NCASO Spring 2014

Complexity à la Mode: Spectral Methods for Complex Systems Part $2H_2(p)\big|_{p=\frac{1}{2}}$

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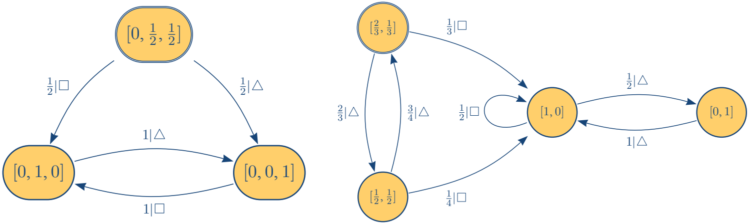
May 22nd 2014

Visualizing Modes



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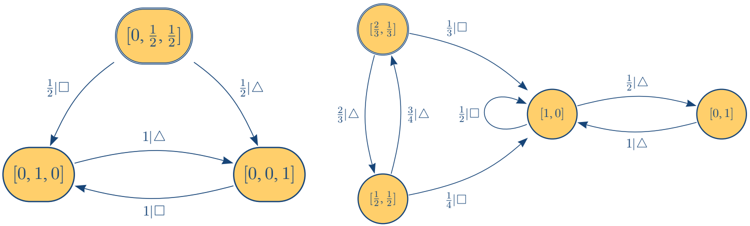

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Spectral methods formalize and empower our intuition.

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- Dynamics, Correlation Functions, and Power Spectra from any HMM presentation of a process

Previously: Power Spectrum of Even Process

$$P_{\mathrm{c}}(\omega) = \langle\pi| T^{(1)} |\mathbf{1}\rangle + 2 \sum_{\lambda \in \Lambda_T} \mathrm{Re} \frac{\langle\pi| T^{(1)} T_\lambda T^{(1)} |\mathbf{1}\rangle}{e^{i\omega} - \lambda}$$

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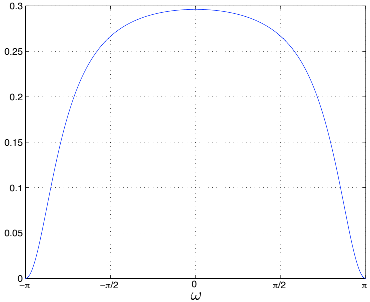

If $\mathcal{A} = \{0, 1\}$ and T is diagonalizable, then the power spectrum is simply:

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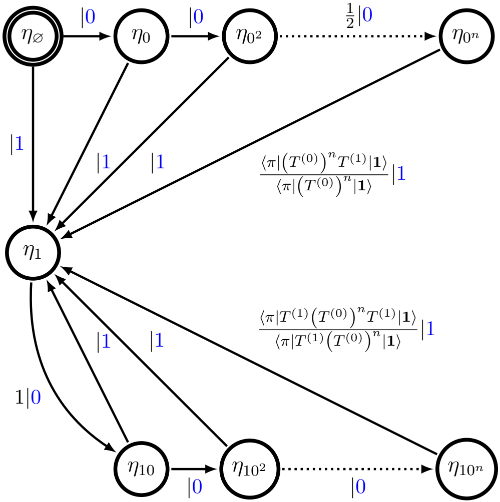
- # of peaks in power spectrum gives lower bound on # of states of any presentation

Power Spectrum of SNS



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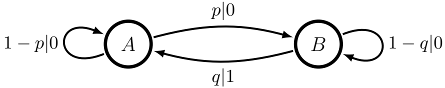

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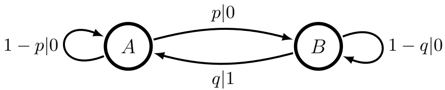
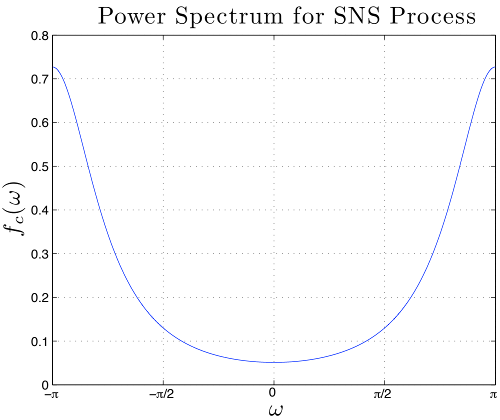
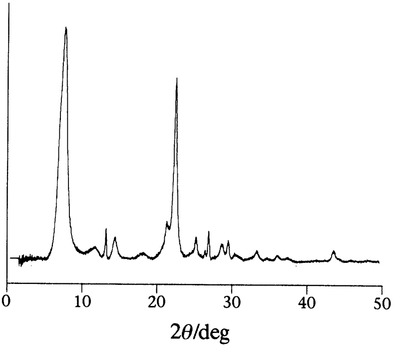




Diffraction Pattern (Power Spectrum of Structure Factors) of Chaotic Crystal



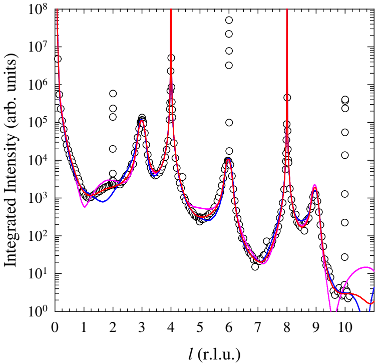
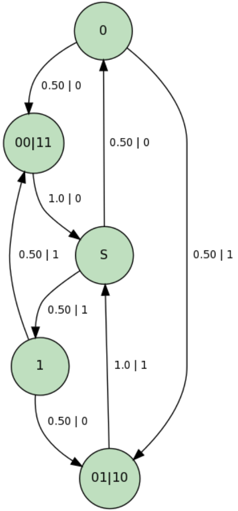


Figure of XRD from typical zeolite beta sample from Treacy et al.

Figure of XRD of multi-layer grapheme on SiC from Hass et al.

RRXOR Process



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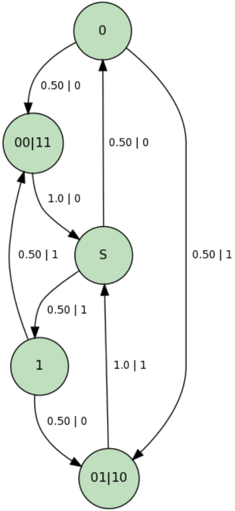

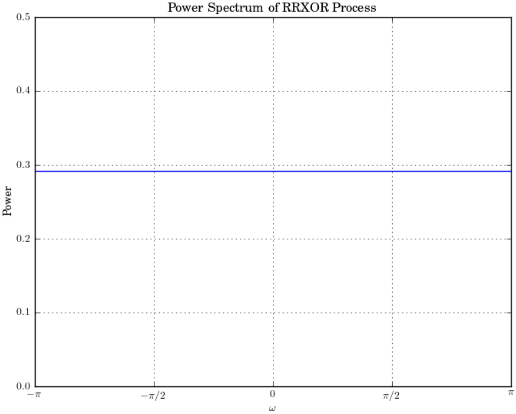

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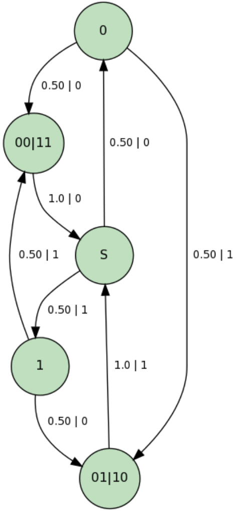

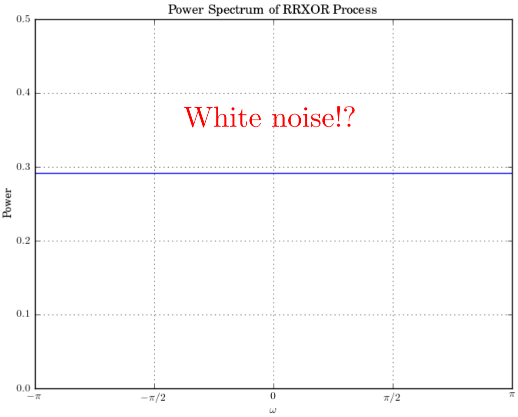

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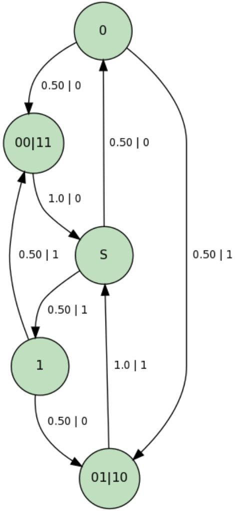

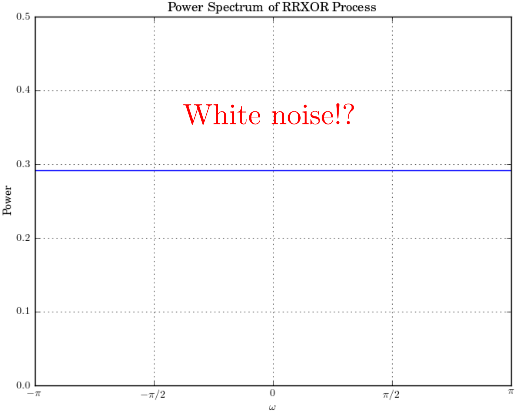

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- Especially as processes become more complex, the interesting behavior gets processed primarily within very high-order correlations.

Definitions

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- For a *stochastic matrix* (i.e., all rows sum to unity), all eigenvalues lie on or within the unit circle; the eigenvalue of unity is guaranteed with $a_1 = g_1 = 1$; single attractor $\to$ $T_1 = |\mathbf{1}\rangle \langle \pi|$.

Reasons for Using Projection Operators

- They play an important role in determining the amplitudes of constituent exponential decays. E.g.:

$$\gamma[\tau] = \sum_{\lambda \in \Lambda_T} \frac{1}{\lambda} \langle \pi | \Big(\sum_{x \in \mathcal{A}} \overline{x} T^{(x)} \Big) T_\lambda \Big(\sum_{x' \in \mathcal{A}} x' T^{(x')} \Big) | \mathbf{1} \rangle \, \lambda^\tau$$

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- They play nicely with non-diagonalizability and (optimistically) also ∞-state extensions.

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The states $\eta \in \boldsymbol{\mathcal{R}}$ of the Mixed State Presentation (MSP) are *distributions* over the states $\boldsymbol{\mathcal{S}}$ of another presentation. Starting from the MSP's start state—the stationary distribution π—the MSP gives the observation-induced evolution of probability density over distributions-over-$\boldsymbol{\mathcal{S}}$.

The initial distribution over MSP—assuming no prior observations of the known process—is denoted δ_π with $\langle \delta_\pi | = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}$.

The transition matrix W of the MSP is a stochastic matrix with $W_1 = |\mathbf{1}\rangle \langle \pi_W|$.

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&= \langle \delta_\pi | W^{L-1} | H(W^{\mathcal{A}}) \rangle ,
\end{aligned}
$$

where

$$
| H(W^{\mathcal{A}}) \rangle \equiv - \sum_{\eta \in \boldsymbol{\mathcal{R}}} | \delta_\eta \rangle \sum_{x \in \mathcal{A}} \langle \delta_\eta | W^{(x)} | \mathbf{1} \rangle \log_2 \left(\langle \delta_\eta | W^{(x)} | \mathbf{1} \rangle \right)
$$

The length-L entropy rate

If W is diagonalizable, then

$$\begin{aligned} \langle \delta_\pi | W^{L-1} &= \langle \delta_\pi | \sum_{\lambda \in \Lambda_W} \lambda^{L-1} W_\lambda \\ &= \sum_{\lambda \in \Lambda_W} \lambda^{L-1} \langle \delta_\pi | W_\lambda \end{aligned}$$

where $\{W_\lambda\}$ can by obtained via

$$W_\lambda = \prod_{\substack{\zeta \in \Lambda_W \\ \zeta \neq \lambda}} \frac{W - \zeta I}{\lambda - \zeta}.$$

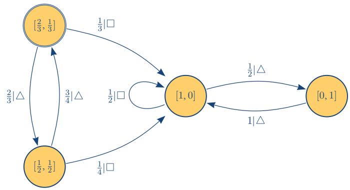
The length-L entropy rate

Hence, for diagonalizable W, the length-L entropy rate can be rewritten as

$$\begin{aligned} h_\mu(L) &= \langle \delta_\pi | \, W^{L-1} \, | H(W^{\mathcal{A}}) \rangle \\ &= \sum_{\lambda \in \Lambda_W} \lambda^{L-1} \, \langle \delta_\pi | \, W_\lambda \, | H(W^{\mathcal{A}}) \rangle \, , \end{aligned}$$

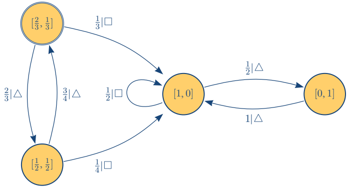
which is a closed-form expression for $h_\mu(L)$ with scalar exponentiation as the only L-dependence.

Example Strictly Sofic Processes



$\mathcal{M}_{\text{msp}}$

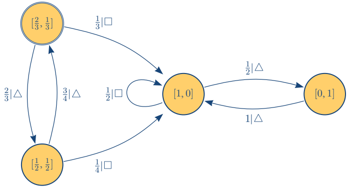
Example Strictly Sofic Processes



$\mathcal{M}_{\text{msp}}$

$$W = \begin{bmatrix} 0 & 2/3 & 1/3 & 0 \\ 3/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Example Strictly Sofic Processes



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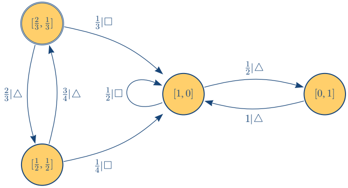
$\mathcal{M}_{\text{msp}}$

$\det(\lambda I - W) =$

$(\lambda - 1)(\lambda + \frac{1}{2})(\lambda^2 - \frac{1}{2}) = 0$

Example Strictly Sofic Processes



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$$\Lambda_W = \left\{1,\ \frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2},\ -\frac{1}{2}\right\}$$

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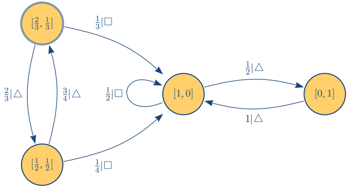

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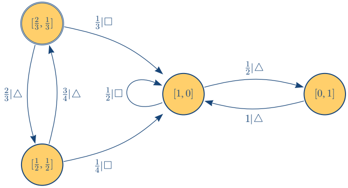
Example Strictly Sofic Processes



$\mathcal{M}_{\text{msp}}$

$$W = \begin{bmatrix} 0 & ^2/_3 & ^1/_3 & 0 \\ ^3/_4 & 0 & ^1/_4 & 0 \\ 0 & 0 & ^1/_2 & ^1/_2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

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$$\langle \delta_\pi | = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

Example Strictly Sofic Processes

λ	$\langle \delta_\pi \vert W_\lambda$			
1	$[0$	0	$\frac{2}{3}$	$\frac{1}{3}]$
$\sqrt{2}/2$	$[\frac{1}{2}$	$\frac{\sqrt{2}}{3}$	$\frac{-2-\sqrt{2}}{6}$	$\frac{-\sqrt{2}-1}{6}]$
$-\sqrt{2}/2$	$[\frac{1}{2}$	$\frac{-\sqrt{2}}{3}$	$\frac{-2+\sqrt{2}}{6}$	$\frac{\sqrt{2}-1}{6}]$
$-1/2$	$[0$	0	0	$0]$

$$|H(W^{\mathcal{A}})\rangle = \begin{bmatrix} \log_2(3) - 2/3 \\ 2 - \frac{3}{4}\log_2(3) \\ 1 \\ 0 \end{bmatrix}$$

Example Strictly Sofic Processes

$$h_\mu(L) = \sum_{\lambda \in \Lambda_W} \lambda^{L-1} \langle \delta_\pi | W_\lambda | H(W^{\mathcal{A}}) \rangle \rangle$$

Example Strictly Sofic Processes

$$
\begin{aligned}
h_\mu(L) &= \sum_{\lambda \in \Lambda_W} \lambda^{L-1} \langle \delta_\pi | W_\lambda | H(W^{\mathcal{A}}) \rangle \\
&= \frac{2}{3} + \left(\frac{\sqrt{2}}{2}\right)^{L-1} \left(\frac{1}{2}\log_2(3) - \frac{\sqrt{2}}{4}\log_2(3) - \frac{2}{3} + \frac{\sqrt{2}}{2}\right) \\
&\quad + \left(-\frac{\sqrt{2}}{2}\right)^{L-1} \left(\frac{1}{2}\log_2(3) + \frac{\sqrt{2}}{4}\log_2(3) - \frac{2}{3} - \frac{\sqrt{2}}{2}\right)
\end{aligned}
$$

Example Strictly Sofic Processes

$$h_\mu(L) = \sum_{\lambda \in \Lambda_W} \lambda^{L-1} \langle \delta_\pi | W_\lambda | H(W^{\mathcal{A}}) \rangle \rangle$$

$$= \frac{2}{3} + \left(\frac{\sqrt{2}}{2}\right)^{L-1} \left(\frac{1}{2}\log_2(3) - \frac{\sqrt{2}}{4}\log_2(3) - \frac{2}{3} + \frac{\sqrt{2}}{2}\right)$$

$$+ \left(-\frac{\sqrt{2}}{2}\right)^{L-1} \left(\frac{1}{2}\log_2(3) + \frac{\sqrt{2}}{4}\log_2(3) - \frac{2}{3} - \frac{\sqrt{2}}{2}\right)$$

$$= \begin{cases} \frac{2}{3} + \left(\frac{\sqrt{2}}{2}\right)^{L-1} \left(-\frac{\sqrt{2}}{2}\log_2(3) + \sqrt{2}\right) & \text{for even } L \\ \frac{2}{3} + \left(\frac{\sqrt{2}}{2}\right)^{L-1} \left(\log_2(3) - \frac{4}{3}\right) & \text{for odd } L \end{cases}$$

Example Strictly Sofic Processes

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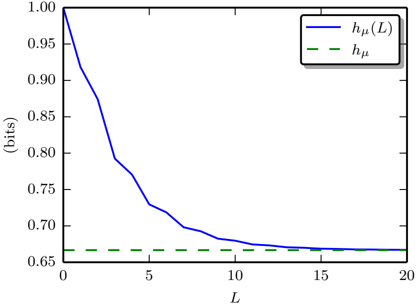

Excess Entropy

For diagonalizable W,

$$\mathbf{E} \equiv \sum_{L=1}^{\infty} [h_\mu(L) - h_\mu]$$

Excess Entropy

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$$\begin{aligned}\mathbf{E} &\equiv \sum_{L=1}^{\infty} [h_\mu(L) - h_\mu] \\ &= \sum_{L=1}^{\infty} \sum_{\substack{\lambda \in \Lambda_W \\ |\lambda|<1}} \lambda^{L-1} \langle \delta_\pi | W_\lambda | H(W^{\mathcal{A}}) \rangle\end{aligned}$$

Excess Entropy

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\mathbf{E} &\equiv \sum_{L=1}^{\infty} \left[h_\mu(L) - h_\mu \right] \\
&= \sum_{L=1}^{\infty} \sum_{\substack{\lambda \in \Lambda_W \\ |\lambda| < 1}} \lambda^{L-1} \langle \delta_\pi | W_\lambda | H(W^{\mathcal{A}}) \rangle \\
&= \sum_{\substack{\lambda \in \Lambda_W \\ |\lambda| < 1}} \langle \delta_\pi | W_\lambda | H(W^{\mathcal{A}}) \rangle \underbrace{\sum_{L=1}^{\infty} \lambda^{L-1}}_{= \sum_{L=0}^{\infty} \lambda^L = \frac{1}{1-\lambda}}
\end{aligned}$$

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&= \sum_{L=1}^{\infty} \sum_{\substack{\lambda \in \Lambda_W \\ |\lambda| < 1}} \lambda^{L-1} \langle \delta_\pi | W_\lambda | H(W^{\mathcal{A}}) \rangle \\
&= \sum_{\substack{\lambda \in \Lambda_W \\ |\lambda| < 1}} \langle \delta_\pi | W_\lambda | H(W^{\mathcal{A}}) \rangle \underbrace{\sum_{L=1}^{\infty} \lambda^{L-1}}_{=\sum_{L=0}^{\infty} \lambda^L = \frac{1}{1-\lambda}} \\
&= \sum_{\substack{\lambda \in \Lambda_W \\ |\lambda| < 1}} \frac{1}{1-\lambda} \langle \delta_\pi | W_\lambda | H(W^{\mathcal{A}}) \rangle
\end{aligned}$$

Alternatively, for any non-diagonalizable W,

$$\begin{aligned}
\mathbf{E} &\equiv \sum_{L=1}^{\infty} \left[h_\mu(L) - h_\mu \right] \\
&= \sum_{L=1}^{\infty} \left[\langle \delta_\pi | \, W^{L-1} \, | H(W^{\mathcal{A}}) \rangle - \langle \delta_\pi | \, W_1 \, | H(W^{\mathcal{A}}) \rangle \right] \\
&= \sum_{L=0}^{\infty} \left[\langle \delta_\pi | \, W^{L} \, | H(W^{\mathcal{A}}) \rangle - \langle \delta_\pi | \, W_1 \, | H(W^{\mathcal{A}}) \rangle \right] \\
&= \sum_{L=0}^{\infty} \langle \delta_\pi | \left[(\underbrace{W - W_1}_{\equiv Q})^L - \delta_{L,0} W_1 \right] | H(W^{\mathcal{A}}) \rangle \\
&= - \underbrace{\langle \delta_\pi | \, W_1 \, | H(W^{\mathcal{A}}) \rangle}_{= \langle \pi_W | H(W^{\mathcal{A}}) \rangle = h_\mu} + \sum_{L=0}^{\infty} \underbrace{\langle \delta_\pi | \, Q^L}_{= \langle \delta_\pi | \mathcal{Q}^L} \, | H(W^{\mathcal{A}}) \rangle \\
&= \langle \delta_\pi | \left(\sum_{L=0}^{\infty} \mathcal{Q}^L \right) | H(W^{\mathcal{A}}) \rangle - h_\mu \\
&= \langle \delta_\pi | \, (I - \mathcal{Q})^{-1} \, | H(W^{\mathcal{A}}) \rangle - h_\mu \\
&= \langle \delta_\pi | \, (I - Q)^{-1} \, | H(W^{\mathcal{A}}) \rangle - h_\mu \; .
\end{aligned}$$

Similar statements result for all of the familiar complexity measures; **T**, **S**, and so on.

Interestingly, they all appear more alike than ever, yielding new insight for how they are all related. E.g.:

$$\mathbf{E} = \langle \delta_\pi | \left(I - Q\right)^{-1} | H(W^{\mathcal{A}}) \rangle - h_\mu ,$$

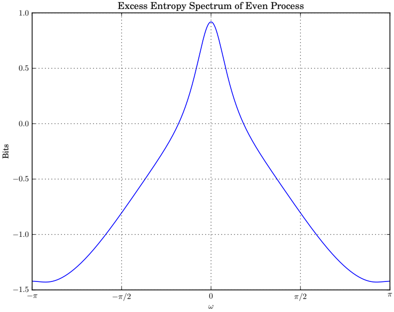
$$\mathbf{S} = \langle \delta_\pi | \left(I - Q\right)^{-1} | H[\eta] \rangle - \mathcal{H} ,$$

and

$$\mathbf{T} = \langle \delta_\pi | \left(I - Q\right)^{-2} | H(W^{\mathcal{A}}) \rangle - h_\mu .$$

Complexity Measures Anew: Complexity Spectra

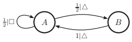
$$\mathbf{E}(\omega) \equiv \mathrm{Re} \langle \delta_\pi | \left(e^{i\omega} I - Q\right)^{-1} | H(W^{\mathcal{A}}) \rangle - h_\mu$$



Thank you!

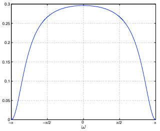
Example: Complexity Measures for a Prototypic Sofic System: The Even Process

$\mathcal{M} = (\mathcal{A} = \{\square, \triangle\}, \boldsymbol{\mathcal{S}} = \{A, B\}, T^{\mathcal{A}})$.

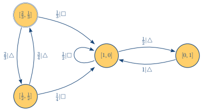


For $\square = 0$ and $\triangle = 1$, we find the continuous part of the power spectrum to be:

$$P_c(\omega) = \frac{1}{3}\left(1 - \frac{1}{5 + 4\cos\omega}\right).$$



$\mathcal{M}_{\text{MSP}} = (\mathcal{A} = \{\square, \triangle\}, \mathcal{R}, W^{\mathcal{A}})$, with $\mathcal{R} = \{(\frac{2}{3}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{2}), (1, 0), (0, 1)\}$.



$$W = \sum_{x \in \mathcal{A}} W^{(x)} = \begin{bmatrix} 0 & 2/3 & 1/3 & 0 \\ 3/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

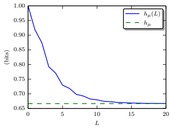
Solving $\det(\lambda I - W) = 0$ gives W's eigenvalues:

$$\Lambda_W = \left\{1, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\frac{1}{2}\right\}.$$

$$\mathcal{H}(L) = \begin{cases} (\log_2(3) - 2/3)\left(\frac{\sqrt{2}}{2}\right)^L & \text{for even } L \\ \frac{2\sqrt{2}}{3}\left(\frac{\sqrt{2}}{2}\right)^L & \text{for odd } L \end{cases}.$$

$$h_\mu(L) = \begin{cases} \frac{2}{3} + \left(\frac{\sqrt{2}}{2}\right)^{L-1}\left(-\frac{\sqrt{2}}{2}\log_2(3) + \sqrt{2}\right) & \text{for even } L \\ \frac{2}{3} + \left(\frac{\sqrt{2}}{2}\right)^{L-1}\left(\log_2(3) - \frac{4}{3}\right) & \text{for odd } L \end{cases}.$$

$h_\mu = 2/3$ bits per step,
$C_\mu = \log_2(3) - \frac{2}{3}$ bits,
$\mathbf{E} = \log_2(3) - \frac{2}{3}$ bits,
$\mathbf{T} = 2\log_2(3)$ bits-symbols,
$\mathbf{S} = 2\log_2(3)$ bits.



In short:

$$\begin{aligned} h_\mu(L) &= H(L) - H(L-1) \\ &= H[X_{L-1} | X_{0:L-1}] \\ &= H\big[X_{L-1} | \big(\mathcal{R}_{L-1} | \mathcal{R}_0 = \pi\big)\big] \\ &= \langle \delta_\pi | \, W^{L-1} \, | H(W^{\mathcal{A}}) \rangle \, , \end{aligned}$$

where
$|H(W^{\mathcal{A}})\rangle = -\sum_{\eta \in \boldsymbol{\mathcal{R}}} |\delta_\eta\rangle \sum_{x \in \mathcal{A}} \langle \delta_\eta | \, W^{(x)} \, |\mathbf{1}\rangle \log_2 \langle \delta_\eta | \, W^{(x)} \, |\mathbf{1}\rangle$.

$$h_\mu(L) = \langle \delta_\pi | \, W_1 \, | H(W^{\mathcal{A}}) \rangle + \sum_{\substack{\lambda \in \Lambda_W \\ |\lambda| < 1}} \lambda^{L-1} \langle \delta_\pi | \, W_\lambda \, | H(W^{\mathcal{A}}) \rangle$$

for diagonalizable W.