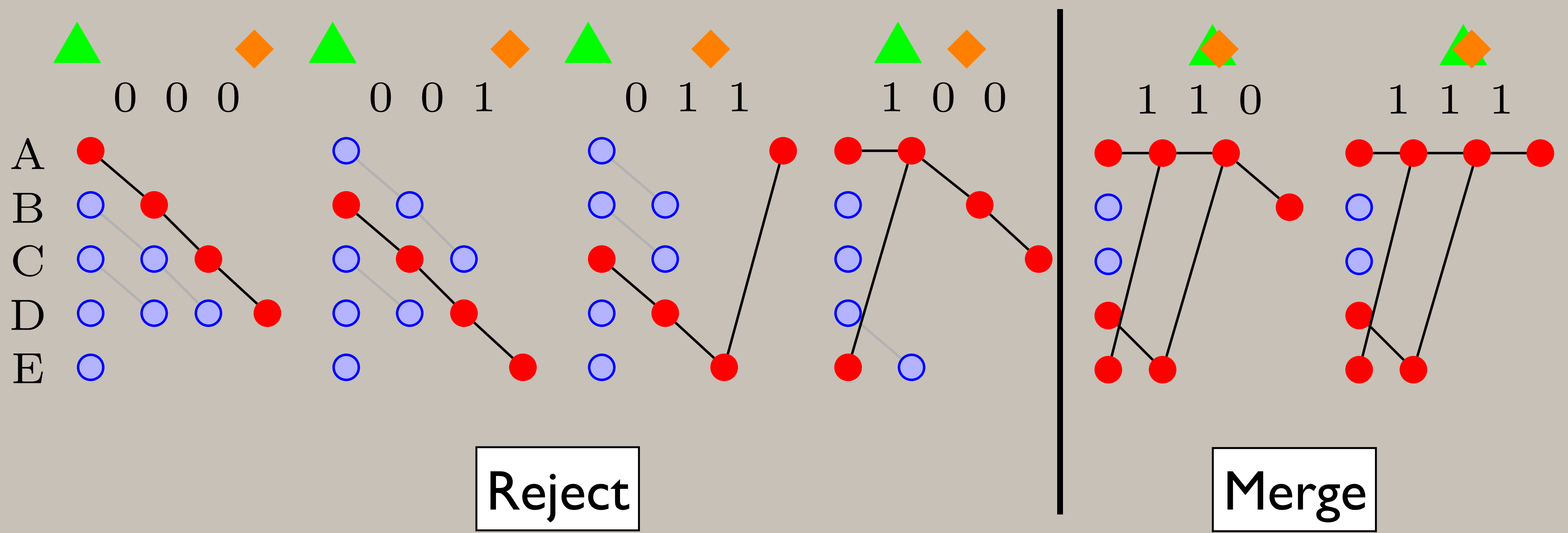

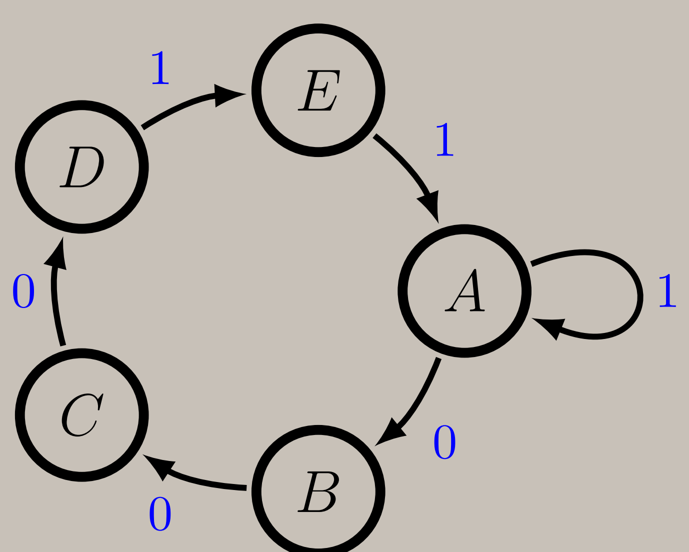

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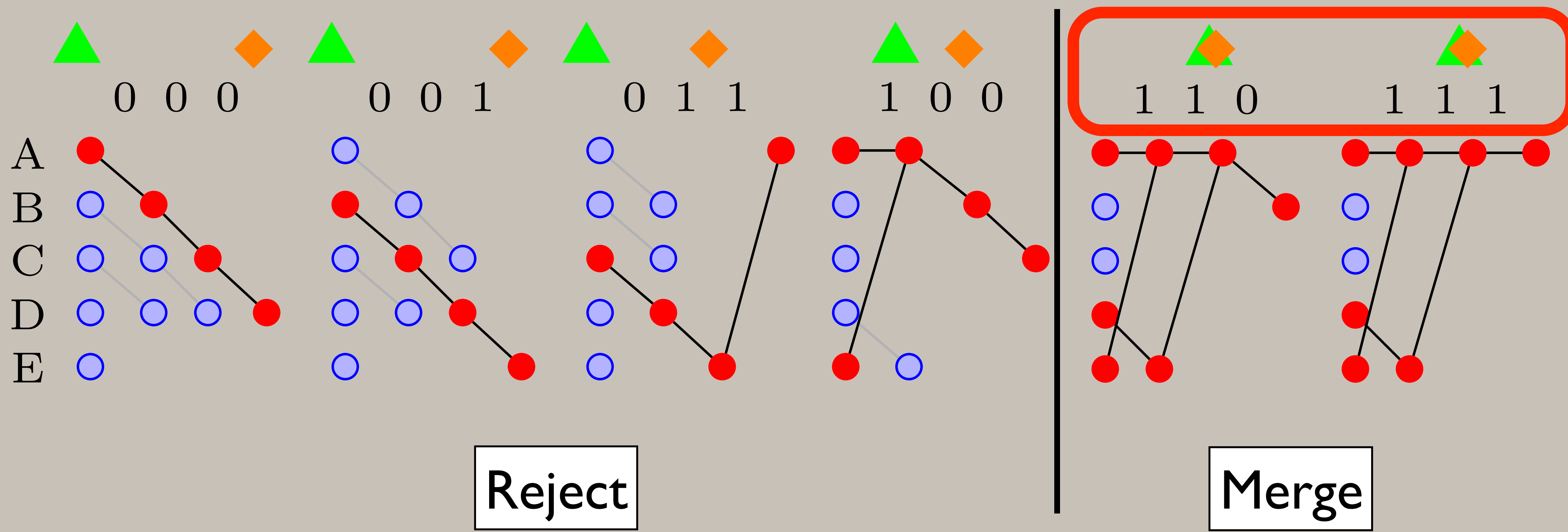

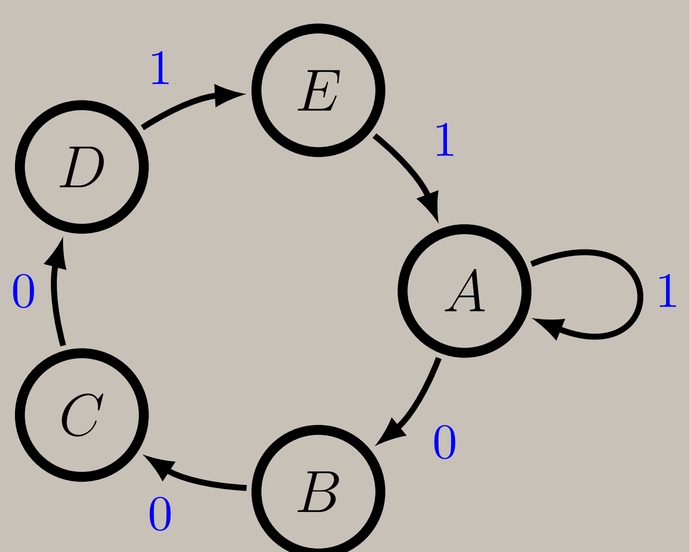

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Conditioning on future ensures a complete path.

CRYPTIC ORDER

Definition

$$k = min\{L \in \mathbb{Z}^+ : H[S_L|\overrightarrow{X}_0] = 0\}$$

$$\underline{\ldots, X_{-R}, X_{-R+1}, \ldots, X_{-1},} X_0, X_1, X_2, \ldots$$

$$S_0$$

How much must we "add back in"?

CRYPTIC ORDER

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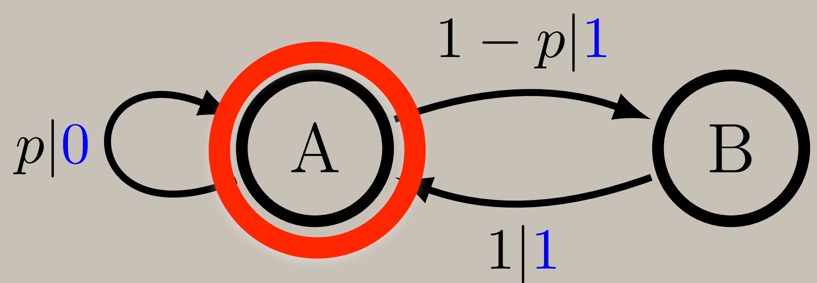
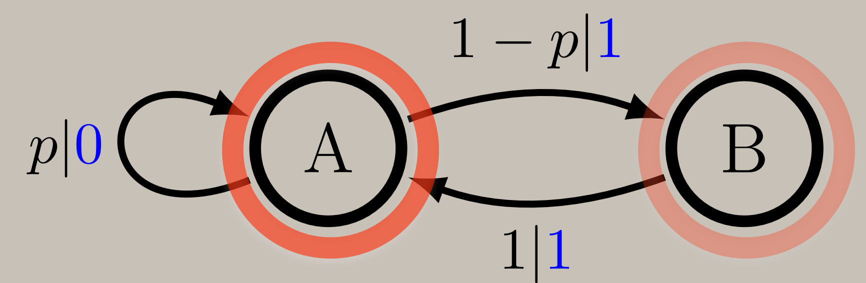
CLASSICAL SYNCHRONIZATION



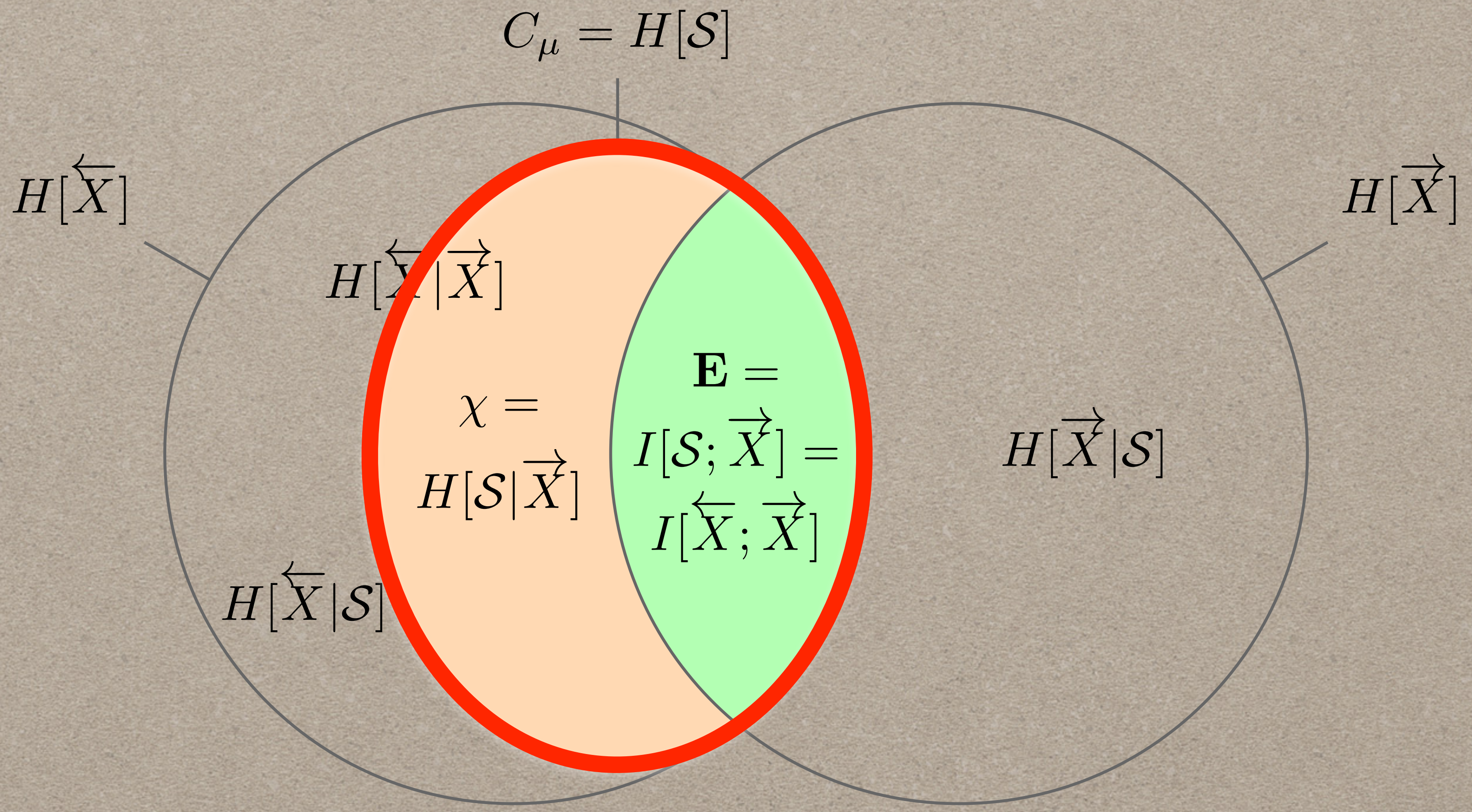


Alice

Bob

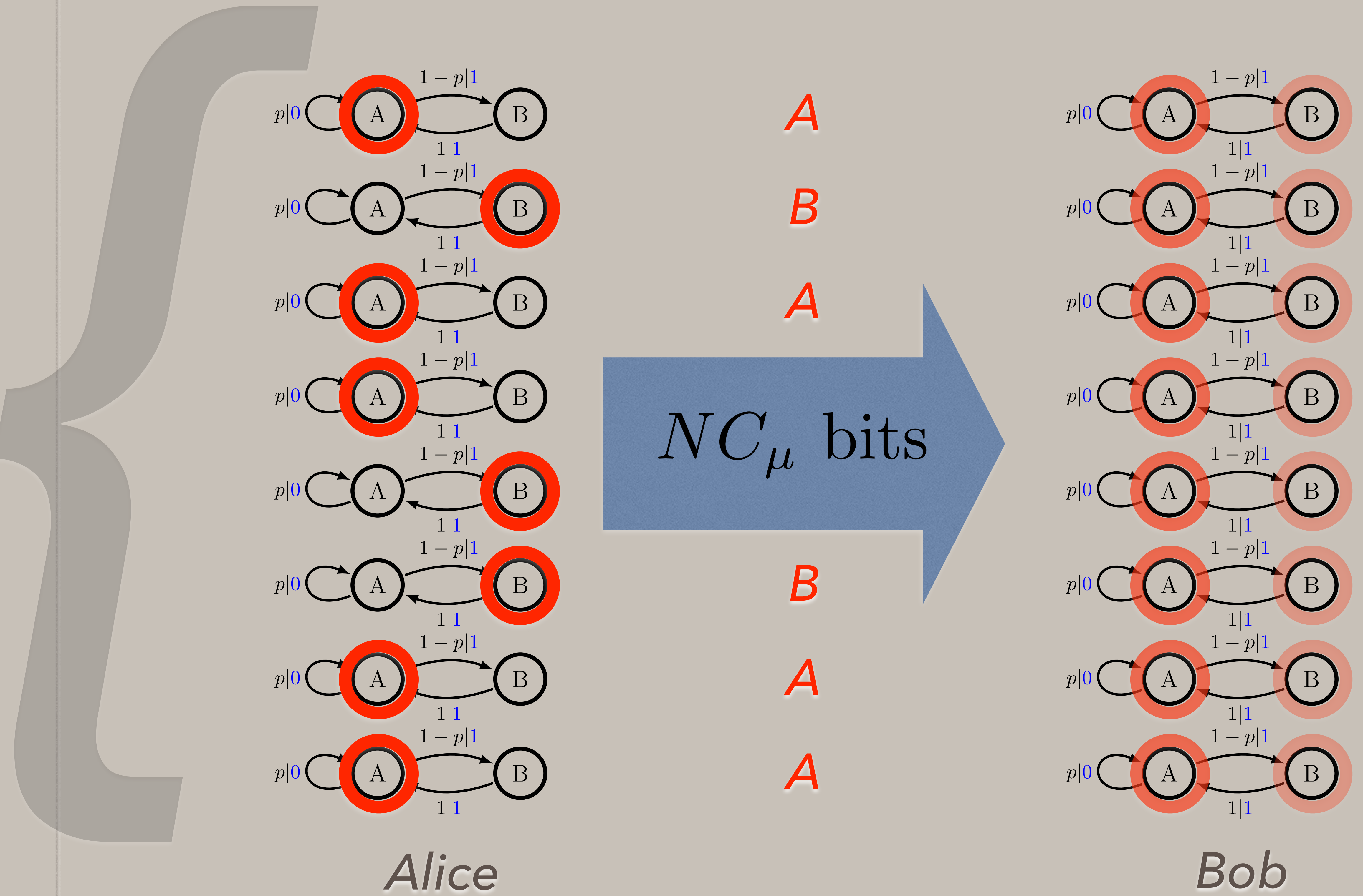
Alice's future prediction:

$$Pr(0) = p$$
$$Pr(1) = 1 - p$$
$$Pr(00) = p^2$$
$$Pr(01) = p(1-p)$$
$$Pr(10) = 0$$
$$Pr(11) = 0$$
$$Pr(000) = p^3$$
$$\ldots$$

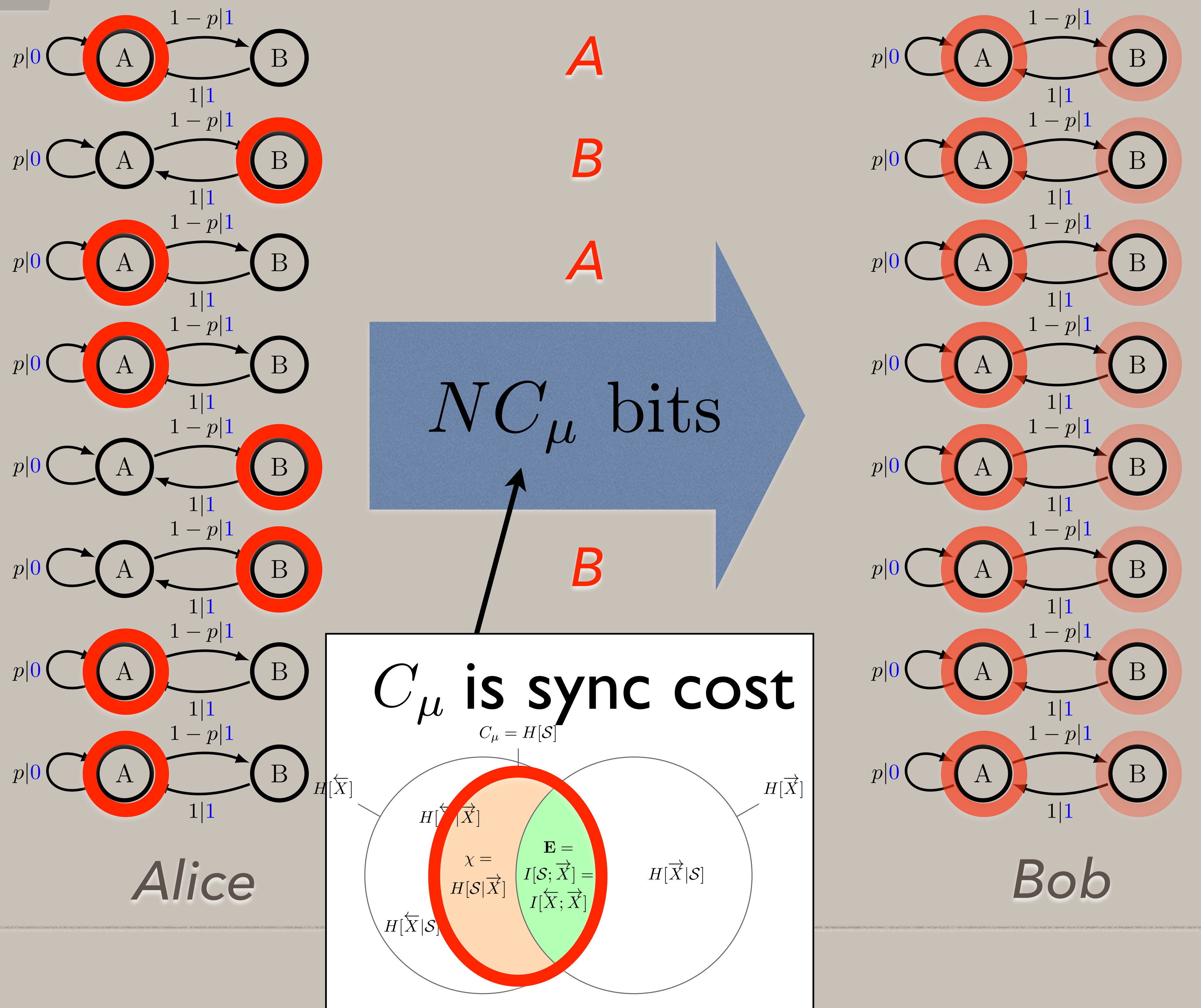

$C_\mu = H[\mathcal{S}]$
$H[\overleftarrow{X}]$
$H[\overrightarrow{X}]$
$H[\overleftarrow{X}|\overrightarrow{X}]$
$\chi =$
$H[\mathcal{S}|\overrightarrow{X}]$
$\mathbf{E} =$
$I[\mathcal{S};\overrightarrow{X}] =$
$I[\overleftarrow{X};\overrightarrow{X}]$
$H[\overrightarrow{X}|\mathcal{S}]$
$H[\overleftarrow{X}|\mathcal{S}]$

$$C_\mu = H(S)$$

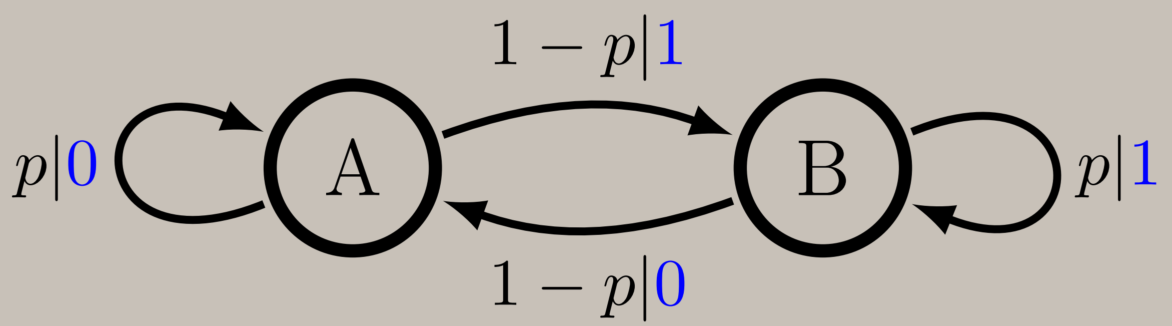
CLASSICAL SYNCHRONIZATION



CLASSICAL SYNCHRONIZATION



QUANTUM REPRESENTATIONS



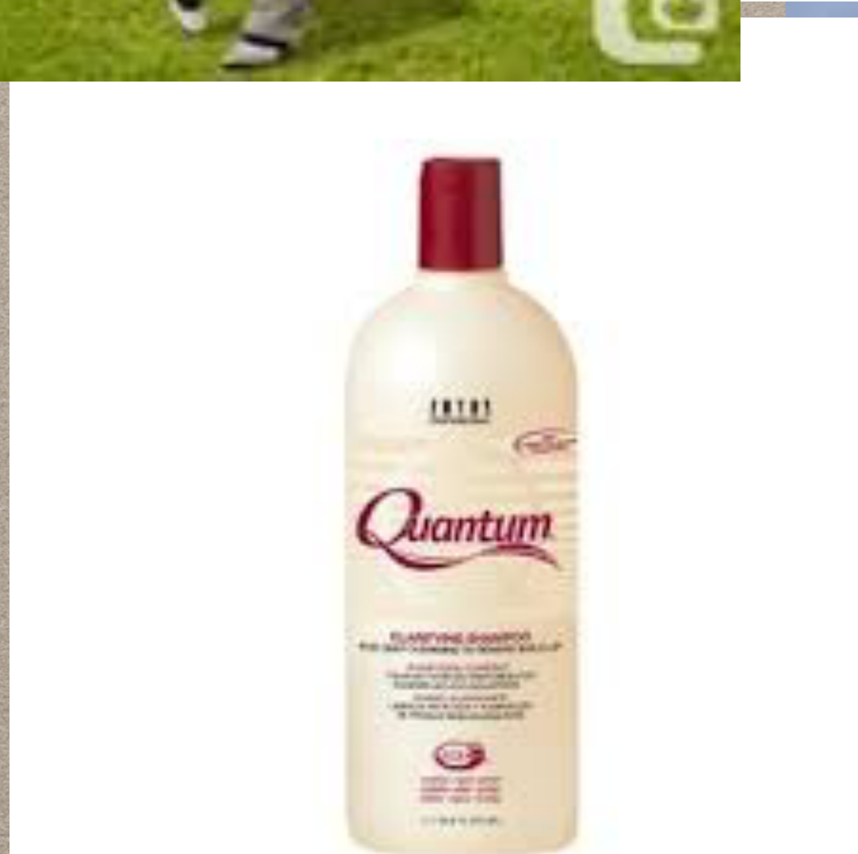
How might we "quantize" this thing?

Is there any benefit?
E.g. what is the quantum communication cost of synchronizing?

Are there any tradeoffs?


QUANTUM
GOLF ACADEMY


Nuka Cola
Quantum
Nuka-Cola Quantum

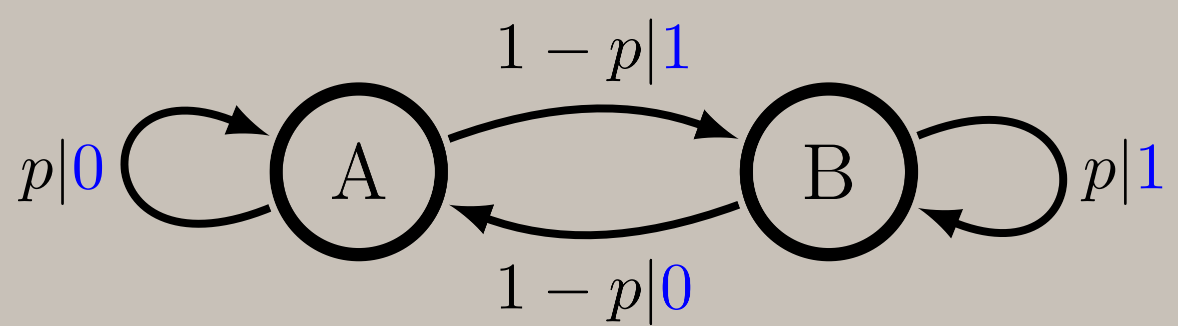

Quantum


QUANTUM
Lazydays

QUANTUM INTUITION

For p ~ 1/2 this is nearly the fair coin.

A ~ B



Can we express this similarity as using non-orthogonal states?

QUANTUM STATES

Map each classical causal state σ_j *to a quantum state*

$$|\eta_j(L)\rangle \equiv \sum_{w^L \in |\mathcal{A}|^L} \sum_{\sigma_k \in \mathcal{S}} \sqrt{Pr(w^L, \sigma_k | \sigma_j)} |w^L\rangle |\sigma_k\rangle$$

QUANTUM STATES

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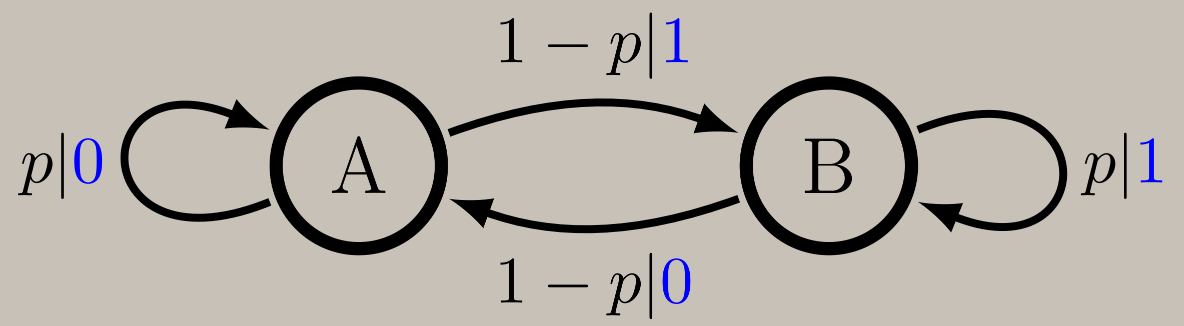

Example (L=1):

$$|\eta_A\rangle = \sqrt{Pr(0|A)}|0\rangle|A\rangle + \sqrt{Pr(1|A)}|1\rangle|B\rangle$$

$$|\eta_B\rangle = \sqrt{Pr(0|B)}|0\rangle|A\rangle + \sqrt{Pr(1|B)}|1\rangle|B\rangle$$

QUANTUM STATES

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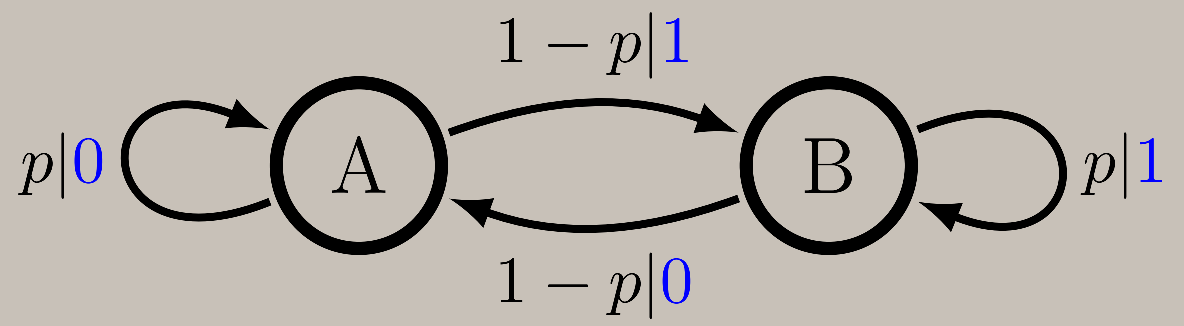
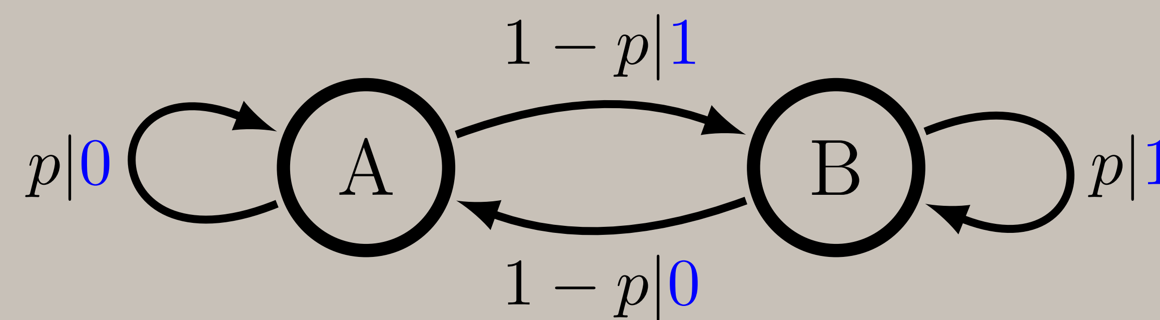


Example (L=1):

$$|\eta_A\rangle = \sqrt{p}|0\rangle|A\rangle + \sqrt{1-p}|1\rangle|B\rangle$$
$$|\eta_B\rangle = \sqrt{1-p}|0\rangle|A\rangle + \sqrt{p}|1\rangle|B\rangle$$

The key is these nontrivial overlaps!

QUANTUM DYNAMICS



$$|\eta_A\rangle = \sqrt{p}|0\rangle|A\rangle + \sqrt{1-p}|1\rangle|B\rangle$$

$$|\eta_B\rangle = \sqrt{1-p}|0\rangle|A\rangle + \sqrt{p}|1\rangle|B\rangle$$

Projective measurement in state-space,

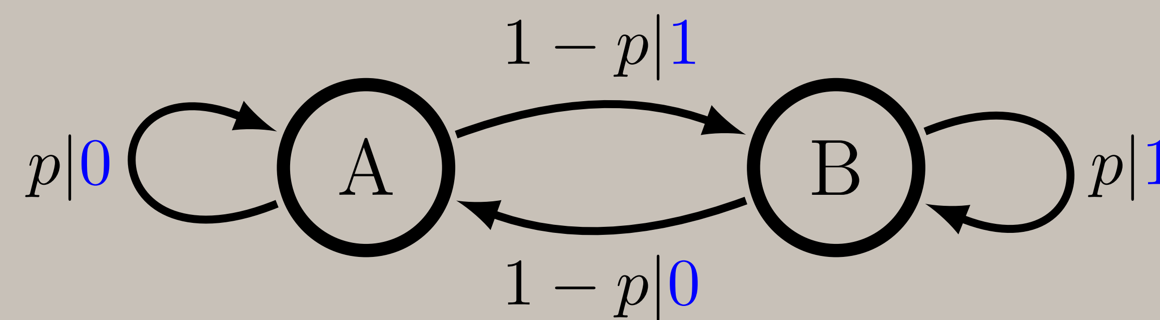
$$Pr(0||\eta_A\rangle) = |\langle 0|\eta_A\rangle|^2 \quad = p \to \text{‘}0\text{‘}, |A\rangle$$

$$Pr(1||\eta_A\rangle) = |\langle 1|\eta_A\rangle|^2 \quad = 1-p \to \text{‘}1\text{‘}, |B\rangle$$

$$Pr(0||\eta_B\rangle) = |\langle 0|\eta_B\rangle|^2 \quad = 1-p \to \text{‘}0\text{‘}, |A\rangle$$

$$Pr(1||\eta_B\rangle) = |\langle 1|\eta_B\rangle|^2 \quad = p \to \text{‘}1\text{‘}, |B\rangle$$

QUANTUM DYNAMICS



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$$|\eta_B\rangle = \sqrt{1-p}|0\rangle|A\rangle + \sqrt{p}|1\rangle|B\rangle$$

Next symbol X_t

Projective measurement in symbol space,

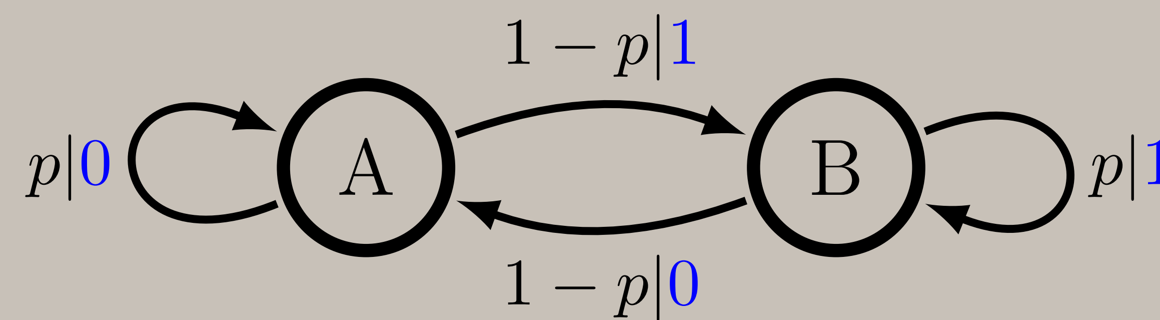
$$Pr(0||\eta_A\rangle) = |\langle 0|\eta_A\rangle|^2 = p \rightarrow \text{‘0‘} \; |A\rangle$$

$$Pr(1||\eta_A\rangle) = |\langle 1|\eta_A\rangle|^2 = 1-p \rightarrow \text{‘1‘} \; |B\rangle$$

$$Pr(0||\eta_B\rangle) = |\langle 0|\eta_B\rangle|^2 = 1-p \rightarrow \text{‘0‘} \; |A\rangle$$

$$Pr(1||\eta_B\rangle) = |\langle 1|\eta_B\rangle|^2 = p \rightarrow \text{‘1‘} \; |B\rangle$$

QUANTUM DYNAMICS



$$|\eta_A\rangle = \sqrt{p}|0\rangle|A\rangle + \sqrt{1-p}|1\rangle|B\rangle$$

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Next symbol X_t

Projective measurement in symbol space,

$$Pr(0||\eta_A\rangle) = |\langle 0|\eta_A\rangle|^2 = p \rightarrow \text{`0`}, |A\rangle$$

$$Pr(1||\eta_A\rangle) = |\langle 1|\eta_A\rangle|^2 = 1-p \rightarrow \text{`1`}, |B\rangle$$

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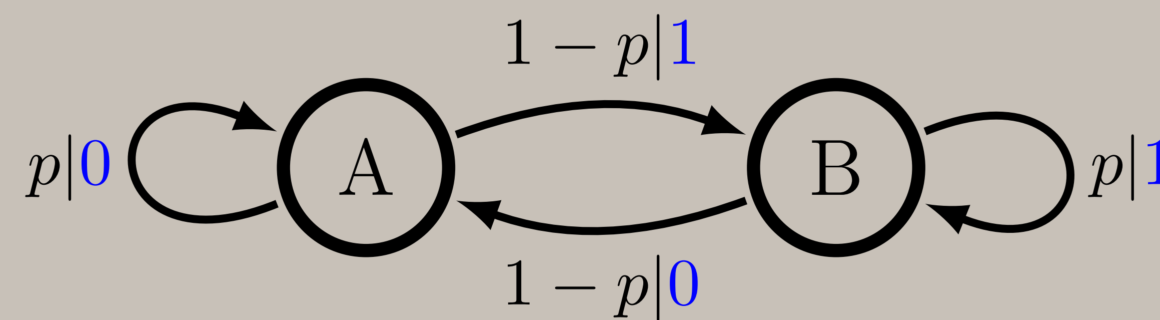
Unifilarity

QUANTUM DYNAMICS



$$|\eta_A\rangle = \sqrt{p}|0\rangle|A\rangle + \sqrt{1-p}|1\rangle|B\rangle$$

$$|\eta_B\rangle = \sqrt{1-p}|0\rangle|A\rangle + \sqrt{p}|1\rangle|B\rangle$$

Next symbol X_t

Projective measurement in symbol-space, then in state-space

$$Pr(0||\eta_A\rangle) = |\langle 0|\eta_A\rangle|^2 = p \rightarrow \text{‘}0\text{‘} \; |A\rangle \rightarrow \text{‘}A\text{‘}, |A\rangle$$

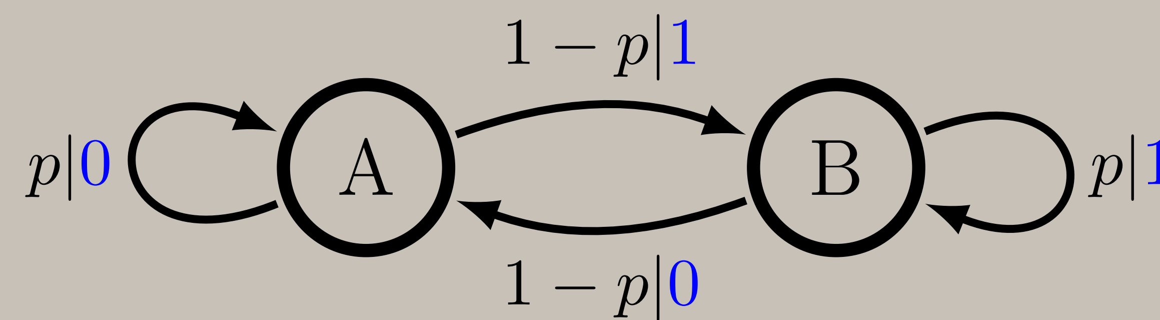
$$Pr(1||\eta_A\rangle) = |\langle 1|\eta_A\rangle|^2 = 1-p \rightarrow \text{‘}1\text{‘} \; |B\rangle \rightarrow \text{‘}B\text{‘}, |A\rangle$$

$$Pr(0||\eta_B\rangle) = |\langle 0|\eta_B\rangle|^2 = 1-p \rightarrow \text{‘}0\text{‘} \; |A\rangle \rightarrow \text{‘}A\text{‘}, |A\rangle$$

$$Pr(1||\eta_B\rangle) = |\langle 1|\eta_B\rangle|^2 = p \rightarrow \text{‘}1\text{‘} \; |B\rangle \rightarrow \text{‘}B\text{‘}, |A\rangle$$

Unifilarity

QUANTUM DYNAMICS



$$|\eta_A\rangle = \sqrt{p}|0\rangle|A\rangle + \sqrt{1-p}|1\rangle|B\rangle$$

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Next symbol X_t

Next state S_{t+1}

Projective measurement in s... space, ...n state-space

$$Pr(0||\eta_A\rangle) = |\langle 0|\eta_A\rangle|^2 = p \rightarrow \text{`0`}\ |A\rangle \rightarrow \text{`}A\text{`}, |A\rangle$$

$$Pr(1||\eta_A\rangle) = |\langle 1|\eta_A\rangle|^2 = 1-p \rightarrow \text{`1`}\ |B\rangle \rightarrow \text{`}B\text{`}, |A\rangle$$

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Unifilarity

QUANTUM DYNAMICS

For general L,

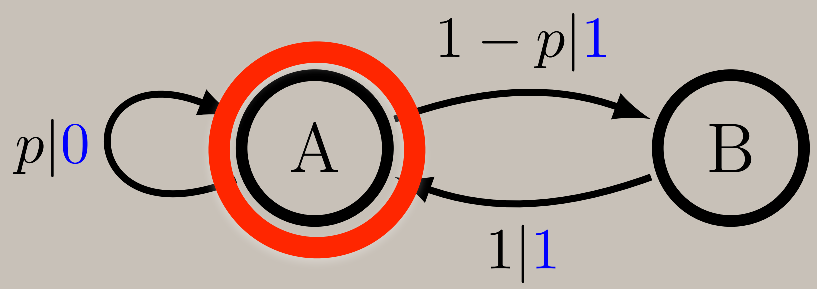
$$|\eta_j(L)\rangle \equiv \sum_{w^L \in |\mathcal{A}|^L} \sum_{\sigma_k \in \mathcal{S}} \sqrt{Pr(w^L, \sigma_k|\sigma_j)}|w^L\rangle|\sigma_k\rangle$$

Reset

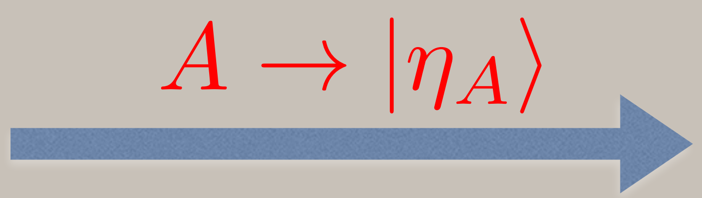
$$Pr(w||\eta_j\rangle) = |\langle w|\eta_j\rangle|^2 = Pr(w|\sigma_j) \rightarrow \text{`}w\text{`}, |\sigma_k\rangle \rightarrow \text{`}\sigma_k\text{`}|\sigma_k\rangle \rightarrow |\eta_k\rangle$$

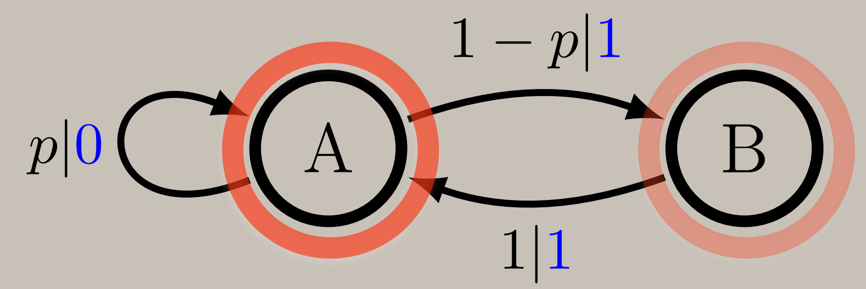
Mechanism reproduces classical process L symbols at a time.

QUANTUM SYNCHRONIZATION



Alice



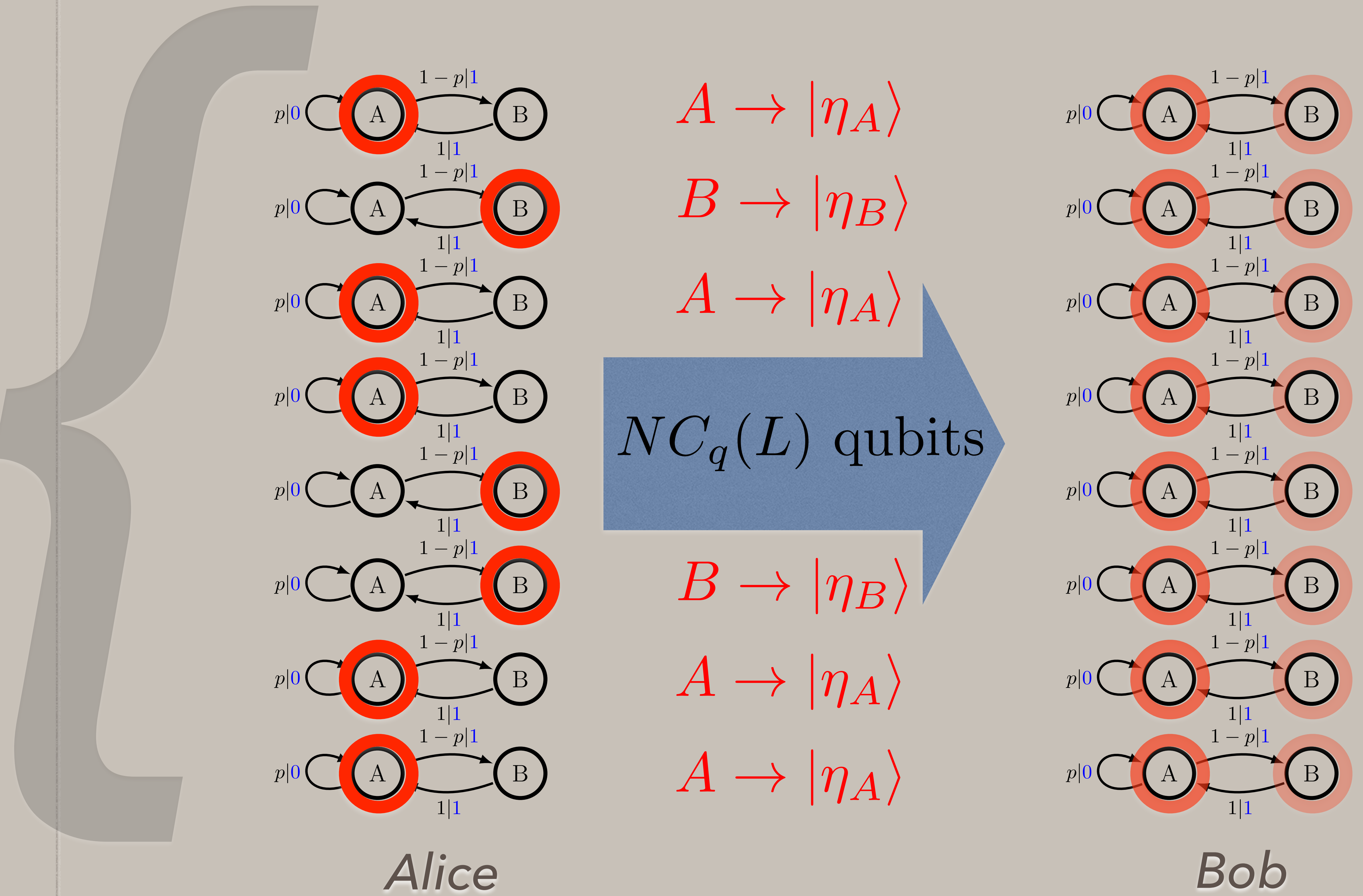


Bob

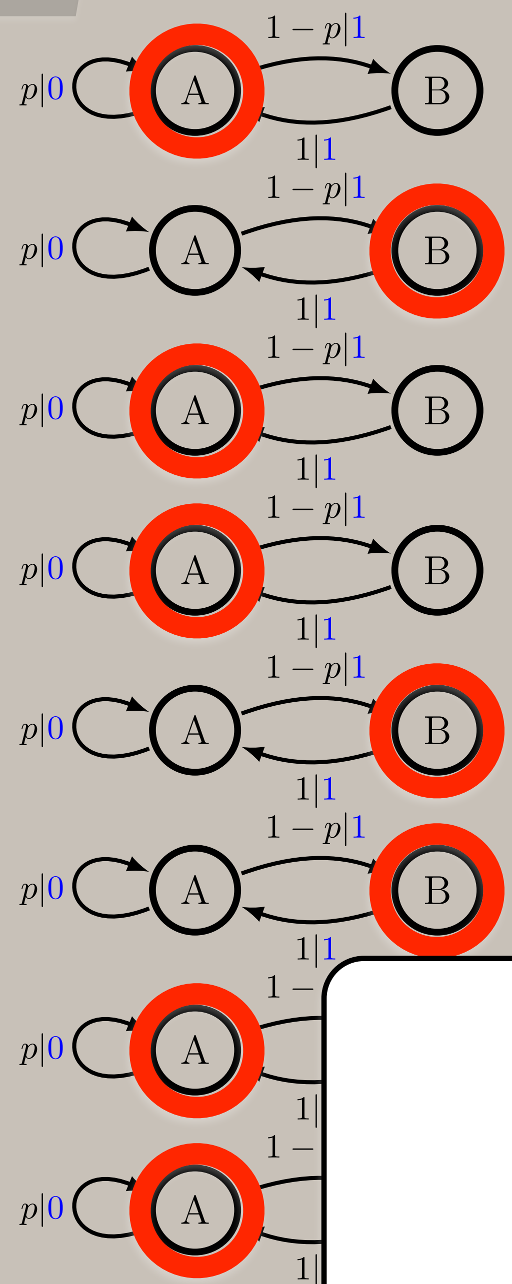
Alice's future prediction:

$$Pr(0) = p$$
$$Pr(1) = 1-p$$
$$Pr(00) = p^2$$
$$Pr(01) = p(1-p)$$
$$Pr(10) = 0$$
$$Pr(11) = 0$$
$$Pr(000) = p^3$$
$$\ldots$$

QUANTUM SYNCHRONIZATION



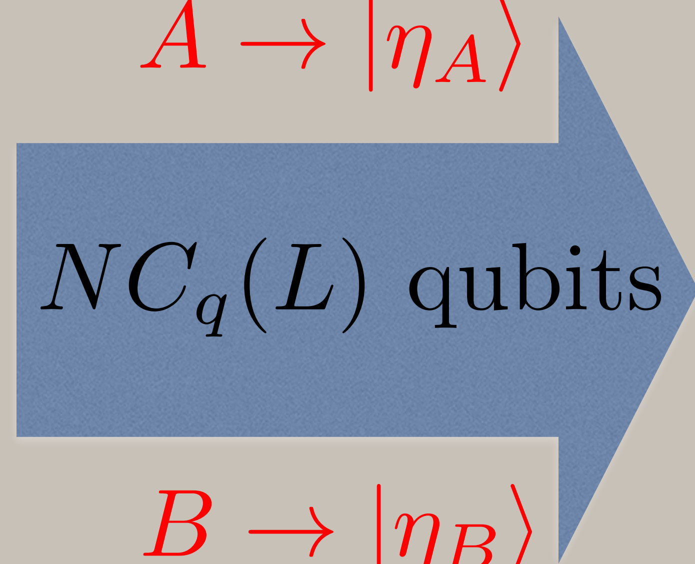
QUANTUM SYNCHRONIZATION



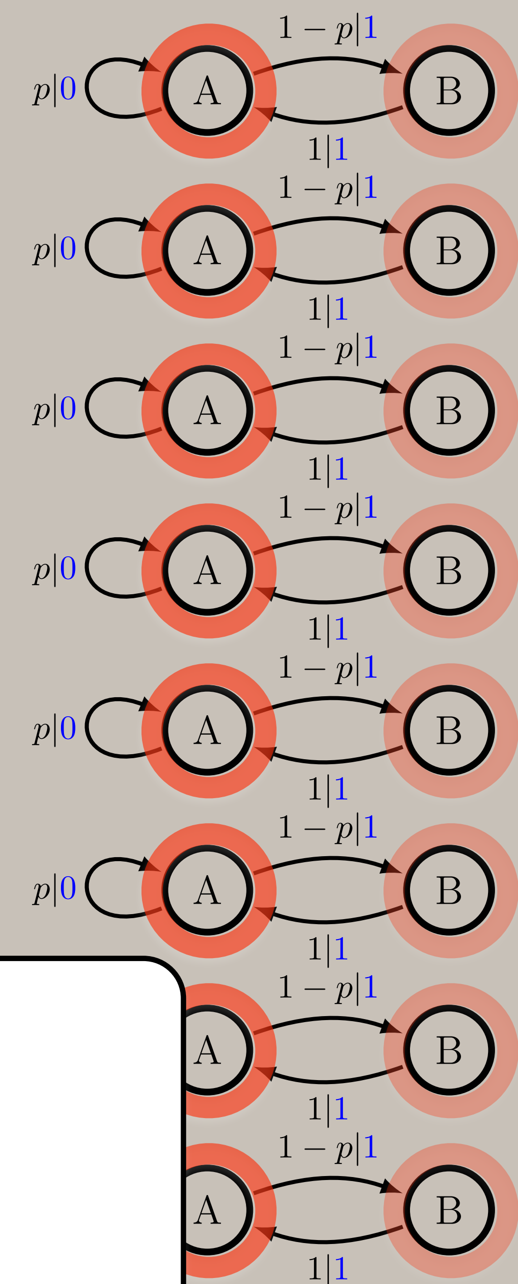
$$A \to |\eta_A\rangle$$

$$B \to |\eta_B\rangle$$

$$A \to |\eta_A\rangle$$



$$B \to |\eta_B\rangle$$



Ali

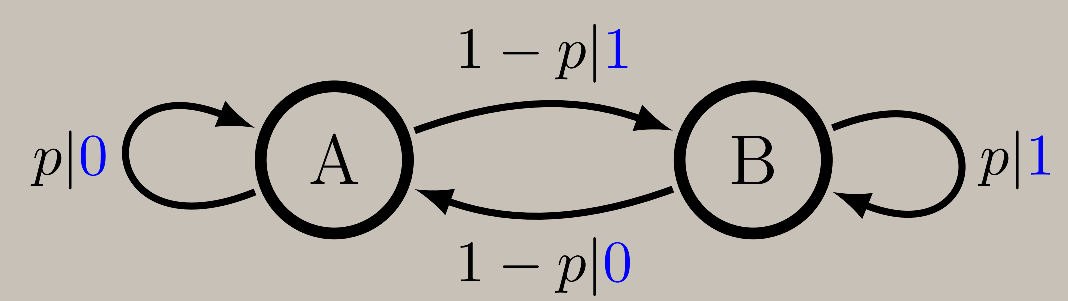
Bob

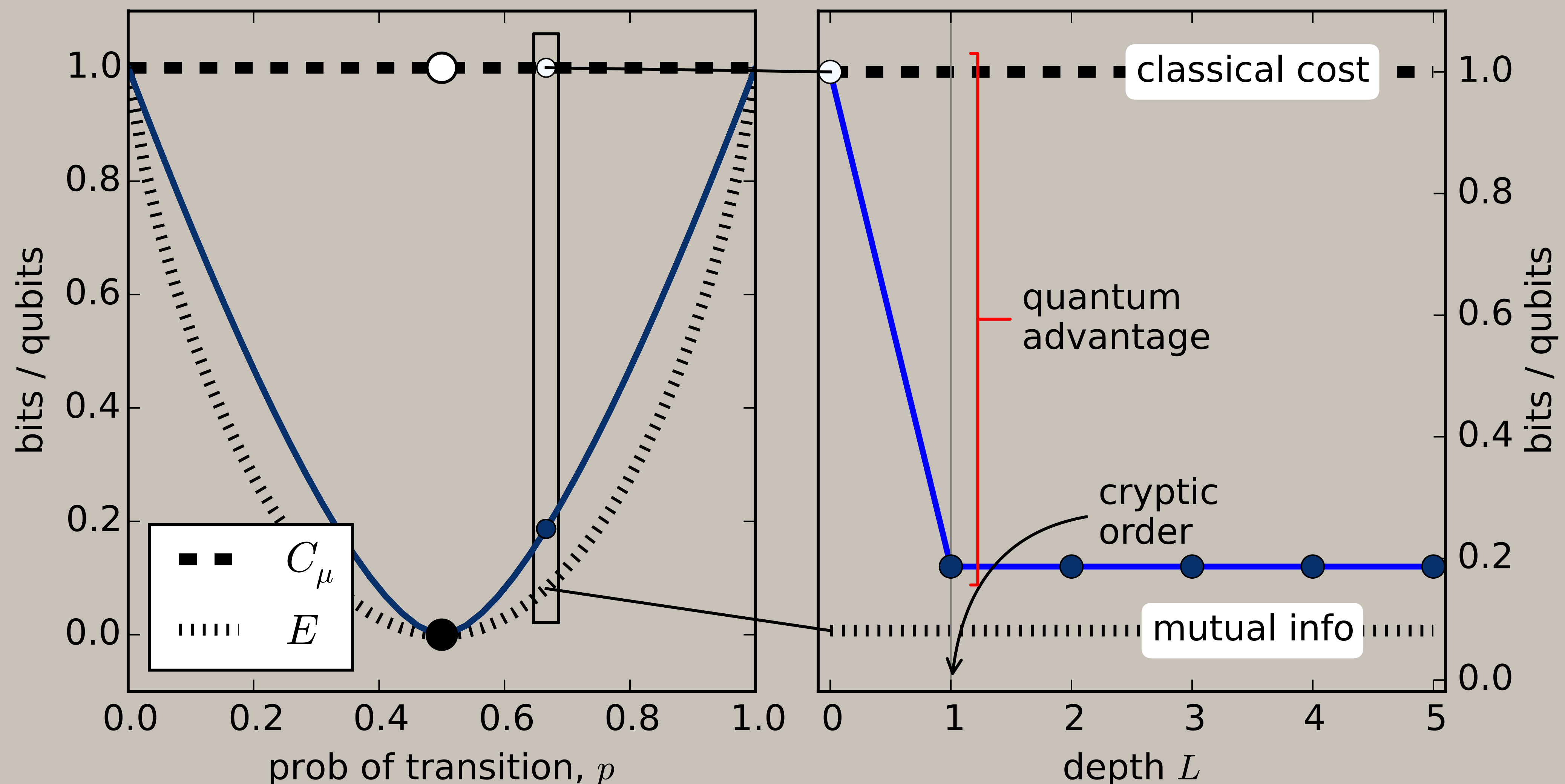
$$C_q(L) = S(\rho(L))$$

$$S(\rho) = \mathrm{tr}\ \rho \log \rho$$

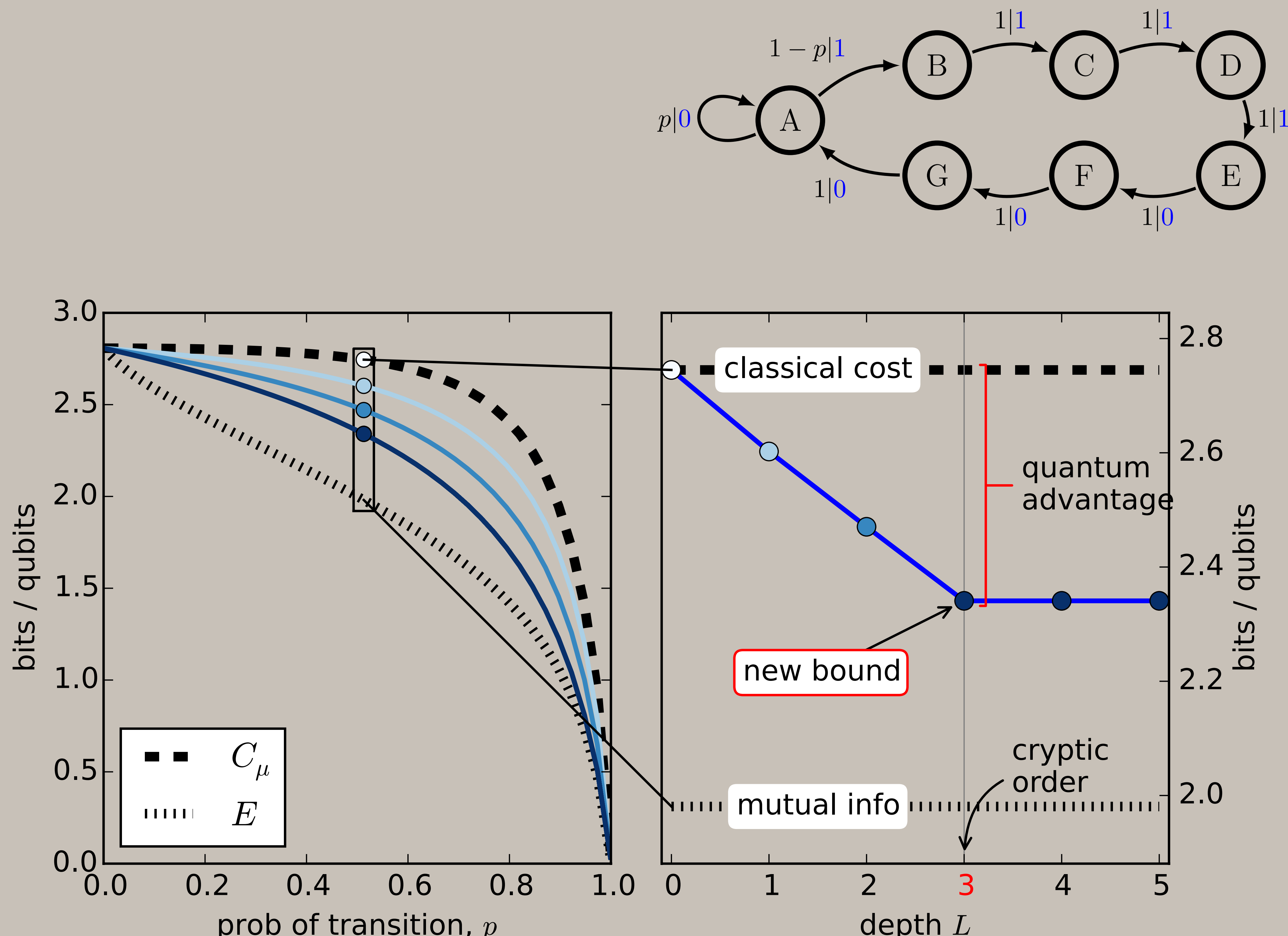
$$\rho(L) = \sum \pi_i |\eta_i(L)\rangle\langle\eta_i(L)|$$

BIASED COINS PROCESS

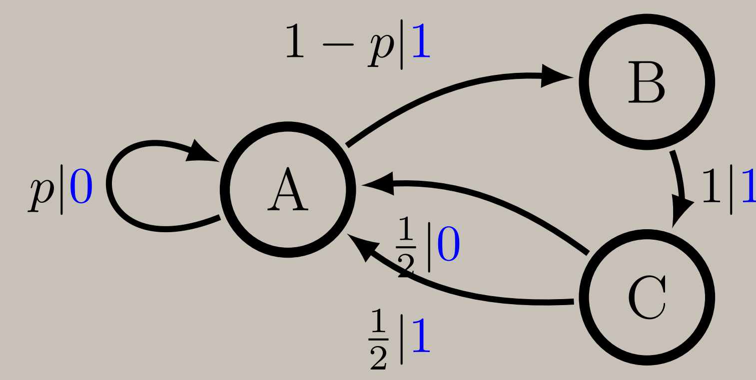


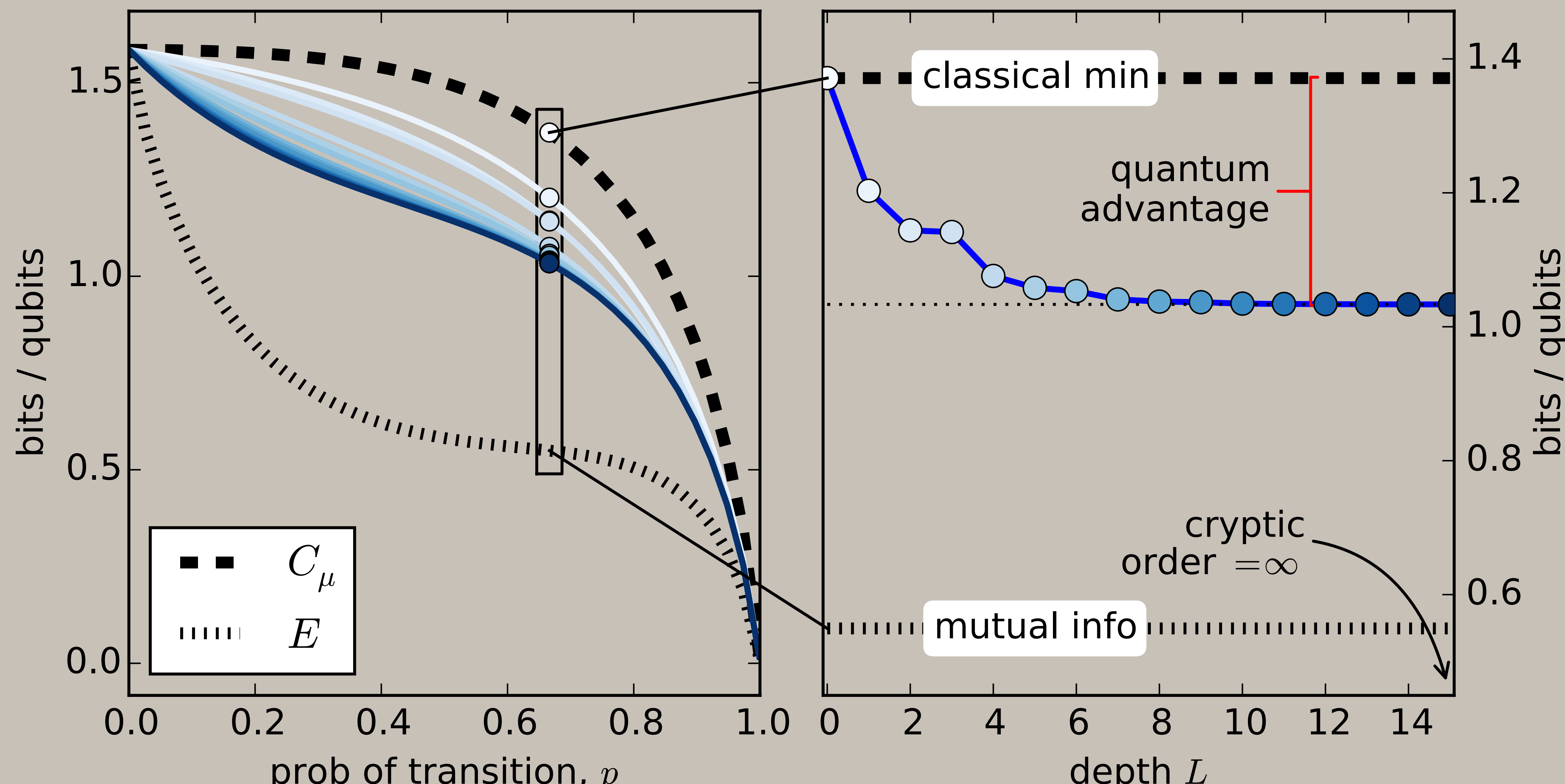


RK GOLDEN MEAN



NEMO PROCESS





EFFICIENT COMPUTATION OF $C_q(L)$

Challenge

Word space grows exponentially.

Many probabilities to evaluate.
\rho lives in exponentially increasing Hilbert space.

Solution

Only track paths until merger.

Record overlaps, not state.

Use overlaps to construct equivalent \rho in R^|S|+

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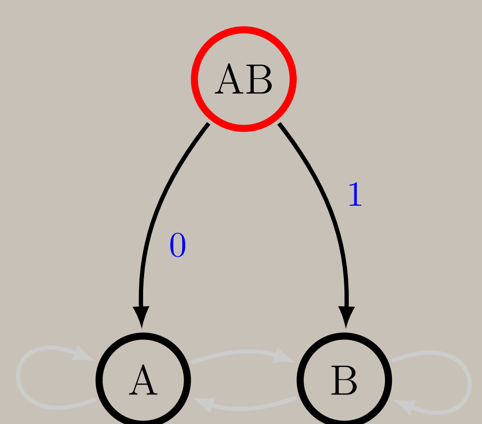
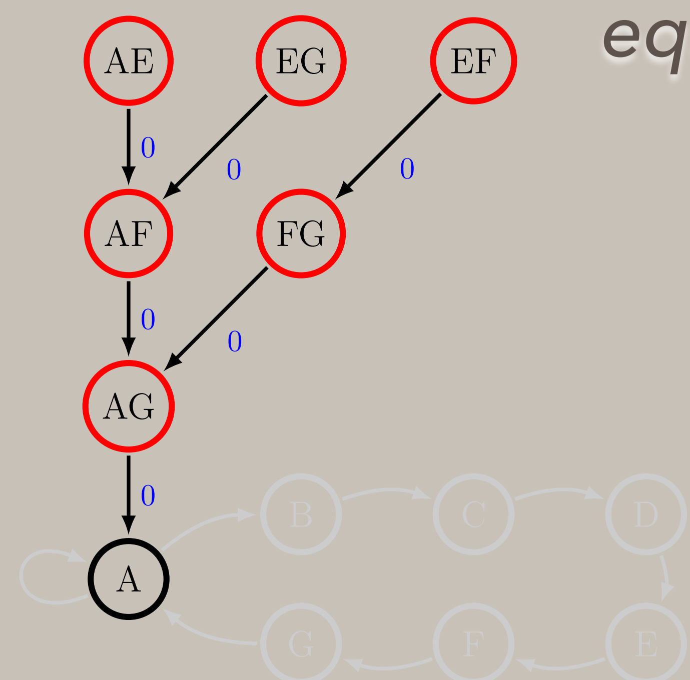
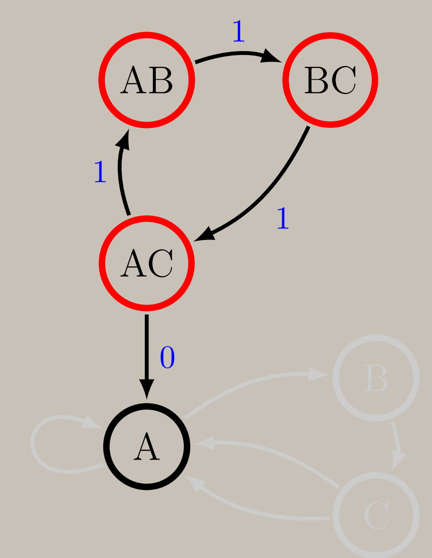


Biased coins

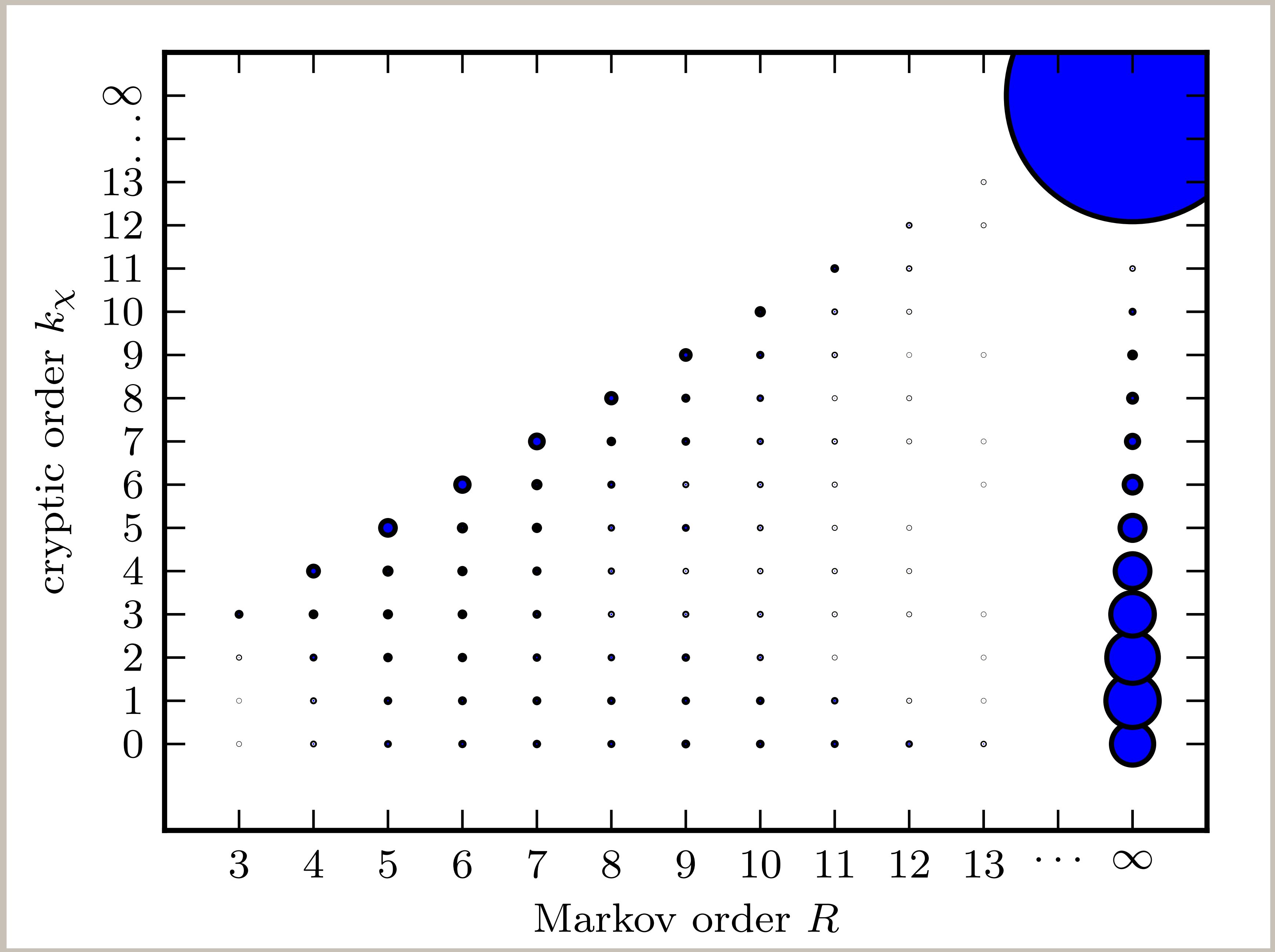


R-k Golden Mean

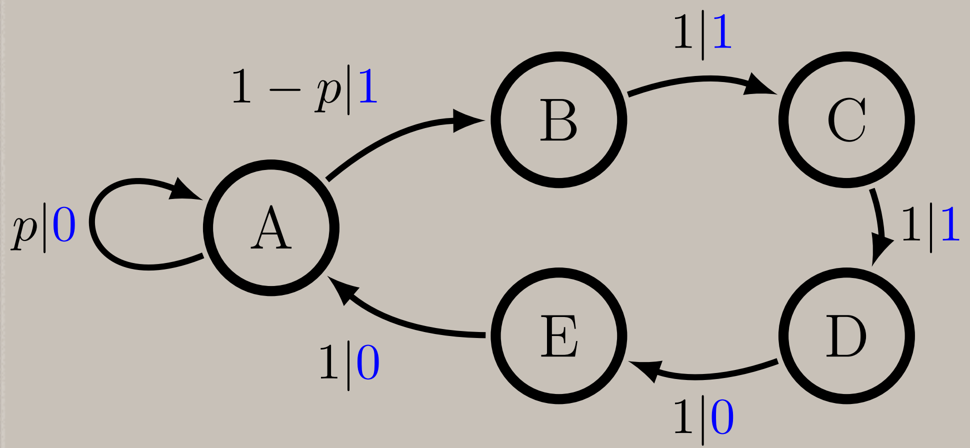


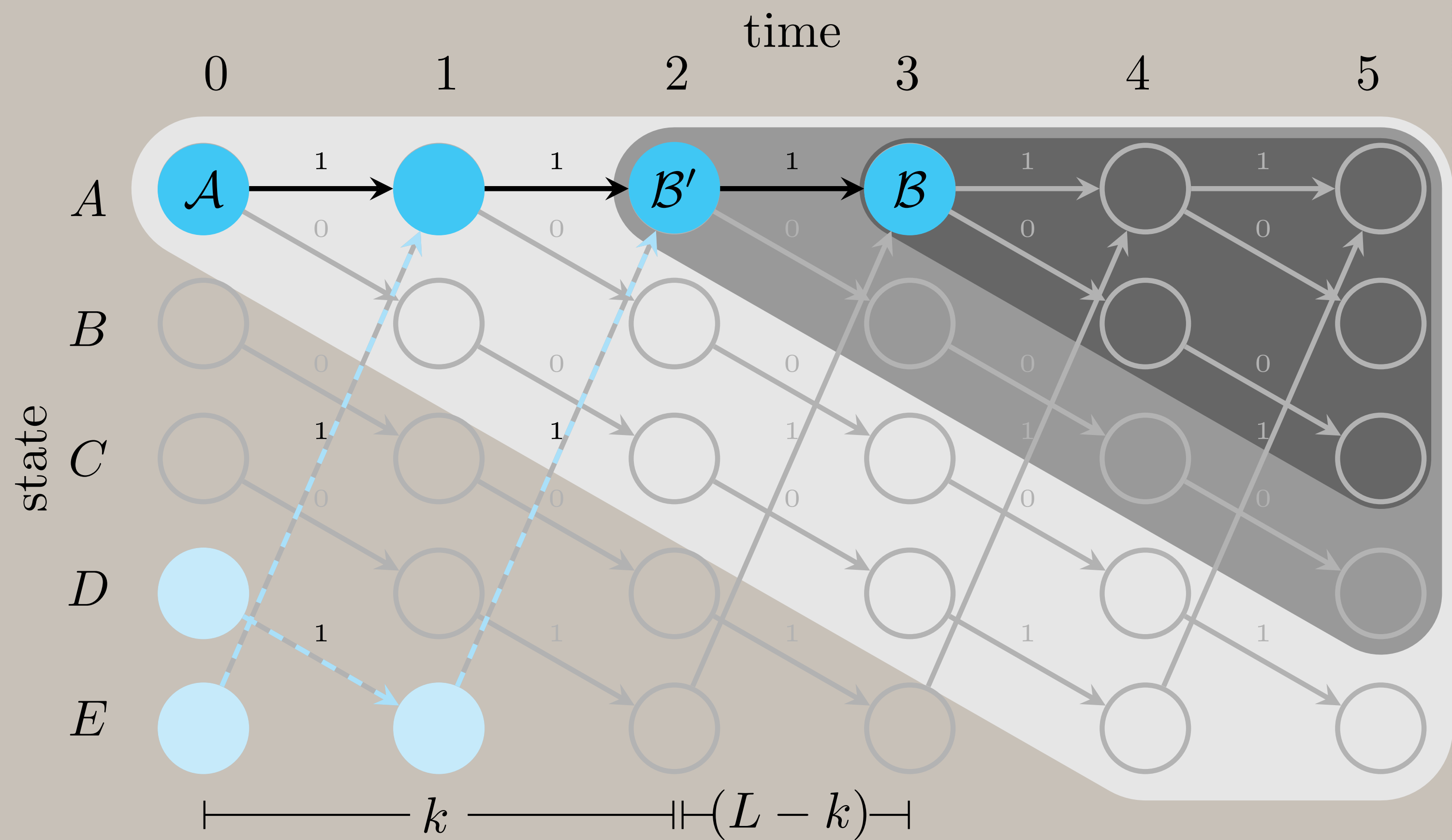
Nemo

WHERE ARE THE CRYPTIC PROCESSES HIDING?



PREDICTION TRADEOFF

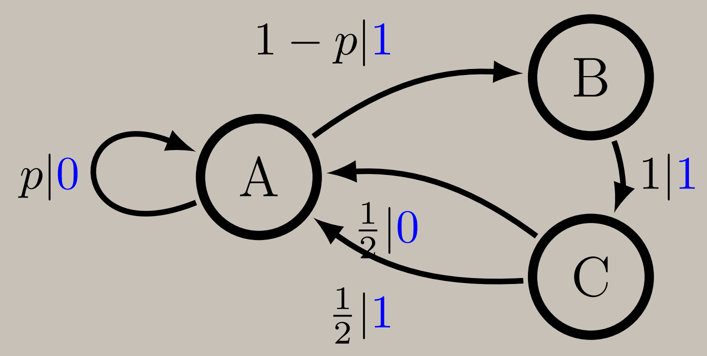




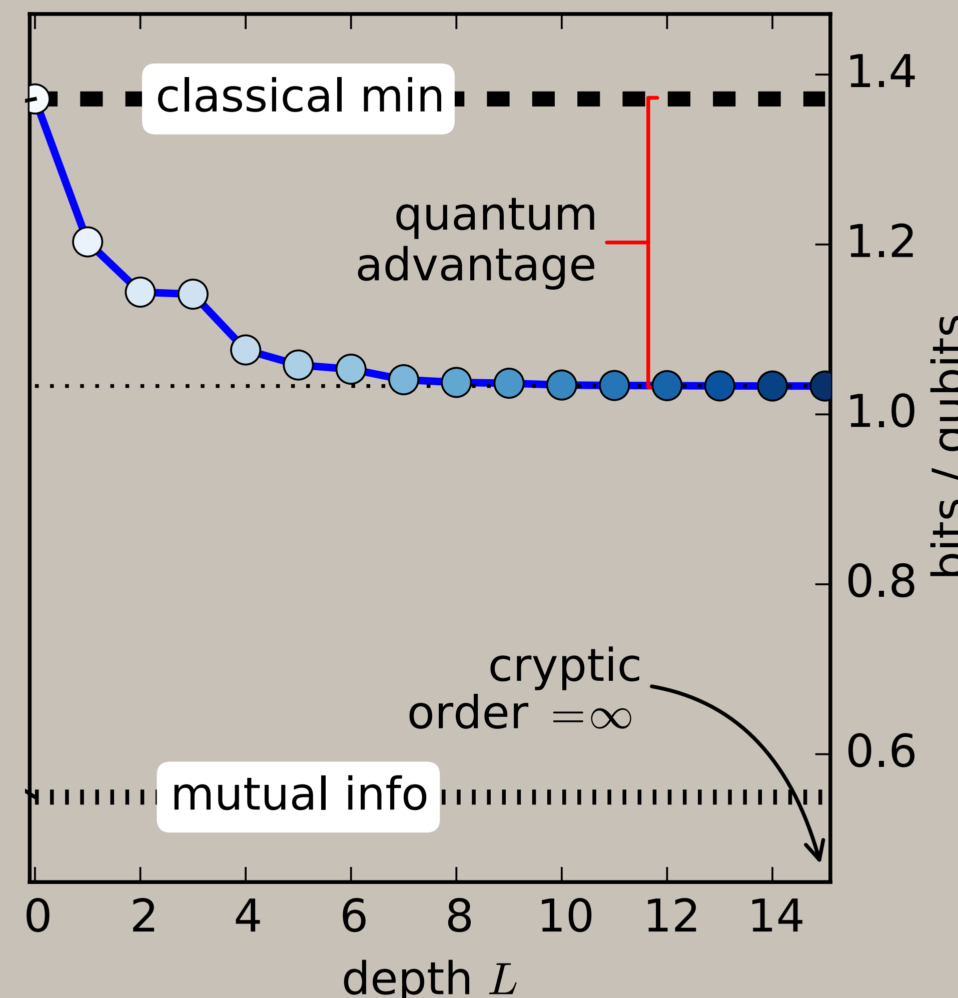
Bob can only make a conditionally equivalent prediction

Protected from overcoding by cryptic order

TAKEAWAYS



- Structure and synchronization
- Quantum advantage
- Cryptic order saturation
- Efficient computation



Structure matters!