Example Dynamical Systems

Reading for this lecture:

NDAC, Sec. 5.0-5.2, 6.0-6.4, 7.0-7.3, & 9.0-9.4

The Pendulum

Example Dynamical Systems ...

1D Flows: Fixed Points

model of static equilibrium

1D Flow: $x \in \mathbb{R}$

$$\dot{x} = F(x)$$

Fixed Points:

$x^* \in \mathbb{R}$ such that

$$\dot{x} \,\big|_{x^*} = 0$$

or

$$F(x^*) = 0$$

Example Dynamical Systems ...

1D Flows: Fixed Points ...

Stability: What is linearized system at x ?

Investigate evolution of perturbations: $x' = x + \delta x$

Local Flow: $\delta \dot{x} = \left. \frac{dF}{dx} \right|_{x(t)} \delta x$

Local Linear System: $\delta \dot{x} = \lambda \, \delta x$

Solution: $\delta x(t) \propto e^{\lambda t} \delta x(0)$

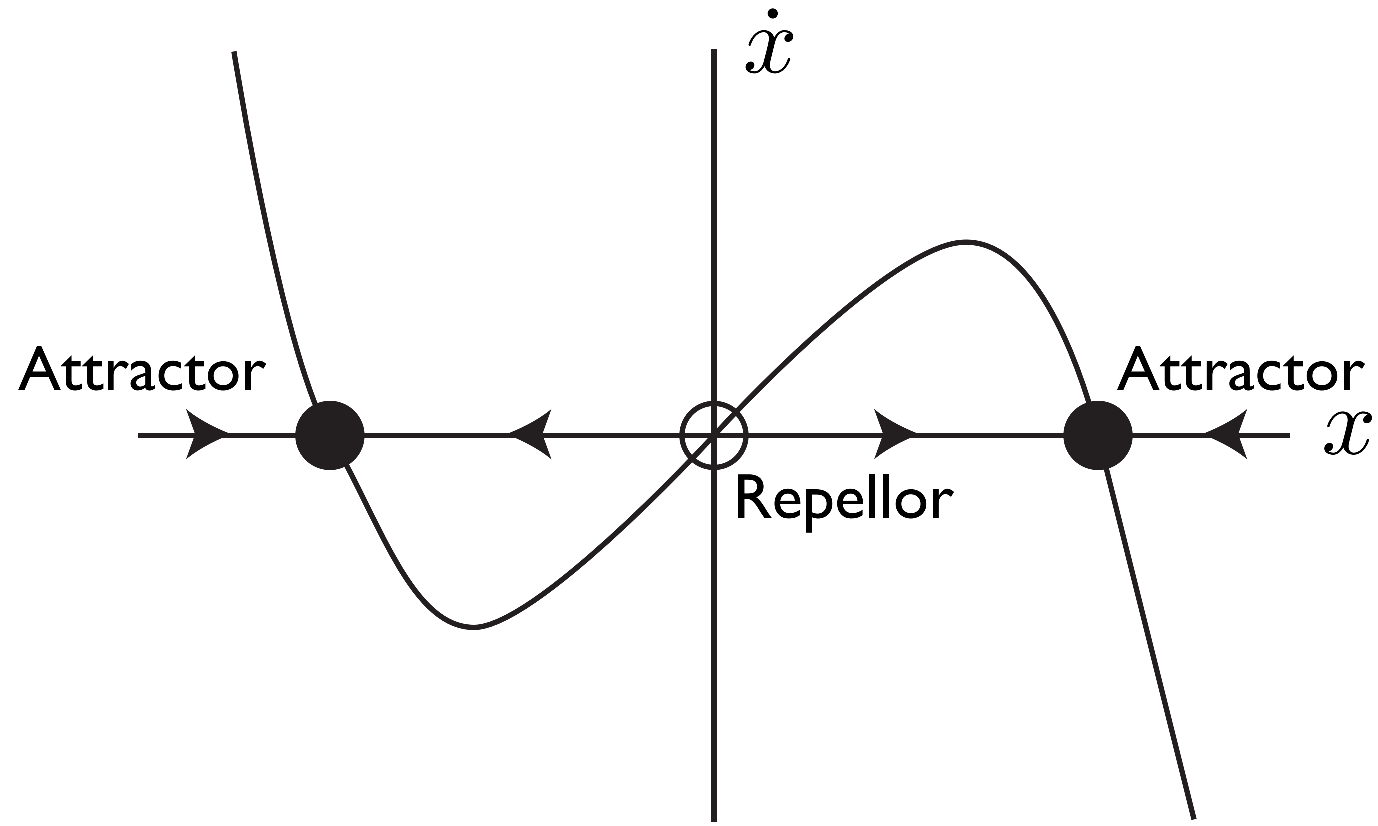
Example Dynamical Systems ...
1D Flows ...
Stability Classification of Fixed Points:

Slope λ of $F(x)$ at x :

1. Stable: $\lambda < 0$

2. Unstable: $\lambda > 0$

3. Neutral: $\lambda = 0$



Example Dynamical Systems ...

2D Flows: Fixed Points

model of static equilibrium

2D Flow: $\vec{x} \in \mathbf{R}^2$

$$\dot{\vec{x}} = \vec{F}(\vec{x})$$

or

$$\begin{aligned} \vec{x} &= (x, y) \\ \vec{F} &= (f, g) \end{aligned}$$

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

Fixed Points:

(x^*, y^*) such that

$$\dot{\vec{x}}|_{(x^*, y^*)} = (0, 0)$$

or

$$0 = f(x^*, y^*)$$

$$0 = g(x^*, y^*)$$

Example Dynamical Systems ...

2D Flows: Fixed Points ...

Stability: What is linearized system at $\vec{x}$?

Investigate evolution of perturbations δx : $\vec{x}' = \vec{x} + \delta\vec{x}$

$$\dot{\vec{x}} = \vec{F}(\vec{x})$$

Local Flow: $\delta\dot{\vec{x}} = \left. \frac{\partial \vec{F}}{\partial \vec{x}} \right|_{\vec{x}(t)} \cdot \delta\vec{x}$

Initial conditions: $x(0) \quad \delta x(0)$

Example Dynamical Systems ...

2D Flows: Fixed Points ...

Local Linear System: $\delta\dot{\vec{x}} = A \cdot \delta\vec{x}$

Jacobian: $A = \dfrac{\partial \vec{F}}{\partial \vec{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$

Solution:

$$\delta\vec{x}(t) \propto e^{At}\delta\vec{x}(0)$$

Example Dynamical Systems ...

2D Flows: Fixed Points (an aside) ...

Solve linear ODEs: Find $\vec{x}(t)$ given

$$\vec{x}(0)$$

$$\dot{\vec{x}} = A\vec{x}$$

Eigenvalues and eigenvectors: λ_j and $\vec{v}_j$:

$$A\vec{v}_j = \lambda_j \vec{v}_j, \;\; j = 1, 2$$

Solution:

$$\vec{x}(t) = \sum_{j=1}^{2} \alpha_j e^{\lambda_j t} \vec{v}_j$$

where calculate α_j so that:

$$\vec{x}(0) = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2$$

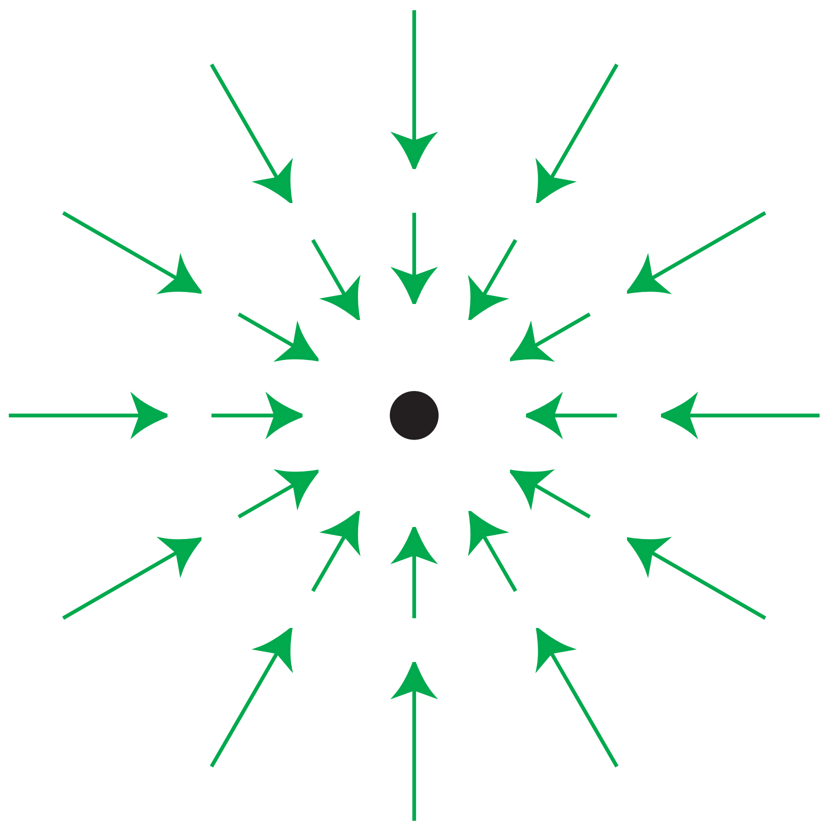
Example Dynamical Systems ...
2D Flows ...
Stability Classification of Fixed Points:

Eigenvalues of Jacobian A at $\vec{x}$: $\lambda_1 \;\&\; \lambda_2 \;\in\; \mathbf{C}$

(Review: *NDAC*, Chapter 5)

Stable fixed point (aka sink, attractor):

$$\Re(\lambda_1), \Re(\lambda_2) < 0$$



Example Dynamical Systems ...

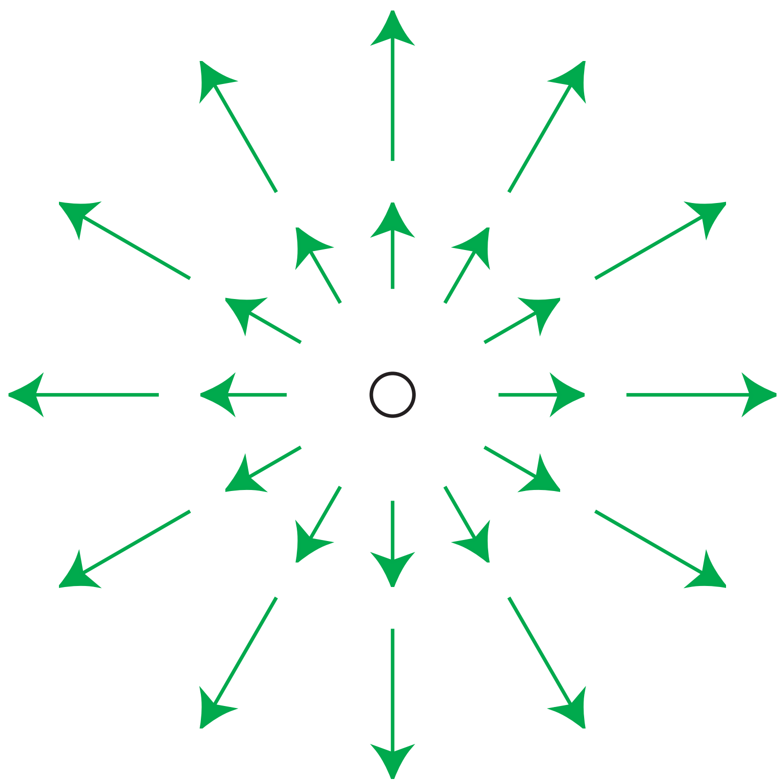
2D Flows ...

Stability Classification of Fixed Points ...

Eigenvalues of Jacobian A at $\vec{x}$: $\lambda_1 \ \& \ \lambda_2 \ \in \ \mathbf{C}$

Unstable fixed point (aka source, repellor):

$$\Re(\lambda_1), \Re(\lambda_2) > 0$$

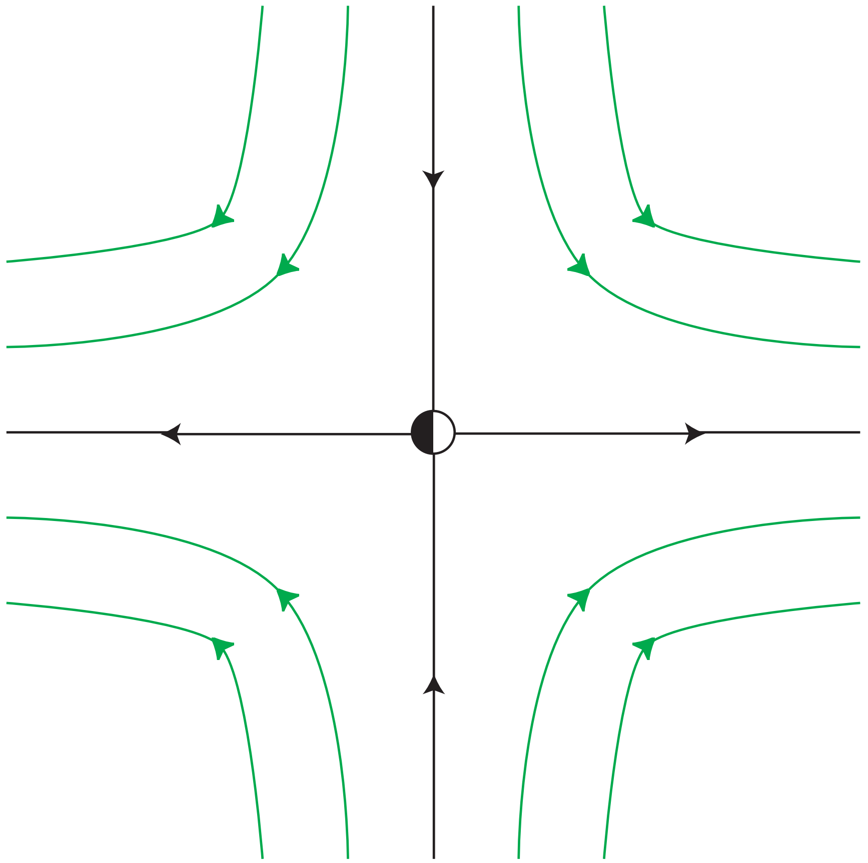


Example Dynamical Systems ...
2D Flows ...
Stability Classification of Fixed Points:

Eigenvalues of Jacobian at $\vec{x}$: λ_1 & $\lambda_2 \in \mathbf{C}$

Saddle fixed point (mixed stability):

$$\Re(\lambda_1) > 0 \;\&\; \Re(\lambda_2) < 0$$

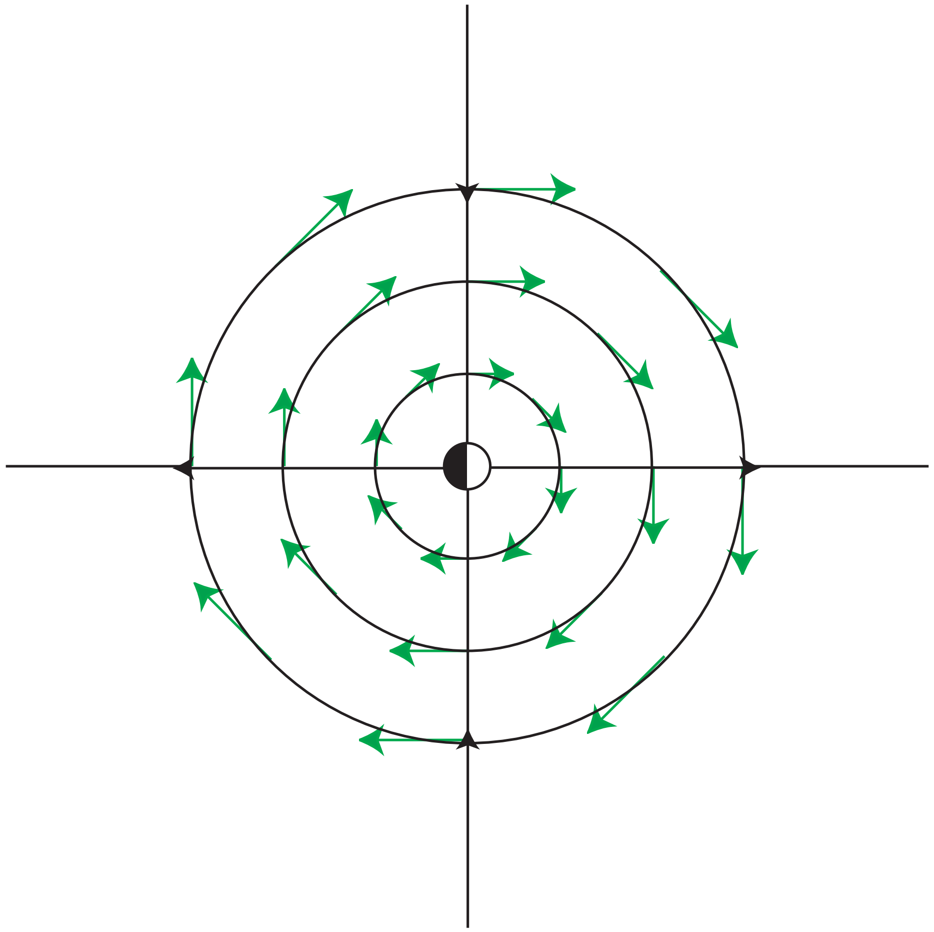


Example Dynamical Systems ...
2D Flows ...
Stability Classification of Fixed Points:

Eigenvalues of Jacobian at $\vec{x}$: $\lambda_1 \;\&\; \lambda_2 \;\in\; \mathbf{C}$

Center:

$$\Re(\lambda_1) = \Re(\lambda_2) = 0$$



Example Dynamical Systems ...

2D Flows ...

Stability Classification of Fixed Points ...

Magnitude of (in)stability: $\mathrm{Det}(A) = \lambda_1 \cdot \lambda_2$

$\mathrm{Det}(A) < 0 : \lambda_1, \lambda_2 \in \mathbf{R},\ \ \lambda_1 > 0 \Rightarrow \lambda_2 < 0$ Saddles

$\mathrm{Det}(A) > 0 :$

Stable: $\mathrm{Tr}(A) < 0$

$\mathrm{Tr}(A) = \lambda_1 + \lambda_2$

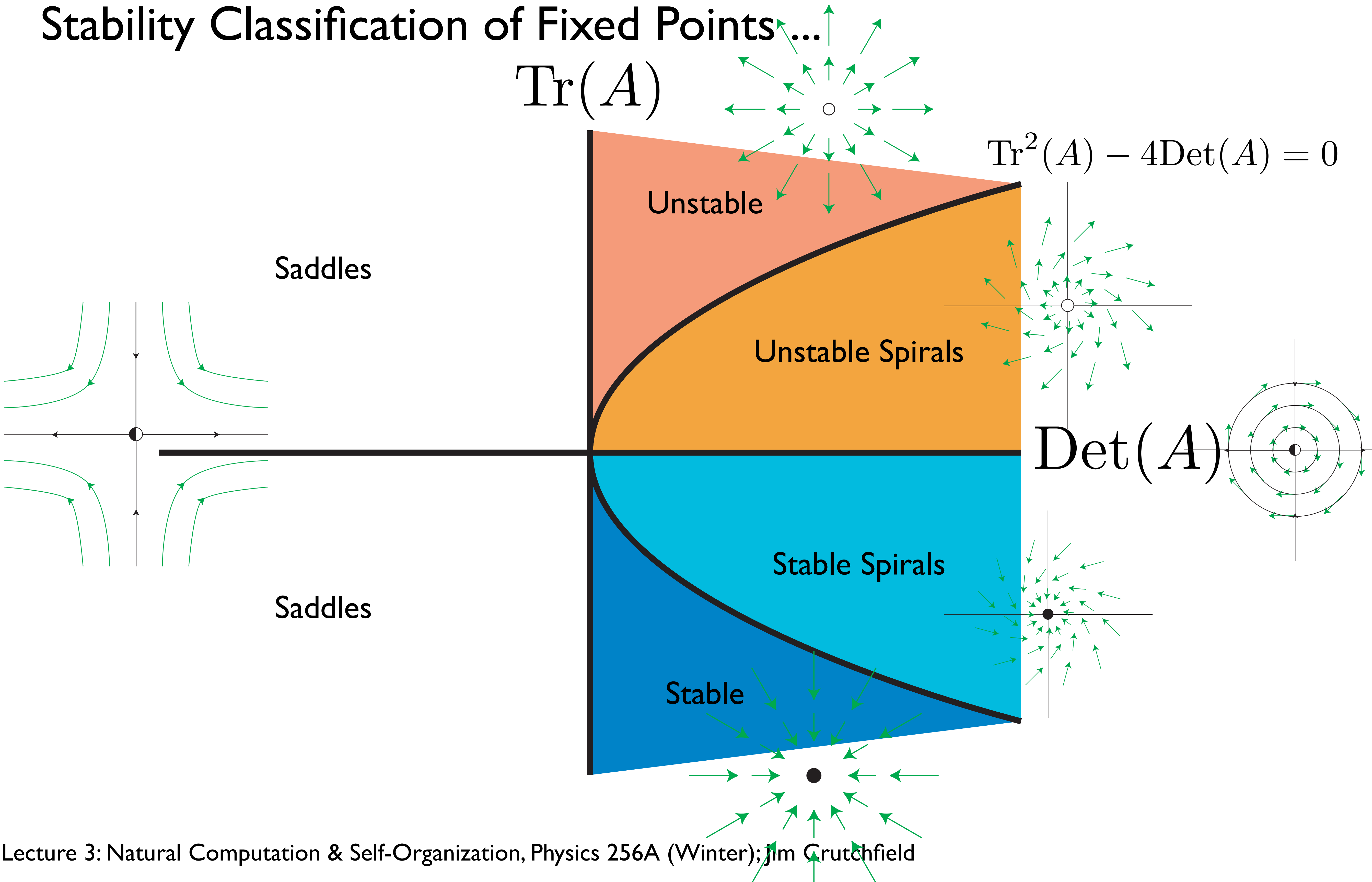
Unstable: $\mathrm{Tr}(A) > 0$

Marginal: $\mathrm{Tr}(A) = 0$

Example Dynamical Systems ...

2D Flows ...

Stability Classification of Fixed Points ...



Example Dynamical Systems ...

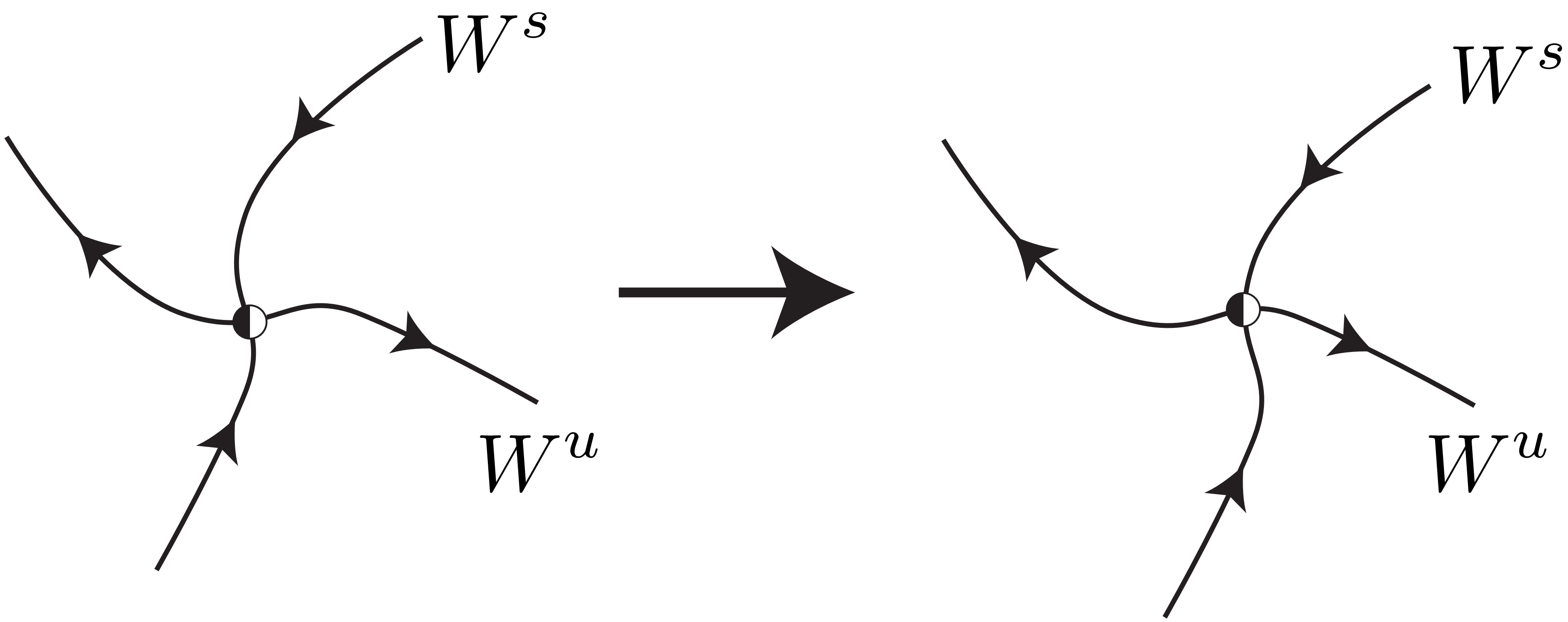
2D Flows ...

Stability Classification of Fixed Points ...

Hyperbolic intersection of W^s and W^u:

Robust, if $\Re(\lambda_i) \neq 0, \forall i$

Fixed point persists under perturbation



Example Dynamical Systems ...

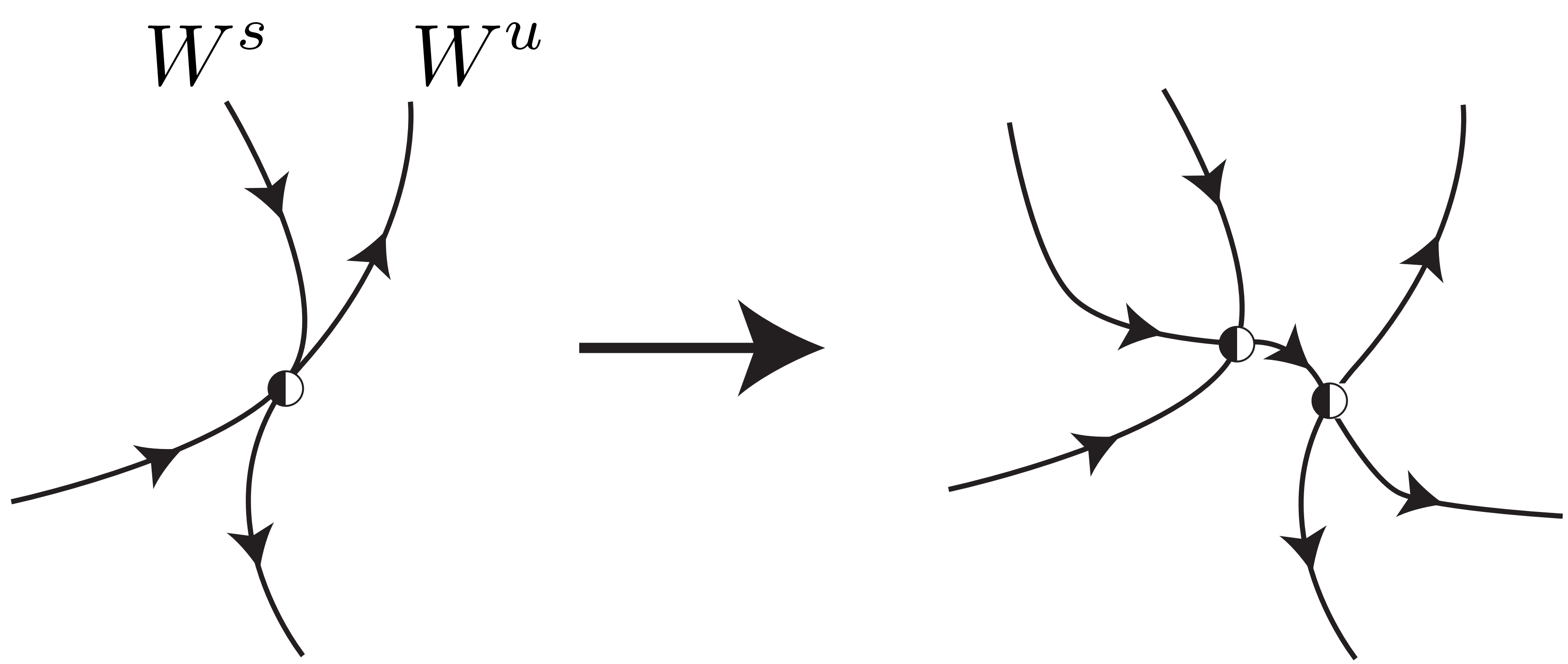
2D Flows ...

Stability Classification of Fixed Points ...

Non-hyperbolic intersection of W^s and W^u:

Fragile

Fixed point changes structure under perturbation



Example Dynamical Systems ...

2D Flows: Limit Cycles

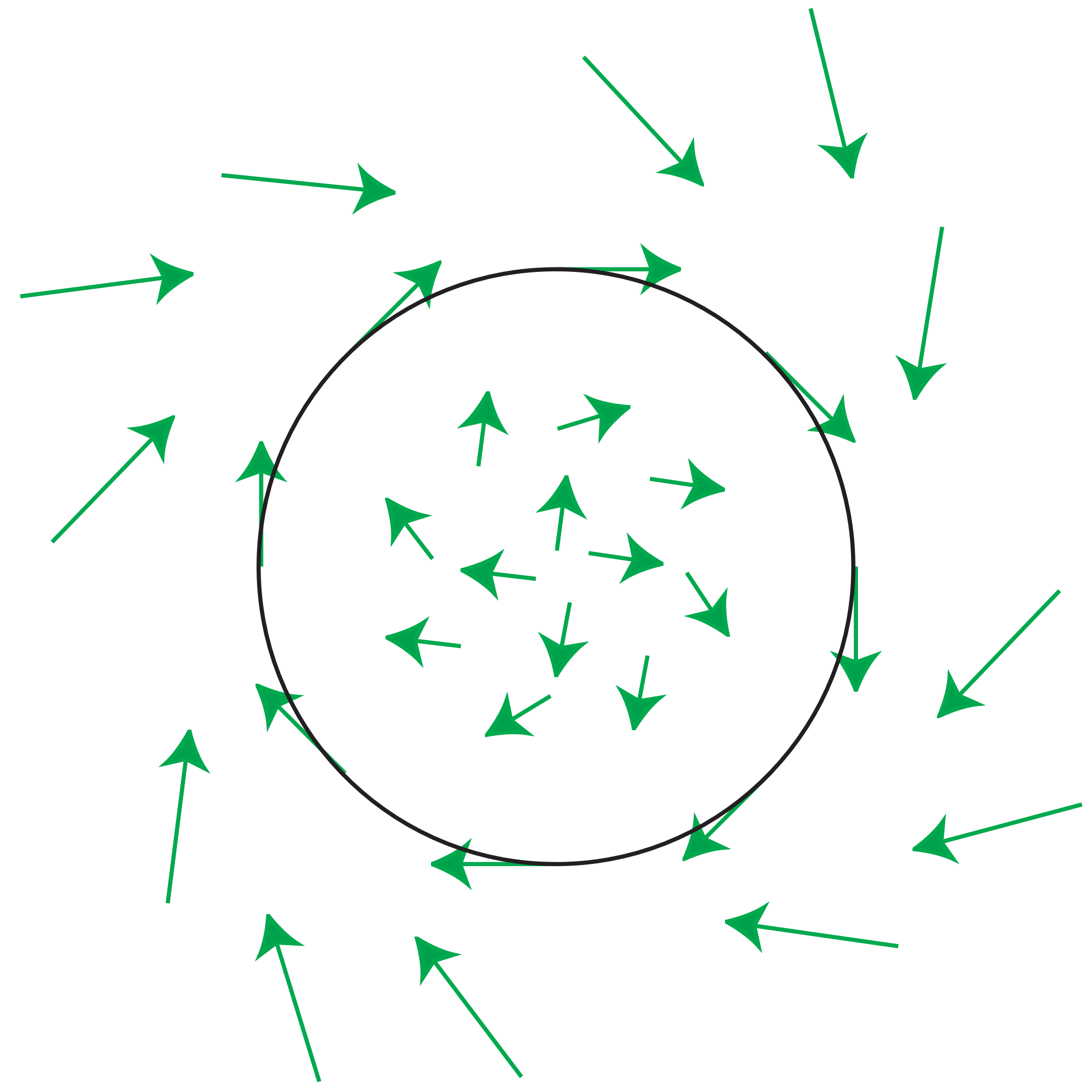
isolated, closed trajectory:

a periodic orbit: $\vec{x}(t) = \vec{x}(t+p), \text{ for all } t$

(p is the period)

model of stable oscillation
this is a new behavior type
not possible in 1D flows

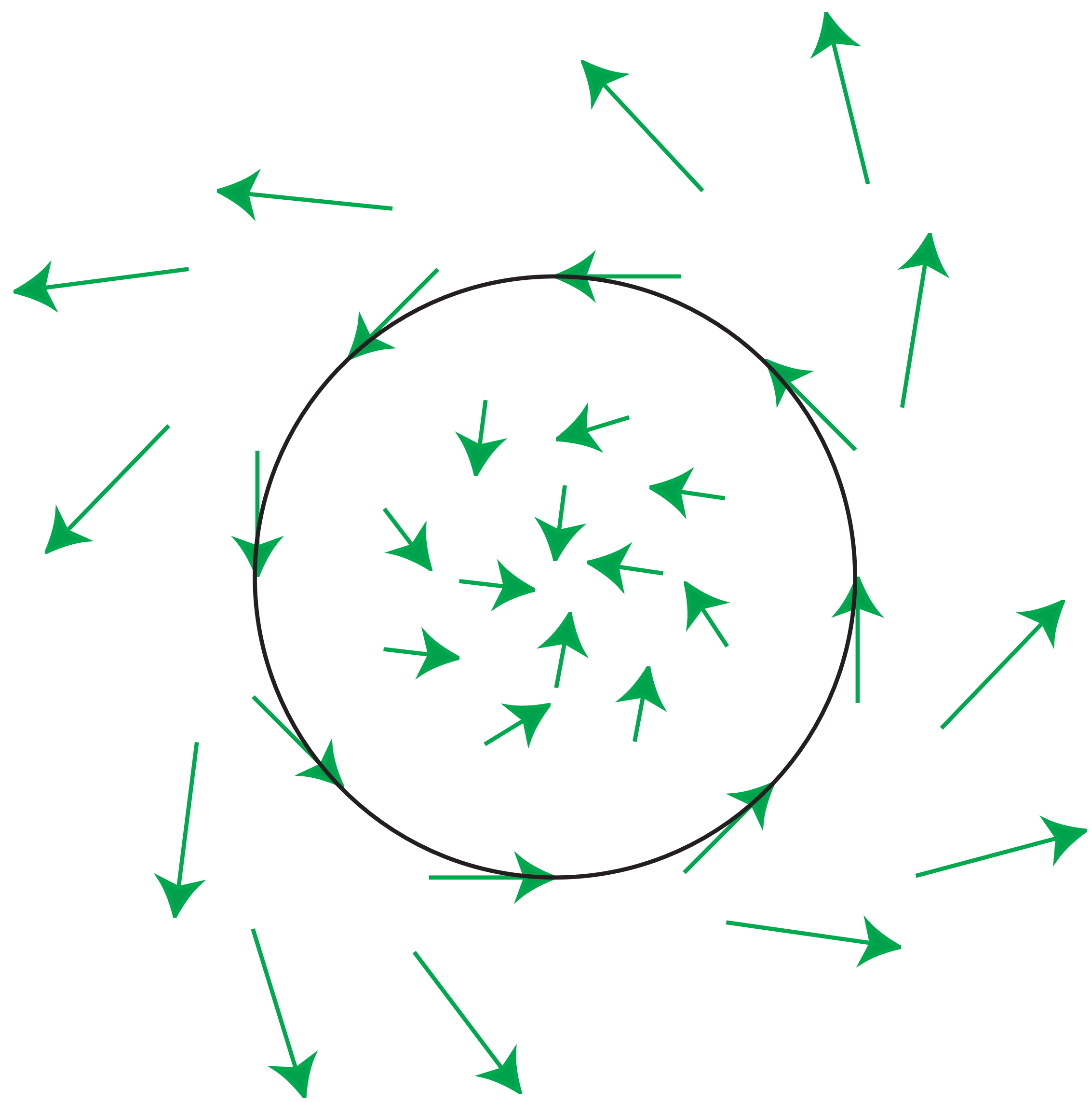
Stable limit cycle



Example Dynamical Systems ...

2D Flows: Limit Cycles ...

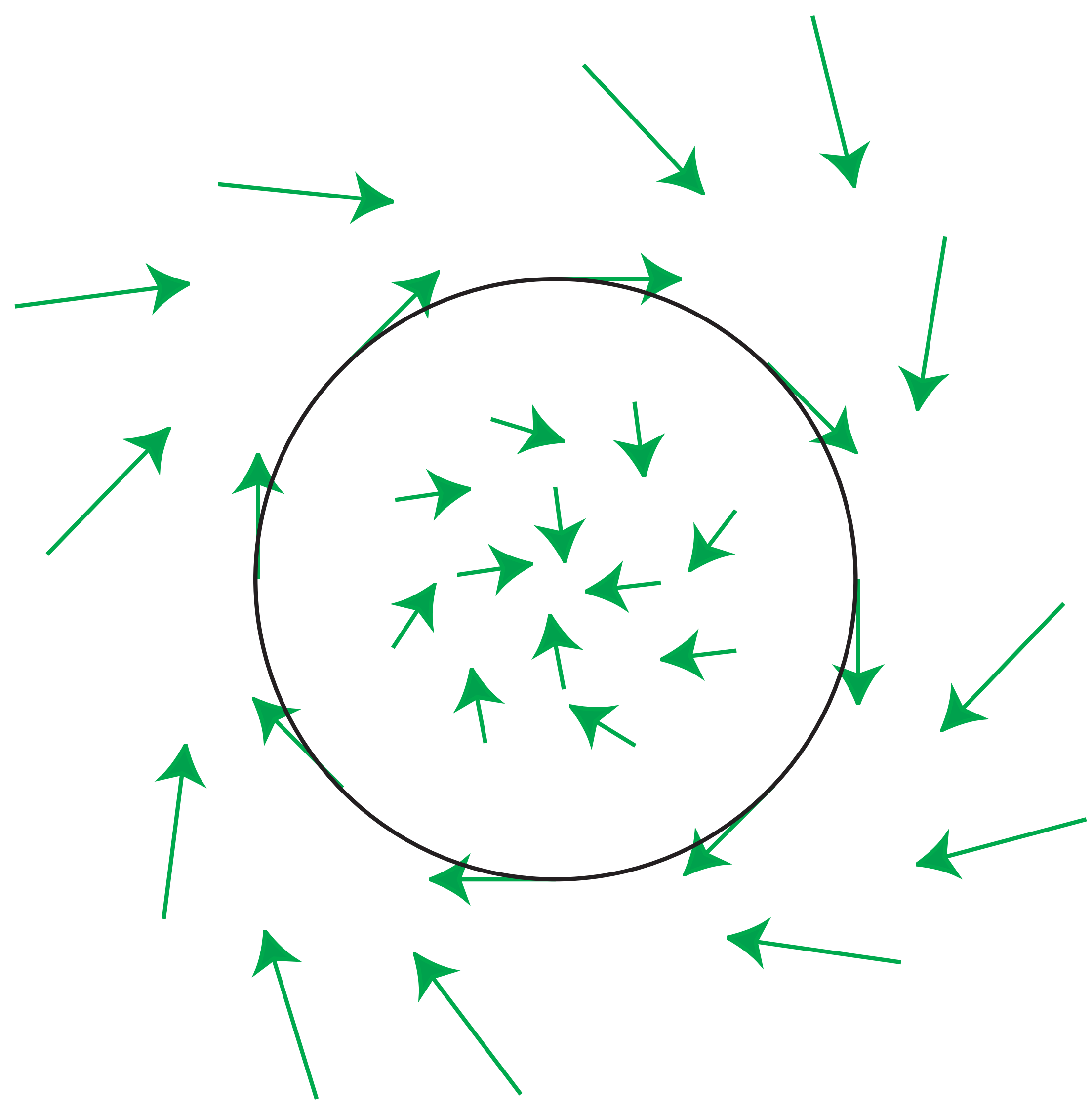
Unstable cycle



Example Dynamical Systems ...

2D Flows: Limit Cycles ...

Saddle cycle



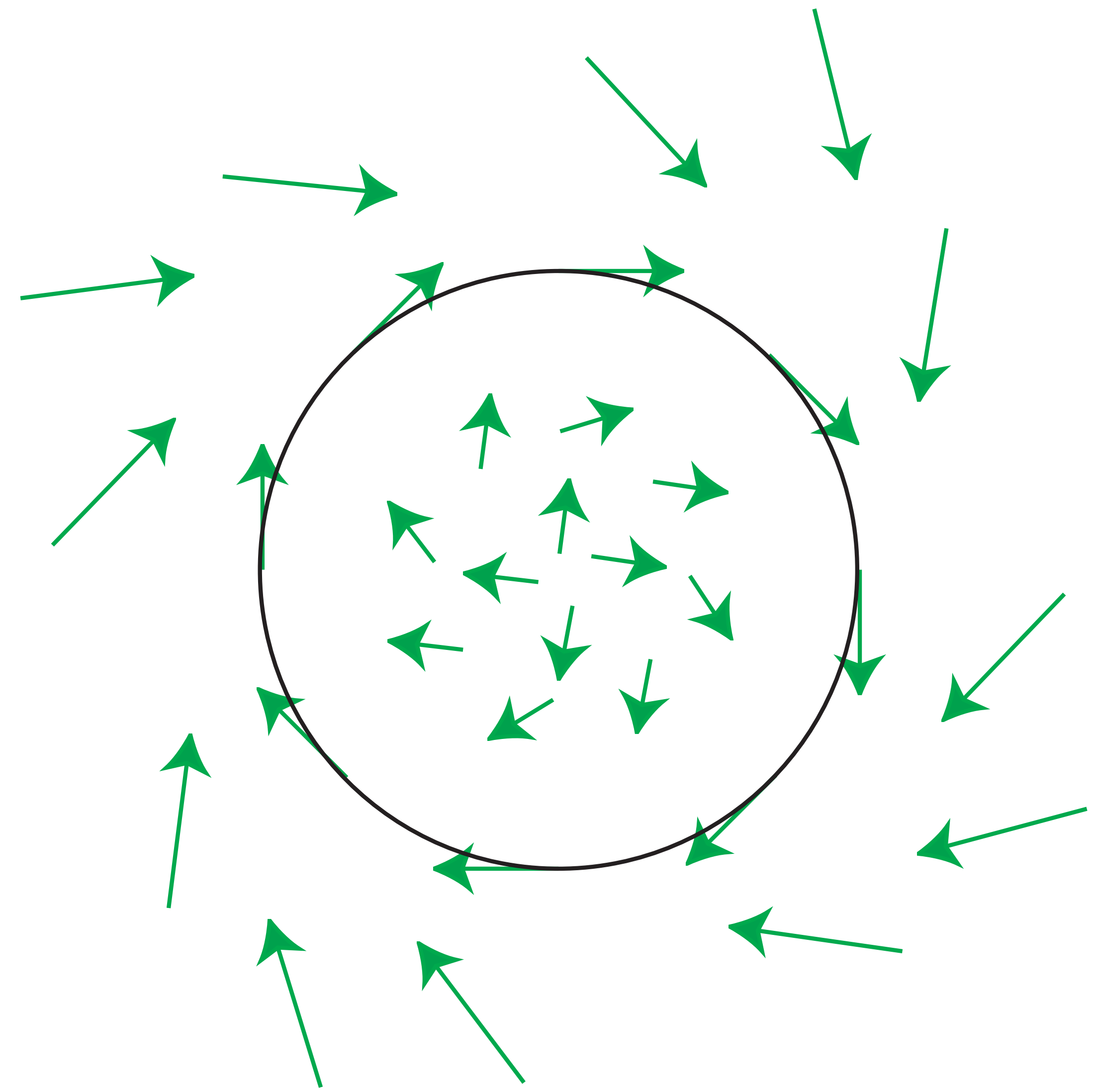
Example Dynamical Systems ...

2D Flows ...

Limit Cycle Examples

Easy in polar coordinates:

$$\begin{aligned} \dot{r} &= r(1 - r^2) \\ \dot{\theta} &= 1 \end{aligned}$$



Example Dynamical Systems ...

2D Flows ...

Limit Cycle Examples ...

Van der Pol Equations:

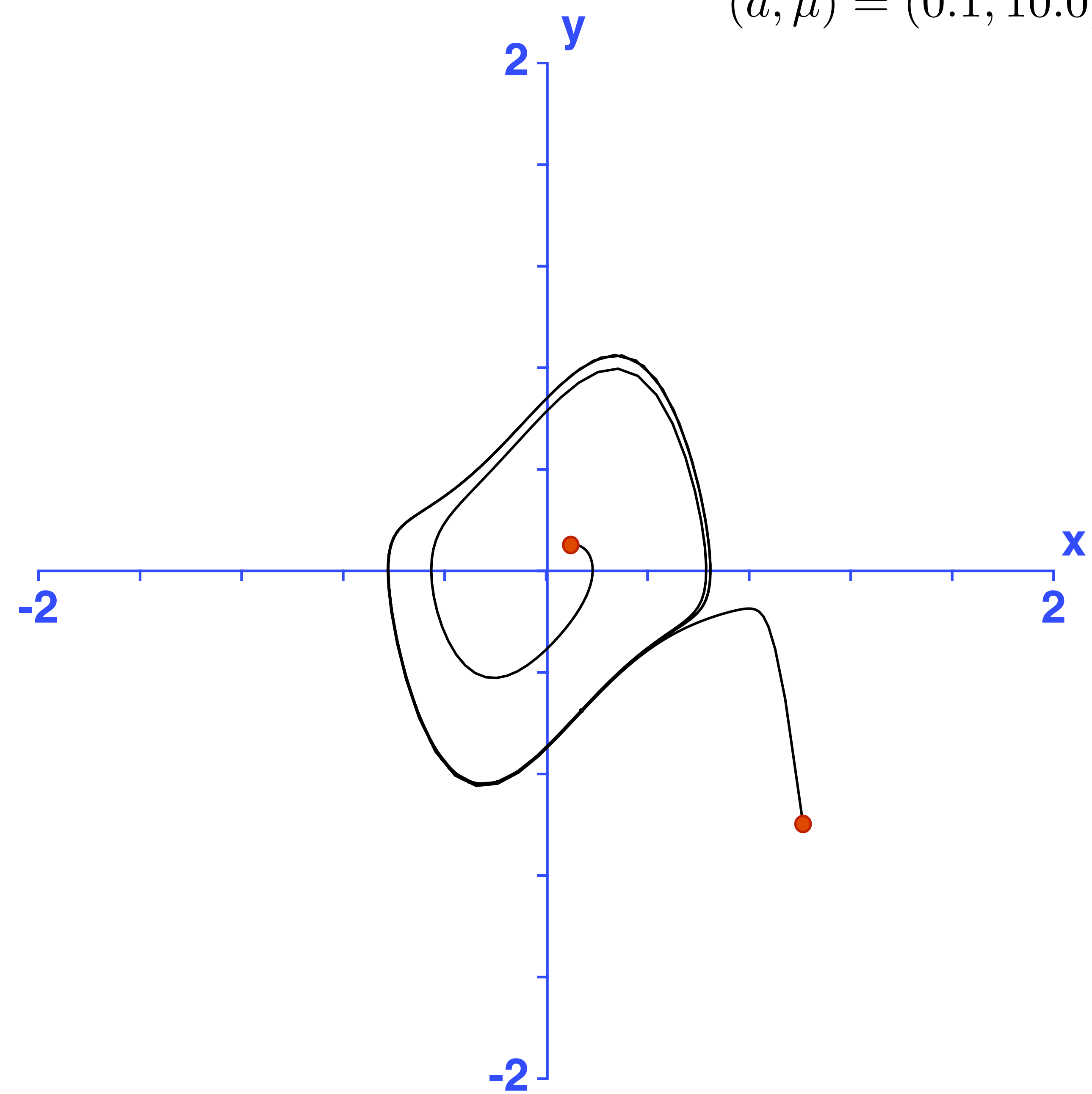
$$\ddot{x} + \mu(x^2 - a)\dot{x} + x = 0$$

or

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \mu y(a - x^2) \end{aligned}$$

Nonlinear damping changes sign:
 Small oscillation: growth
 Large oscillation: damped

$(a, \mu) = (0.1, 10.0)$



Example Dynamical Systems ...

2D Flows ...

Limit cycle existence
(requires real work to show!)

Systems that can't have stable oscillations:

1. Simple harmonic oscillator
2. Gradient systems: $\dot{\vec{x}} = -\nabla V(\vec{x})$
3. Lyapunov systems

Example Dynamical Systems ...

2D Flows ...

Limit cycle existence

(requires real work to show!)

How to find limit cycles?

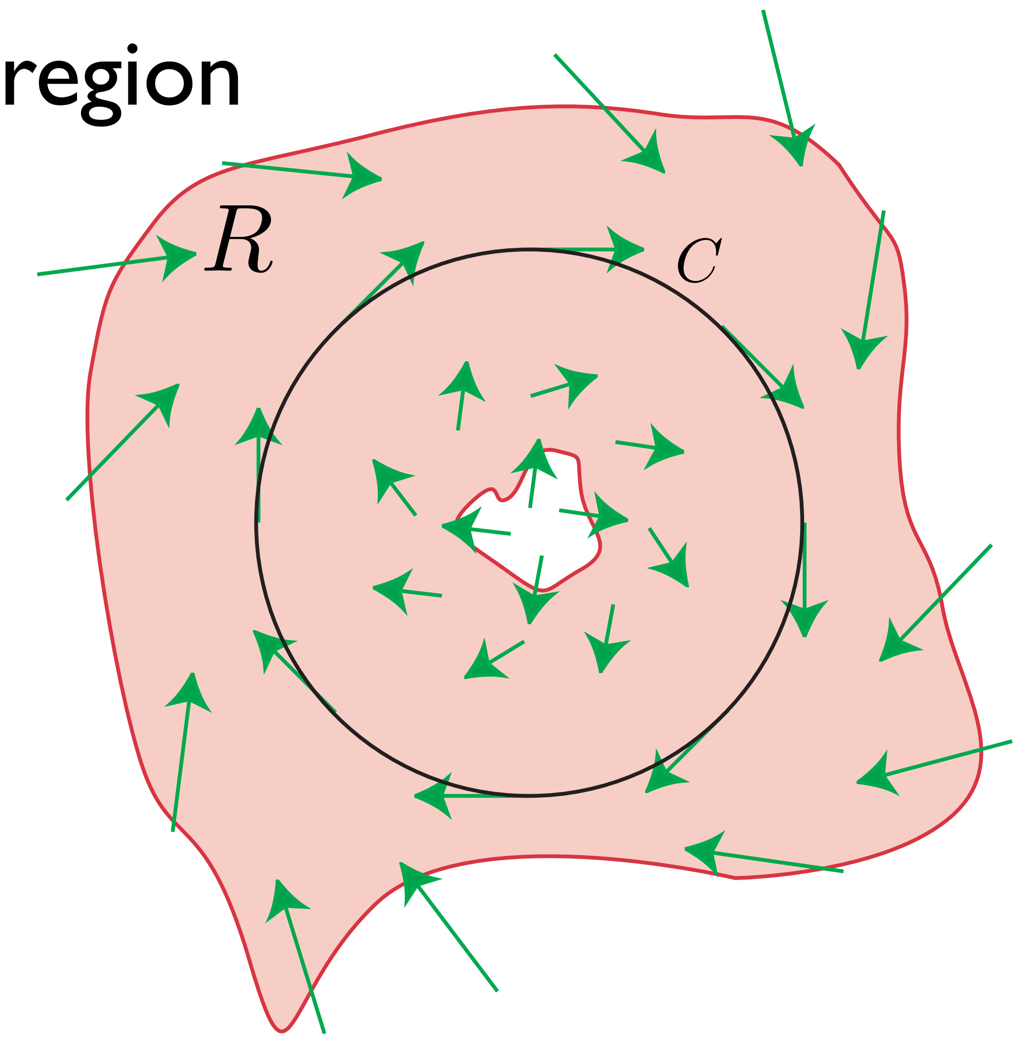
Poincaré-Bendixson Theorem:

(a) trajectory confined to trapping region

(b) no fixed points

then have limit cycle C

somewhere inside R.



Example Dynamical Systems ...

3D Flows:

Fixed points

Limit cycles

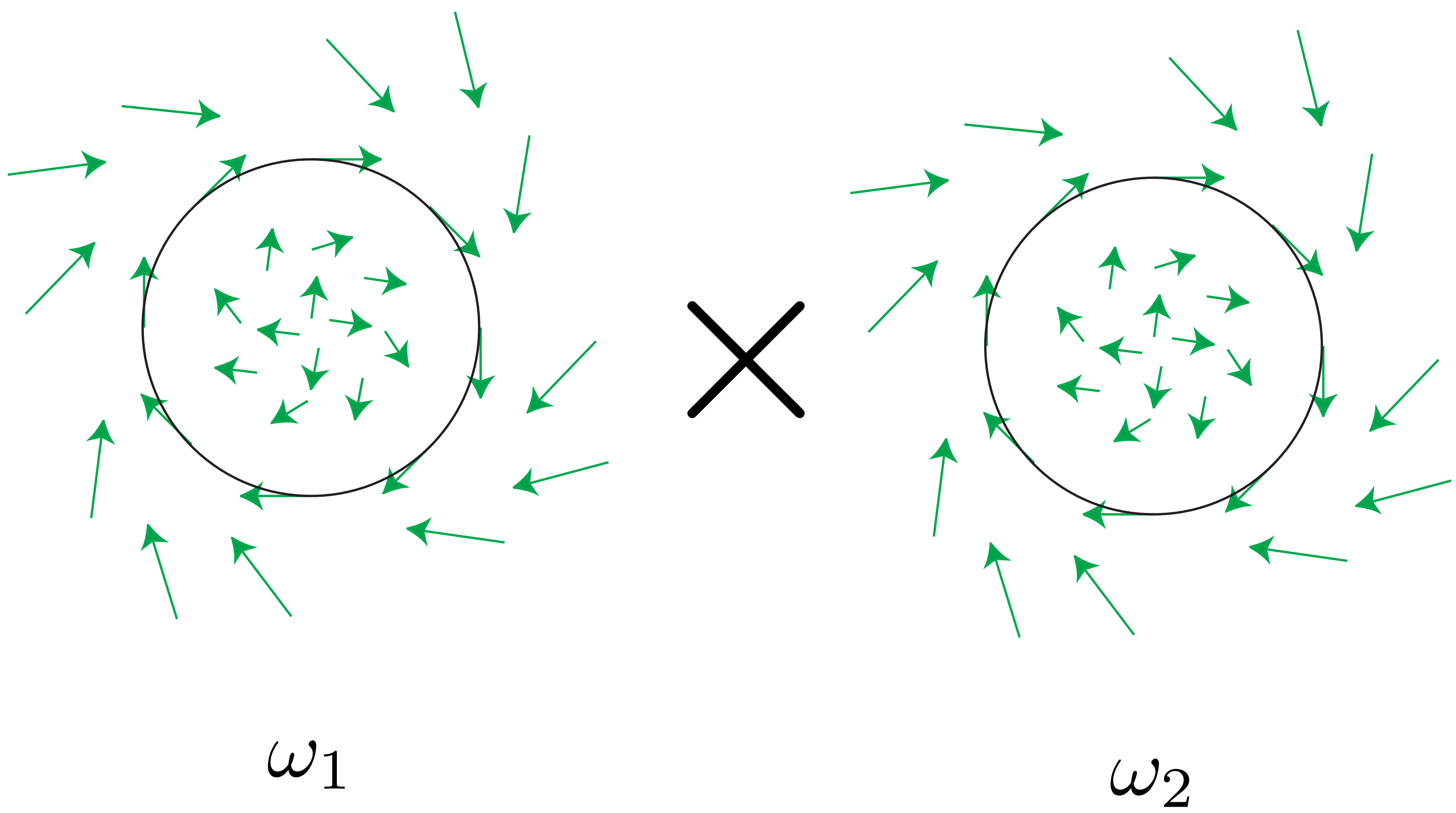
and ... ?

Example Dynamical Systems ...

3D Flows: Quasiperiodicity

product of two limit cycles:

two irrational frequencies $\omega_1 \neq \omega_2$

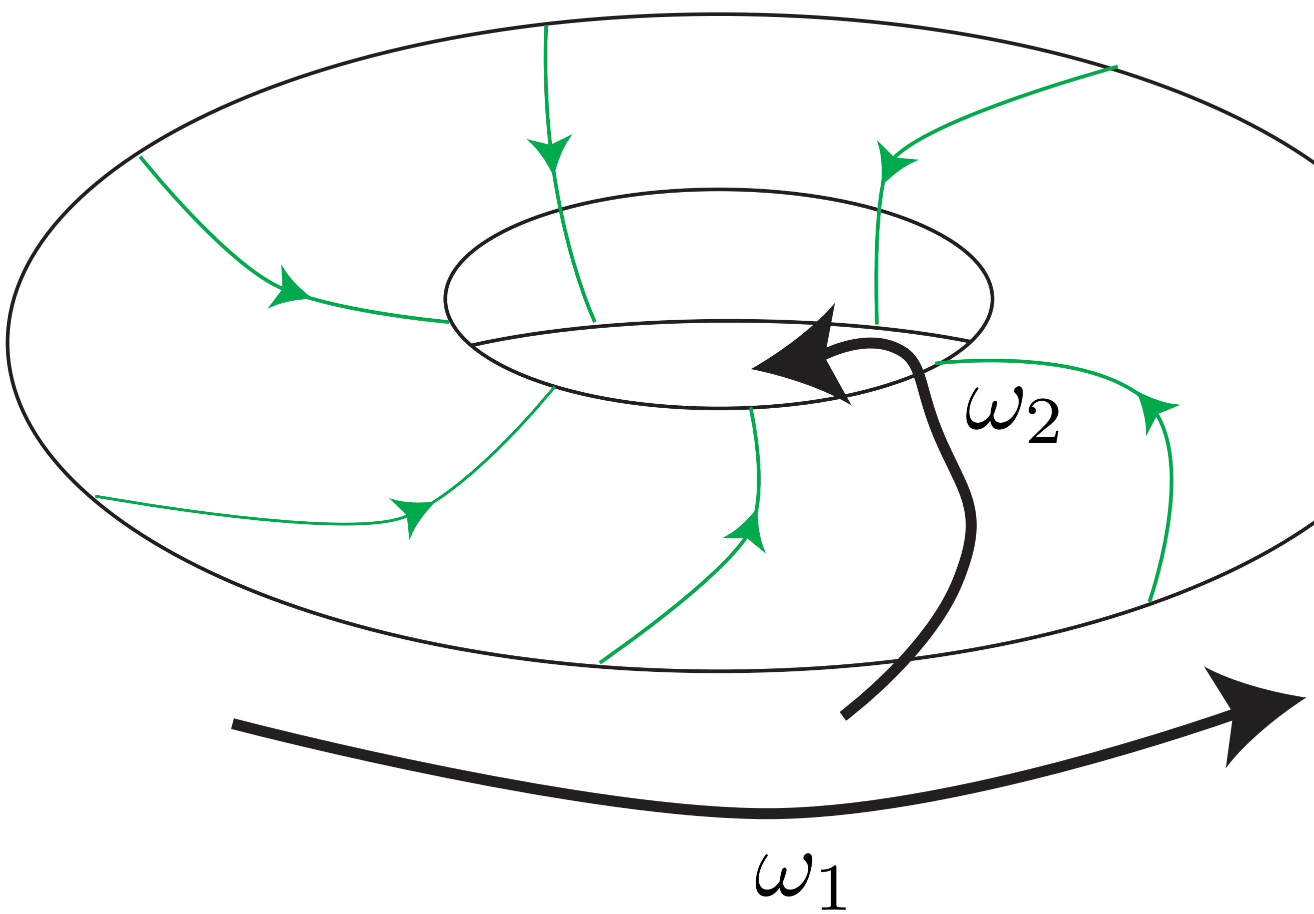


Example Dynamical Systems ...

3D Flows: Quasiperiodicity ...

a new kind of behavior *not possible* in 1D or 2D

Torus attractor



$\omega_1 \neq \omega_2$

Example Dynamical Systems ...

3D Flows: Chaos

recurrent instability

one way to do this:
Orbit reinjection near unstable fixed point

not possible in lower D flows

a new behavior type

Example Dynamical Systems ...

3D Flows: Chaos ...

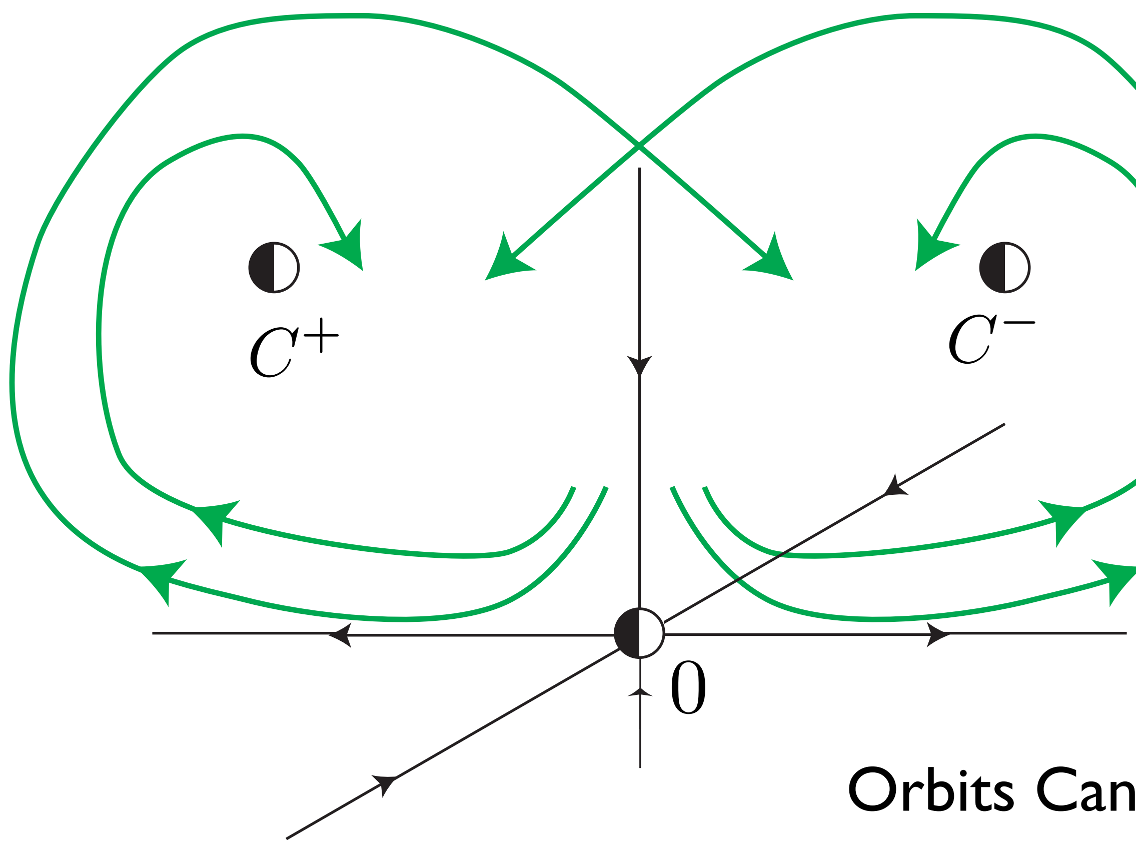
A topological construction:

saddle fixed point at origin: 0

1D unstable manifold: $\dim(W^u(\mathbf{0})) = 1$

2D stable manifold: $\dim(W^s(\mathbf{0})) = 2$

two fixed points: C^+ & C^-



Orbits Cannot Cross: Need 3D!

Does any ODE implement this flow design?

Example Dynamical Systems ...

3D Flows: Chaos ...

Does any ODE implement this design?

Yes, the Lorenz equations:

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned}$$

Parameters: $\sigma, r, b > 0$

Exercise: Show fixed point at the origin can be a saddle, with 2 stable and 1 unstable directions

Exercise: Show there is a symmetry $(x, y) \to (-x, -y)$

Example Dynamical Systems ...

3D Flows: Chaos ...

Lorenz ODE properties:

Trajectories stay in a bounded region near origin

No stable fixed points or stable limit cycles inside

Volume shrinks to zero (everywhere inside):

$$\dot{V} = \int_{\text{region}} dV \; \nabla \cdot \vec{F}(\vec{x})$$

$$\nabla \cdot \vec{F}(\vec{x}) = \mathrm{Tr}(A) = -\sigma - 1 - b$$

$$\dot{V} = -(\sigma + 1 + b)V$$

$$V(t) = e^{-(\sigma+1+b)t}$$

Region volume shrinks exponentially fast!

What does the invariant set look like?

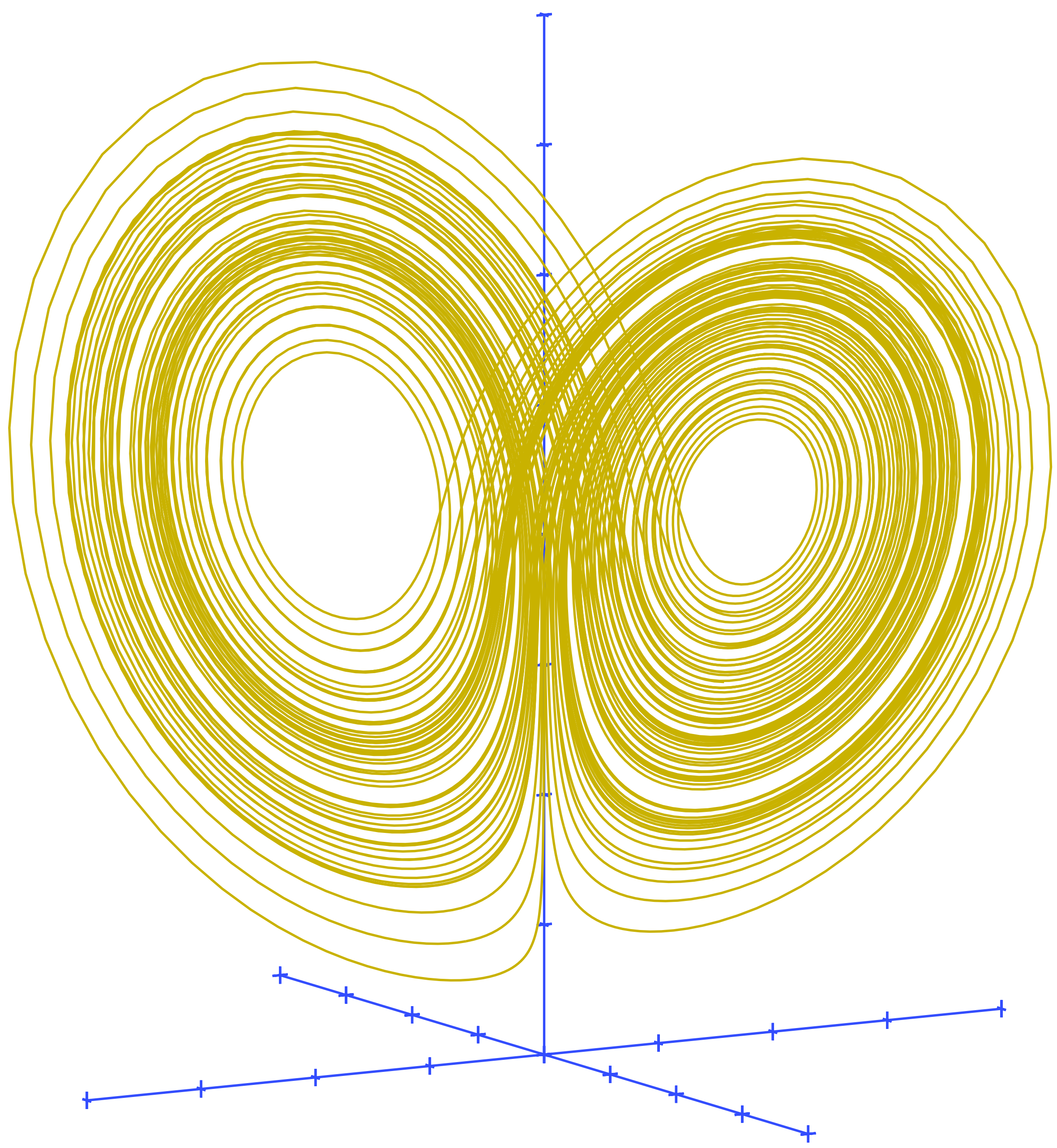
Example Dynamical Systems ...

3D Flows: Chaos ...

Lorenz simulation demo:

- fixed point:
- limit cycle:
- chaotic attractor:

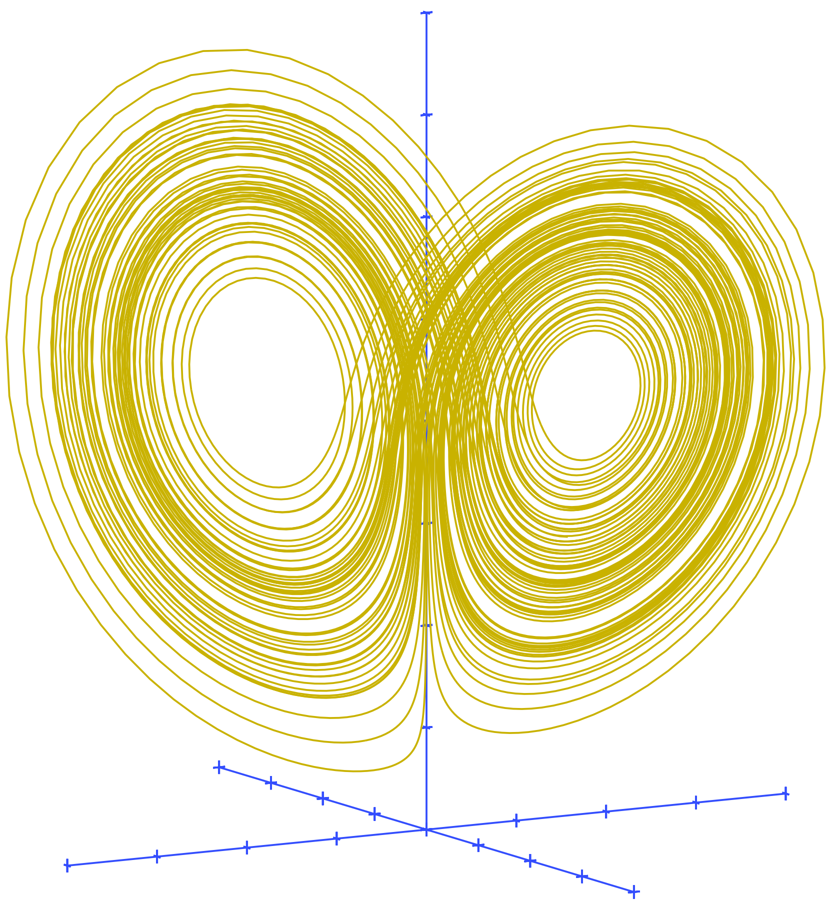
$$(\sigma, r, b) = (10, 28, 8/3)$$



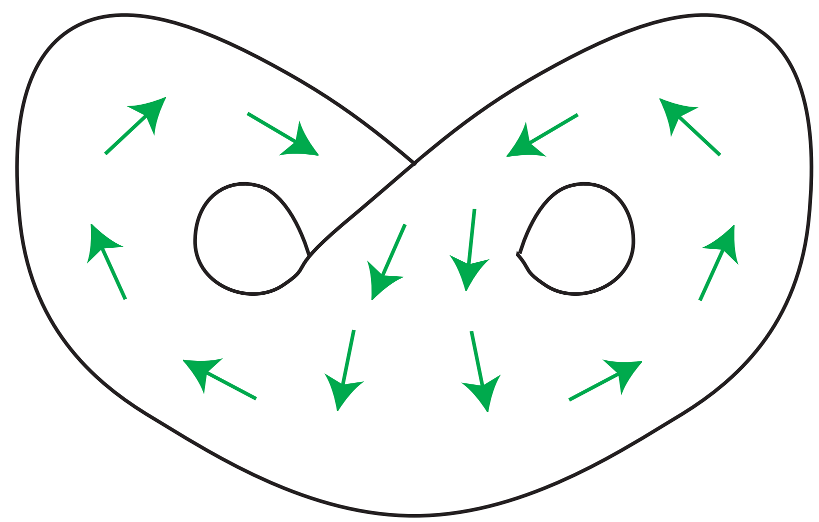
Example Dynamical Systems ...

3D Flows: Chaos ...

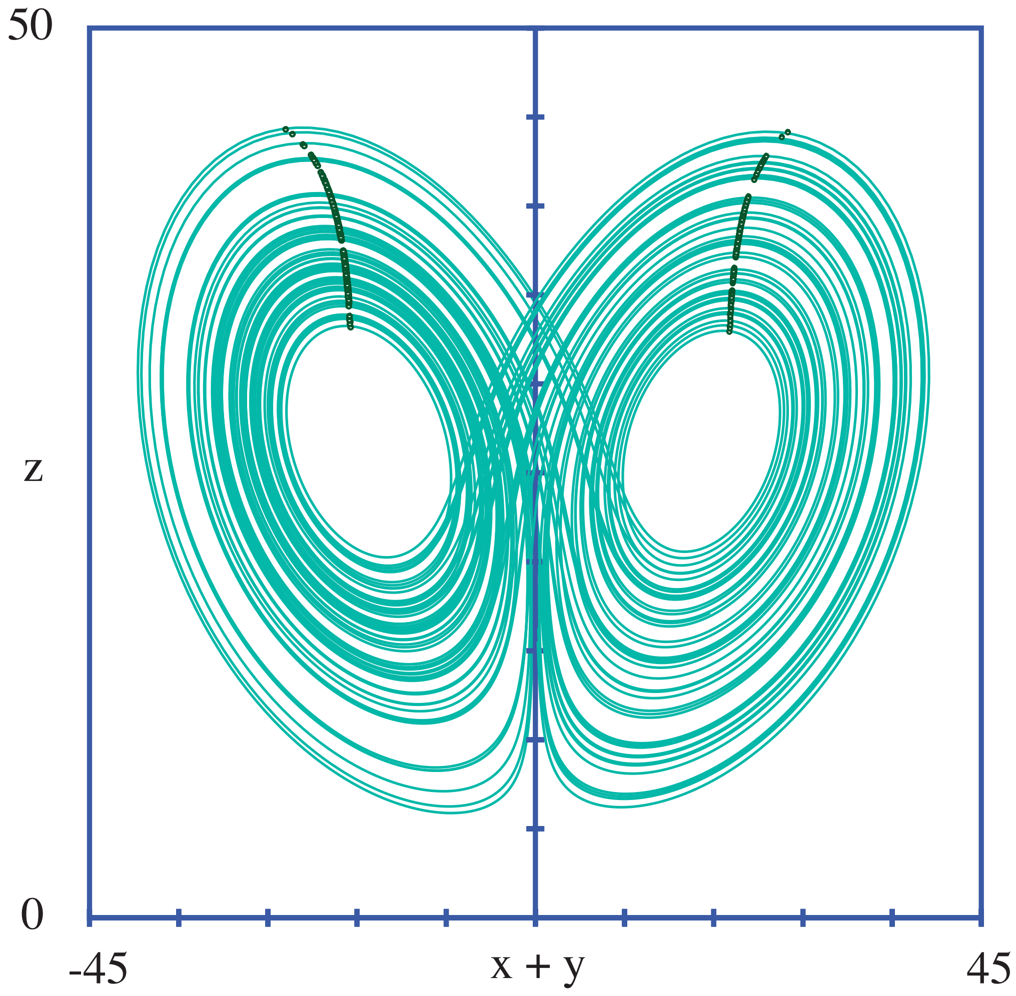
Lorenz attractor structure



Branched manifold



From Continuous-Time Flows to Discrete-Time Maps:



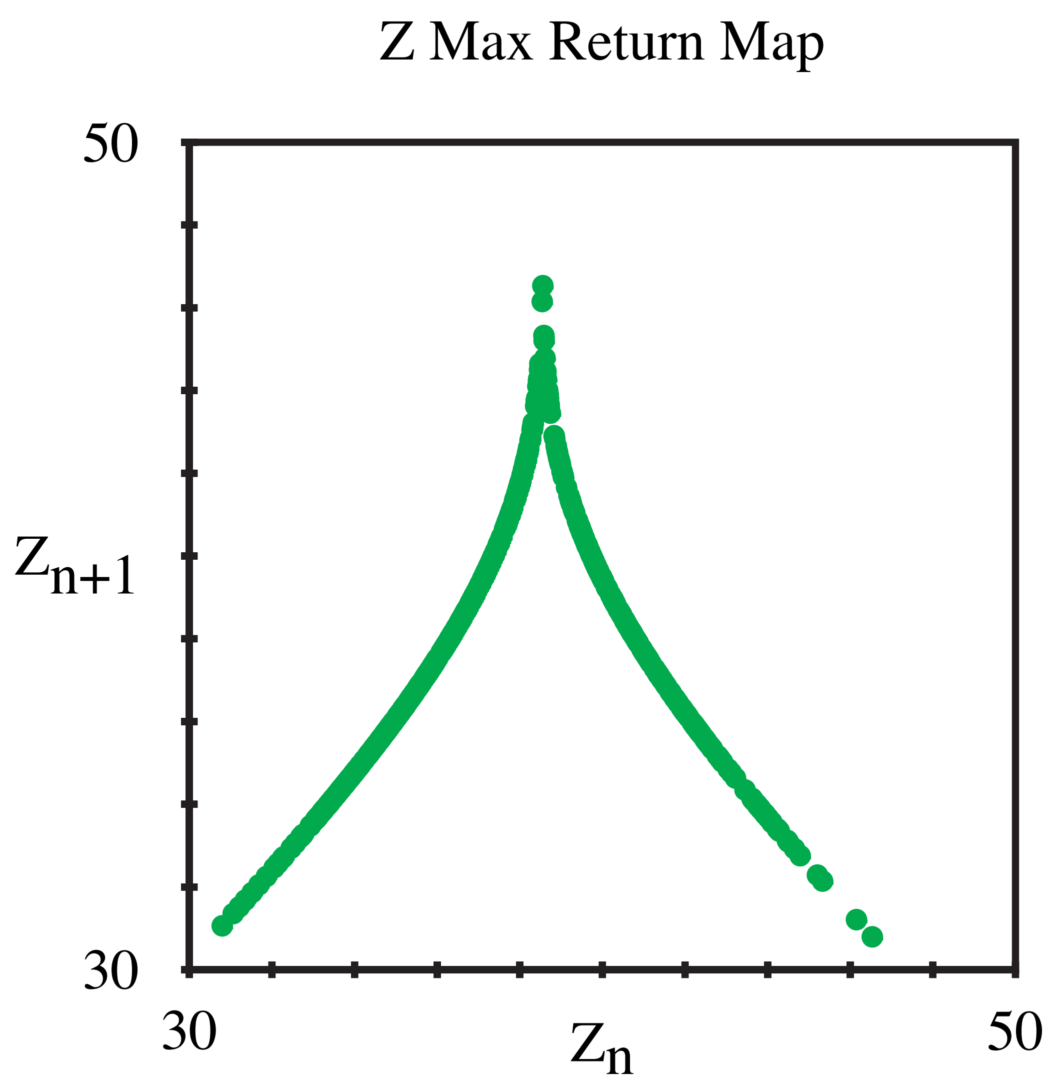
Series of z-maxima: $\widehat{z}_1, \widehat{z}_2, \widehat{z}_3, \ldots$

What happens if you plot
$\widehat{z}_{n+1}$ versus $\widehat{z}_n$?

Example Dynamical Systems ...

From Continuous-Time Flows to Discrete-Time Maps:

Max-z Return Map: $z_{n+1} = f(z_n)$

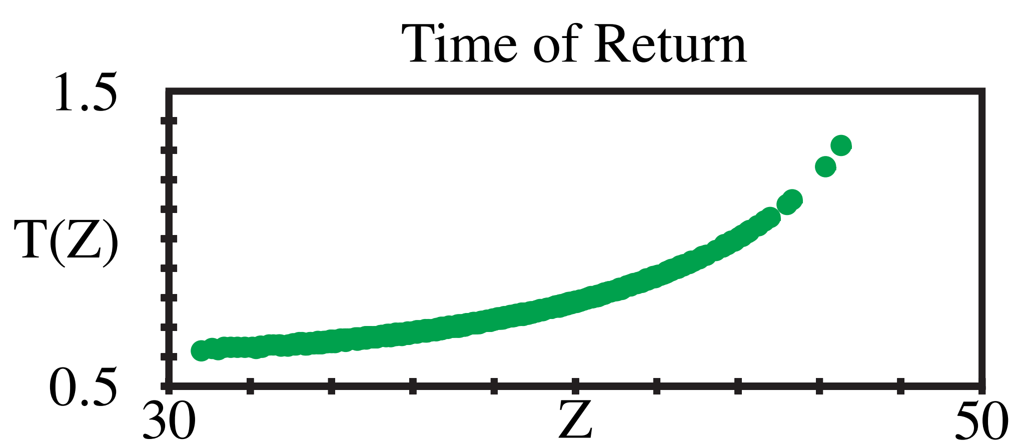


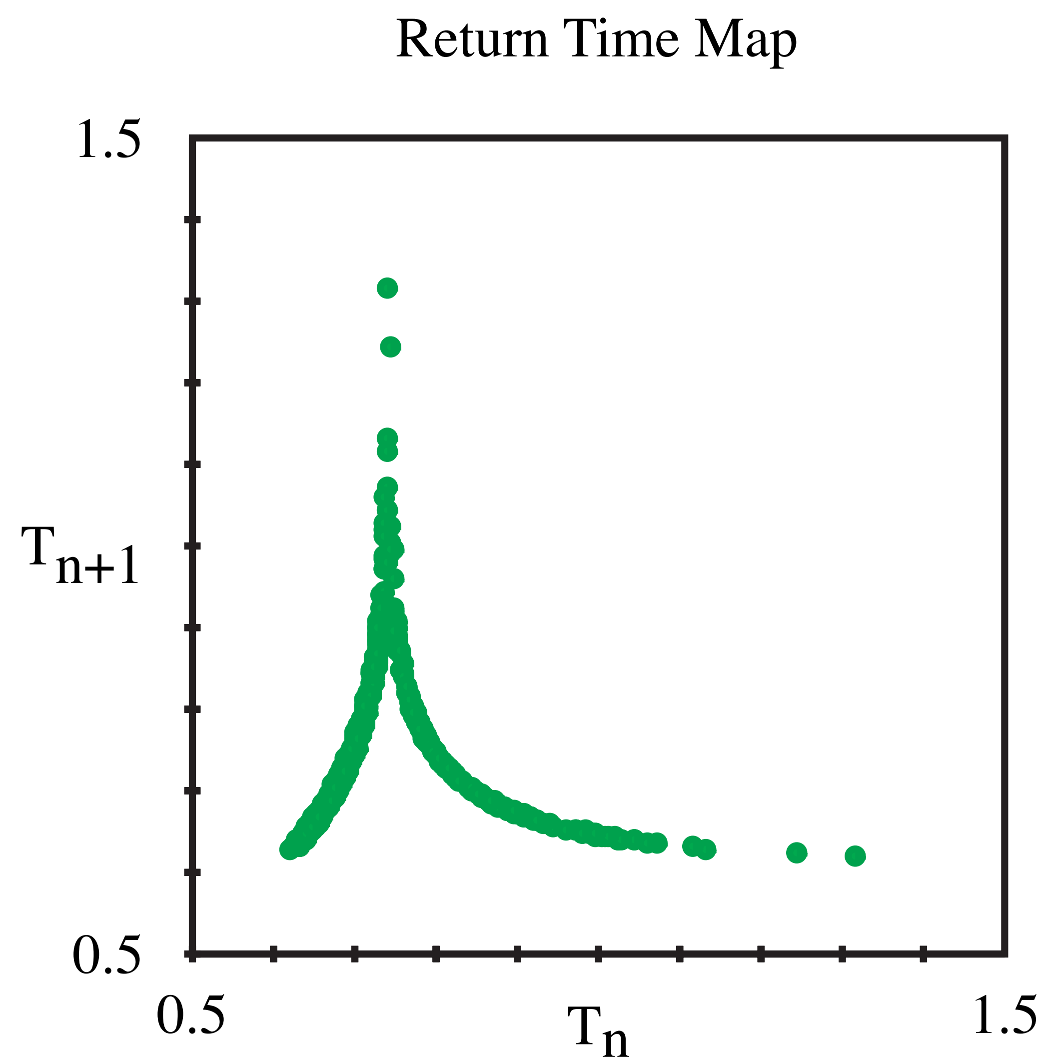
Example Dynamical Systems ...

From Continuous-Time Flows to Discrete-Time Maps:

Time of Return Function: $T(z_n)$

Return Time Map: $T_{n+1} = h(T_n)$





Example Dynamical Systems ...

3D Flows ...

Lorenz reduces to a cusp 1D map:

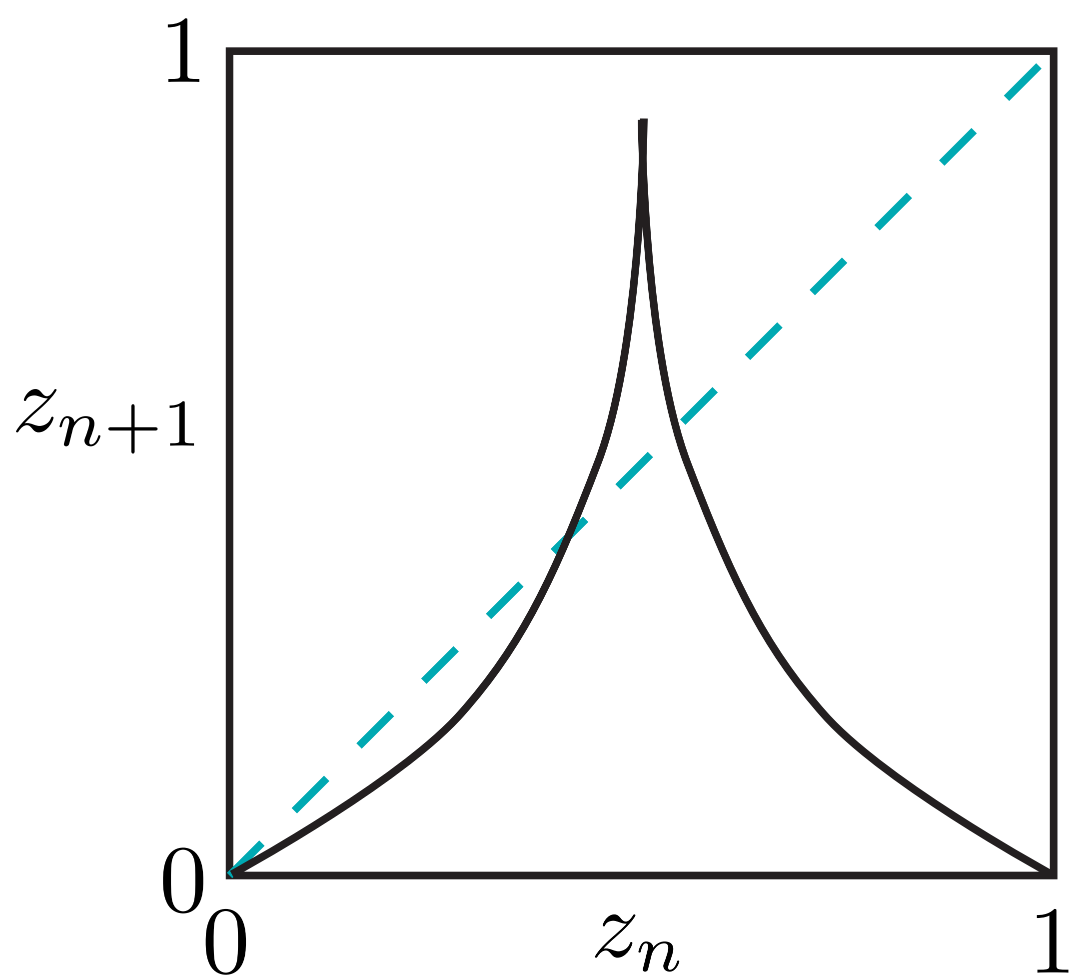
normalize to $z_n \in [0, 1]$

$$z_{n+1} = a(1 - |1 - 2z_n|^b)$$

Parameters:

height: $a > 0$

peak sharpness: $0 < b < 1$



Example Dynamical Systems ...

3D Flows ...

Rössler equations

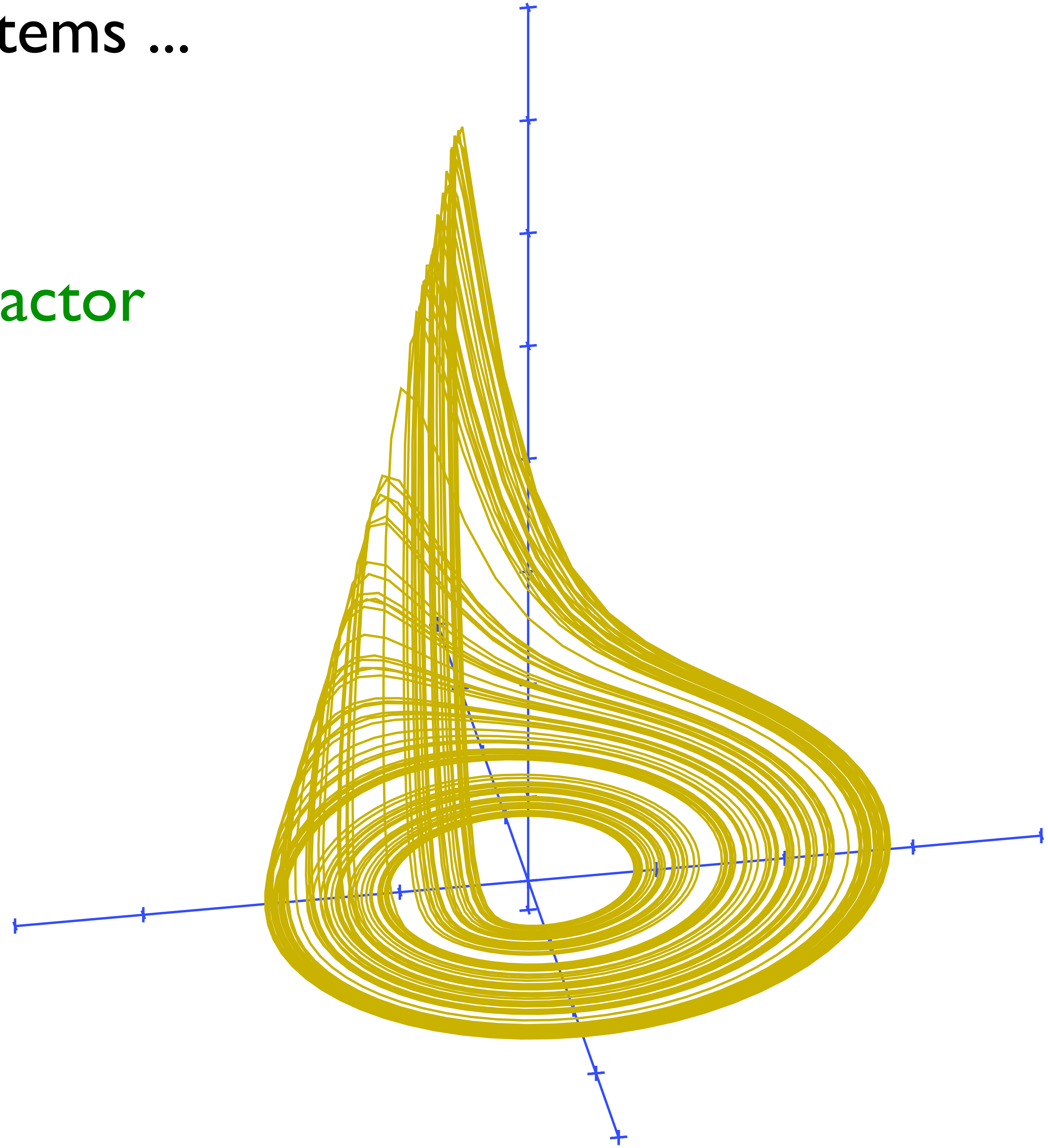
$$
\begin{aligned}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
\end{aligned}
$$

Parameters: $a, b, c > 0$

Example Dynamical Systems ...

3D Flows ...

Rössler chaotic attractor

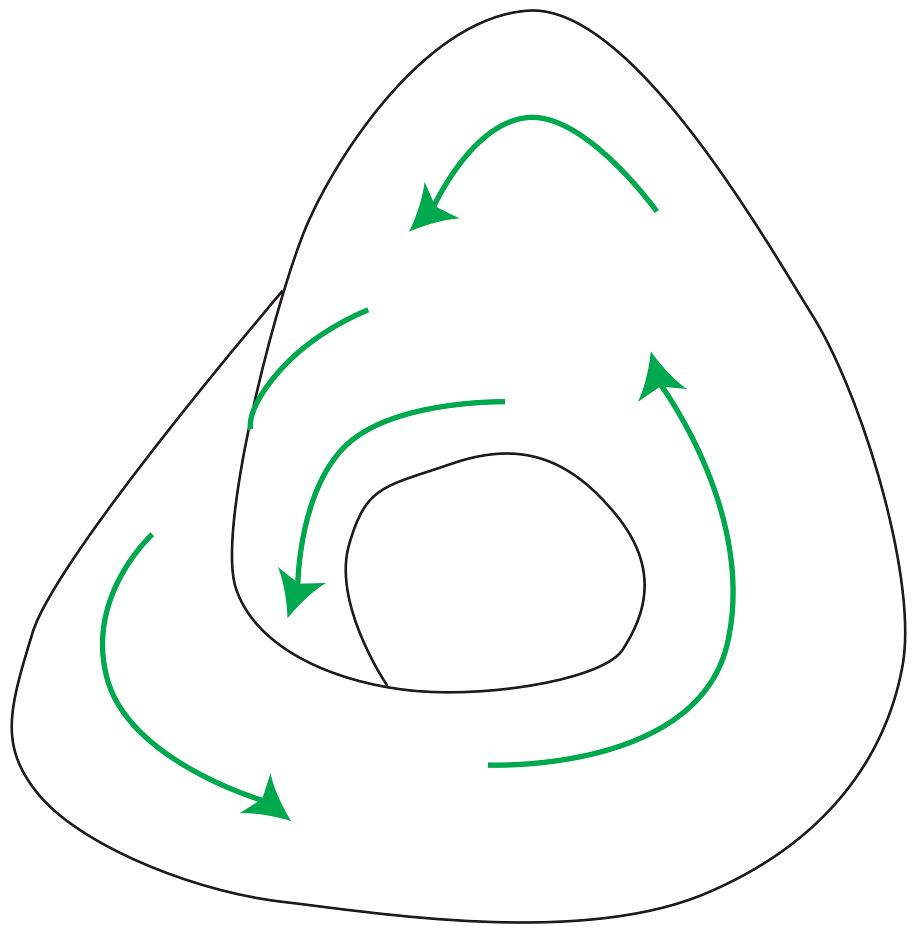


Parameters: $(a, b, c) = (0.2, 0.2, 5.7)$

Example Dynamical Systems ...

3D Flows ...

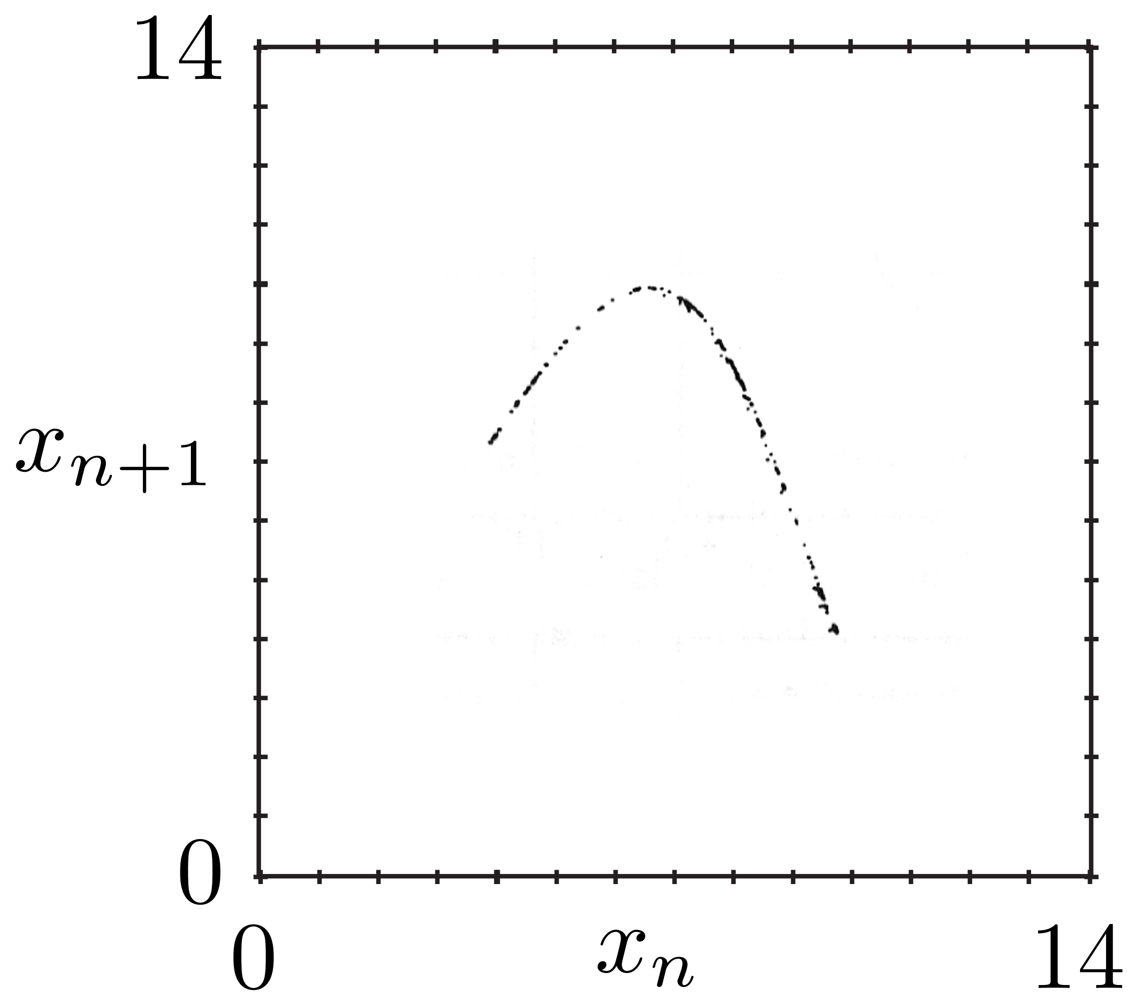
Rössler branched manifold



Example Dynamical Systems ...

3D Flows ...

Rössler maximum-x return map: $x_{n+1} = f(x_n)$



Example Dynamical Systems ...

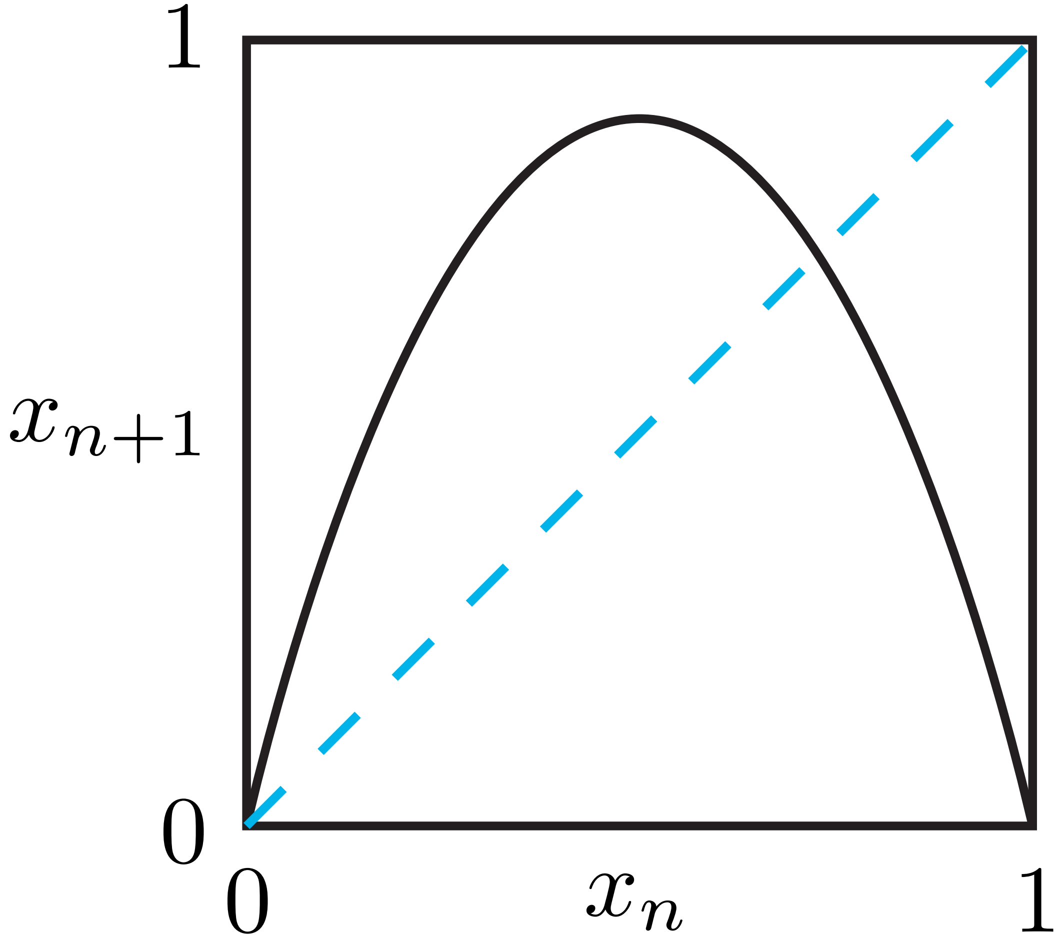
3D Flows ...

When normalized to $x_n \in [0, 1]$

get the Logistic Map:

$$x_{n+1} = r x_n (1 - x_n)$$

Parameter (height): $r \in [0, 4]$



Example Dynamical Systems ...

Classification of Possible Behaviors

Dimension	Attractor
1	Fixed point
2	Fixed point, Limit cycle
3	Fixed Point, Limit Cycle, Torus, Chaotic
4	Above + Hyperchaos
5	Above + ?

Example Dynamical Systems ...

Lorenz:

$$\begin{aligned}\dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz\end{aligned} \qquad \sigma, r, b > 0$$

Rössler:

$$\begin{aligned}\dot{x} &= -y-z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x-c)\end{aligned}$$

Cusp Map: $z_n \in [0,1] \quad a > 0, 0 < b < 1$

$$z_{n+1} = a(1 - |1 - 2z_n|^b)$$

Logistic map:

$$x_{n+1} = rx_n(1-x_n) \qquad x_n \in [0,1] \qquad r \in [0,4]$$

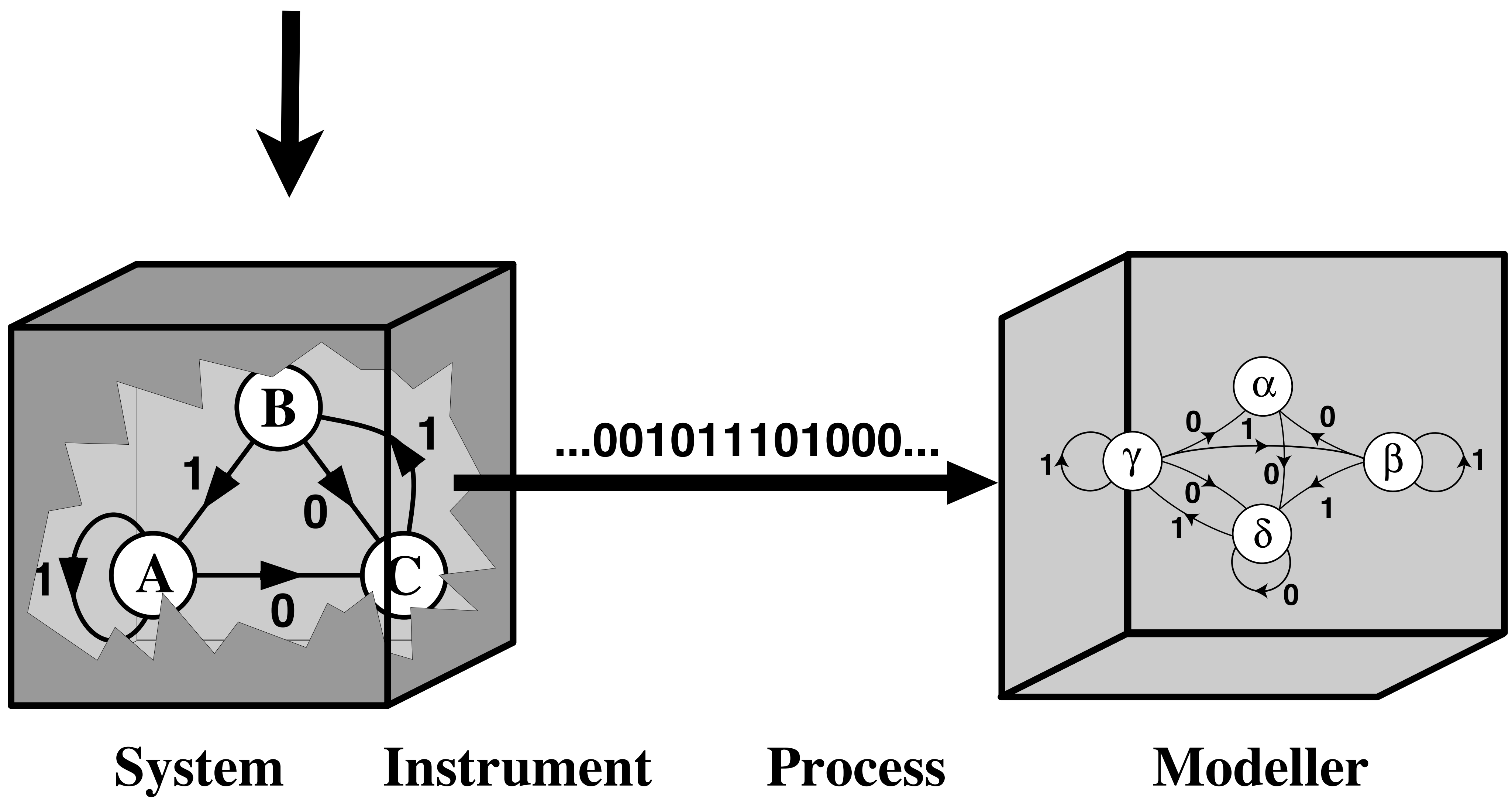
Play with these!

The Big Picture

Global view of the state space structures:
The attractor-basin portrait

The Learning Channel

You Are Here



Example Dynamical Systems ...

Reading for next lecture:

NDAC, Chapter 3.