The Role of the Future in the Analysis of Chaotic Dynamical Systems

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Lecture 32: Natural Computation & Self-Organization, Physics 256B (Spring 2014); Jim Crutchfield

Setting the Scene: Processes

$$\mathcal{P} = (\mathbf{X}, \mu) : \quad \mathbf{X} \subseteq \mathcal{A}^{\mathbb{Z}}, \sigma(\mathcal{P}) = \mathcal{P}$$

$$\cdots \quad X_{-3} \; X_{-2} \; X_{-1} \quad X_0 \quad X_1 \quad X_2 \quad X_3 \quad \cdots$$

Additional properties:

- Ergodic
- Stationary
- Discrete

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Setting the Scene: Processes

$$\mathcal{P} = (\mathbf{X}, \mu) : \quad \mathbf{X} \subseteq \mathcal{A}^{\mathbb{Z}}, \sigma(\mathcal{P}) = \mathcal{P}$$

X_0

$$\cdots \quad X_{-3} \; X_{-2} \; X_{-1} \; X_0 \; X_1 \; X_2 \; X_3 \quad \cdots$$

present

Additional properties:

- Ergodic
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Setting the Scene: Processes

$$\mathcal{P} = (\mathbf{X}, \mu) : \quad \mathbf{X} \subseteq \mathcal{A}^{\mathbb{Z}}, \sigma(\mathcal{P}) = \mathcal{P}$$

$$X_{:0} \qquad X_0$$

$$\cdots \; X_{-3} \; X_{-2} \; X_{-1} \; X_0 \; X_1 \; X_2 \; X_3 \; \cdots$$

past present

Additional properties:

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Setting the Scene: Processes

$$\mathcal{P} = (\mathbf{X}, \mu) : \quad \mathbf{X} \subseteq \mathcal{A}^{\mathbb{Z}}, \sigma(\mathcal{P}) = \mathcal{P}$$

$X_{:0}$	X_0	$X_{1:}$
$\cdots \; X_{-3} \; X_{-2} \; X_{-1}$	X_0	$X_1 \; X_2 \; X_3 \; \cdots$
past	present	future

Additional properties:

- Ergodic
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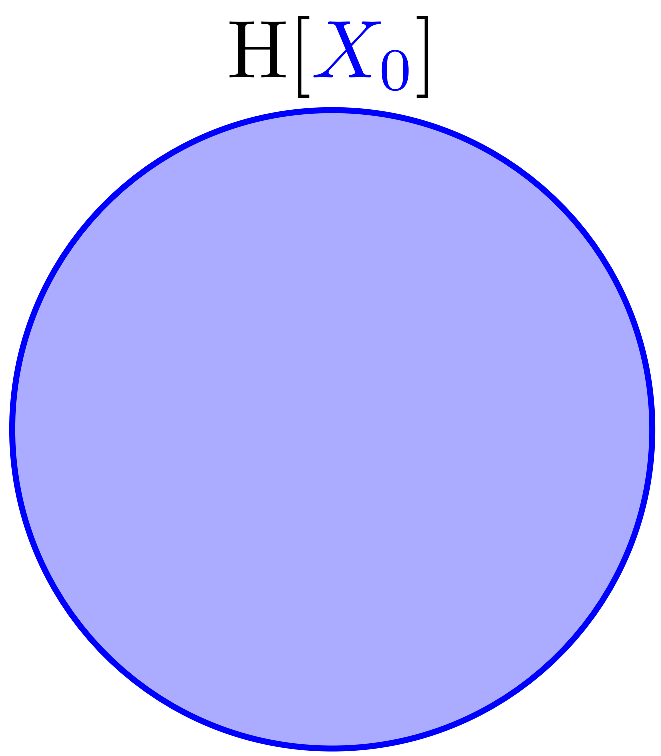
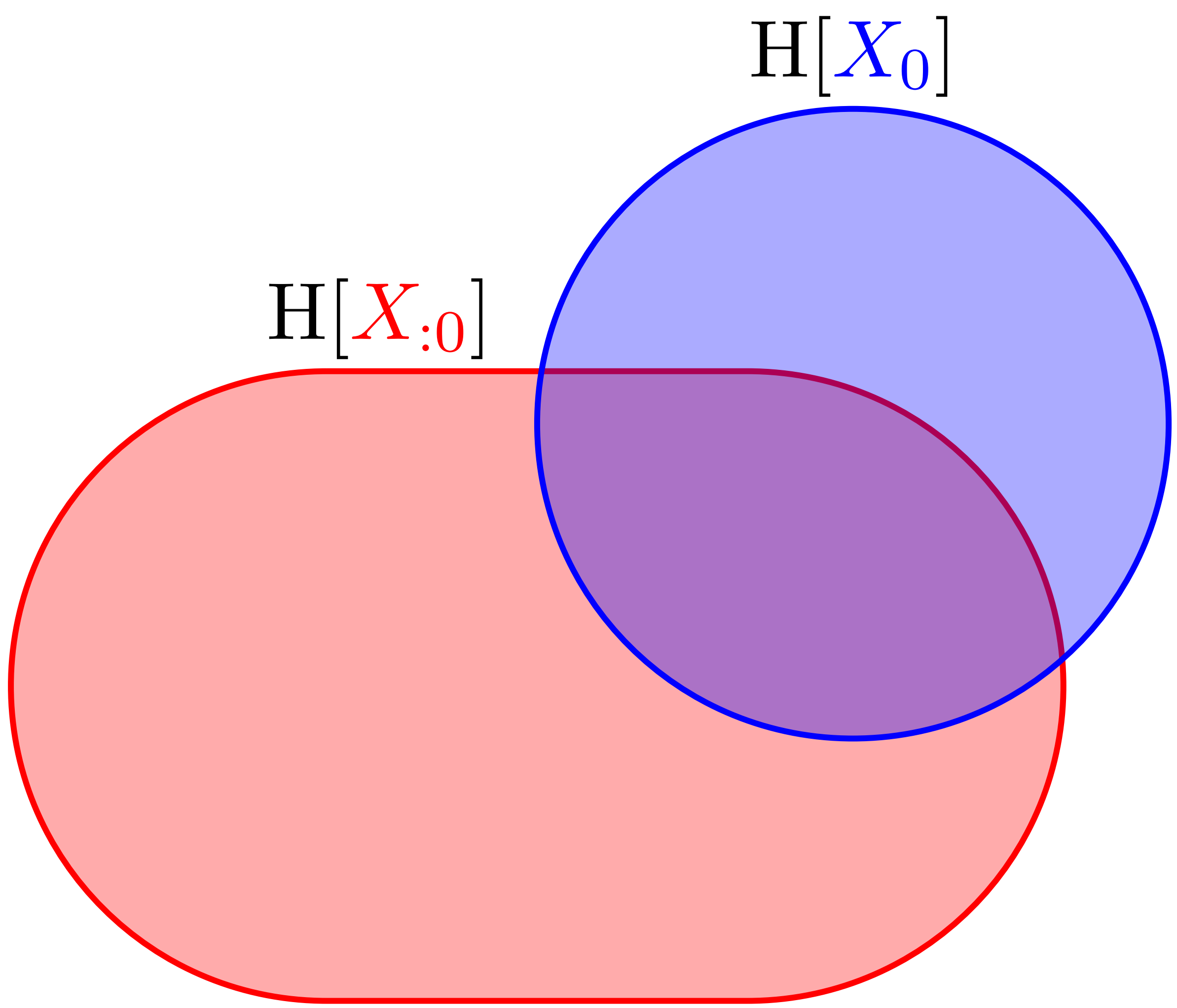
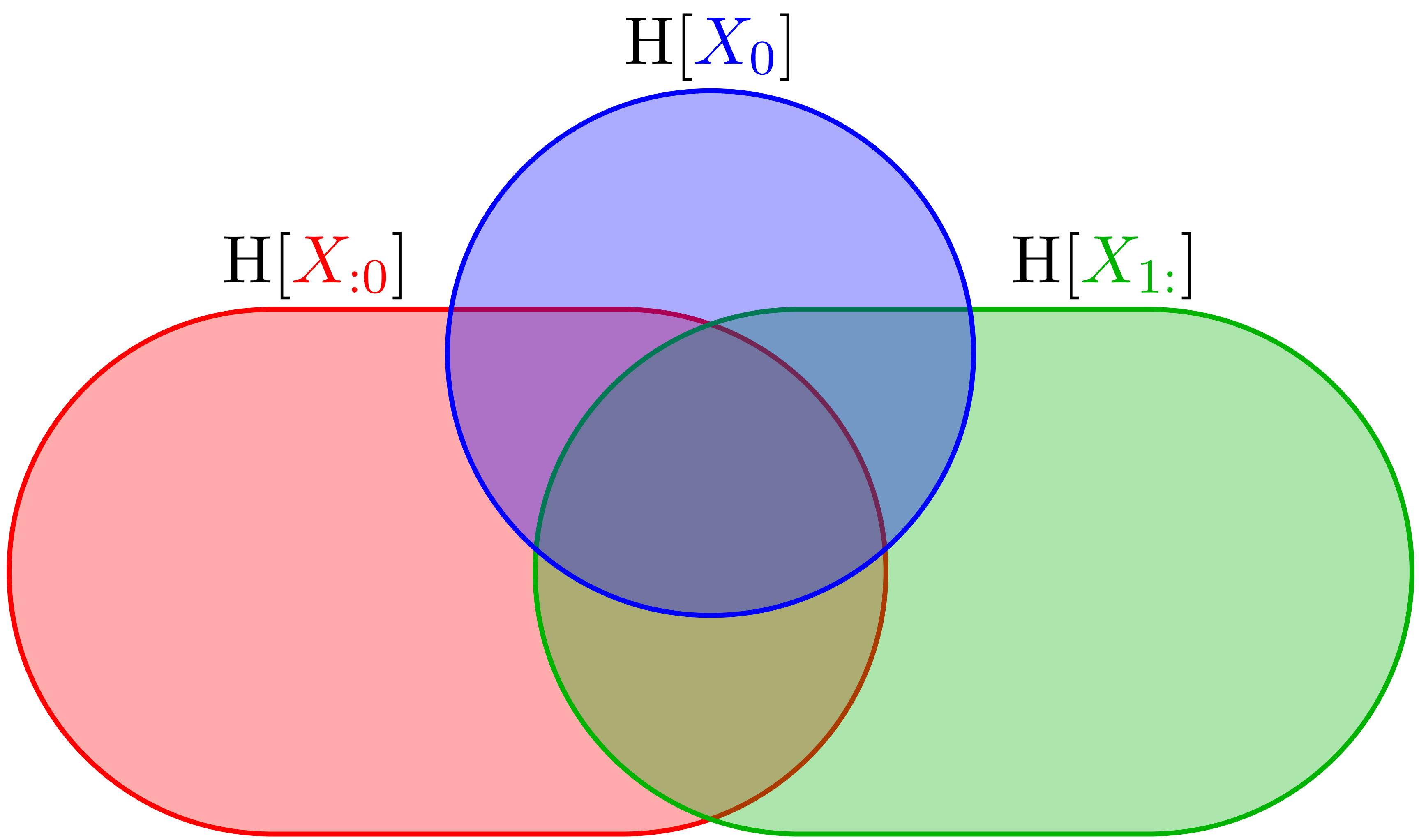
Past, Present & Future



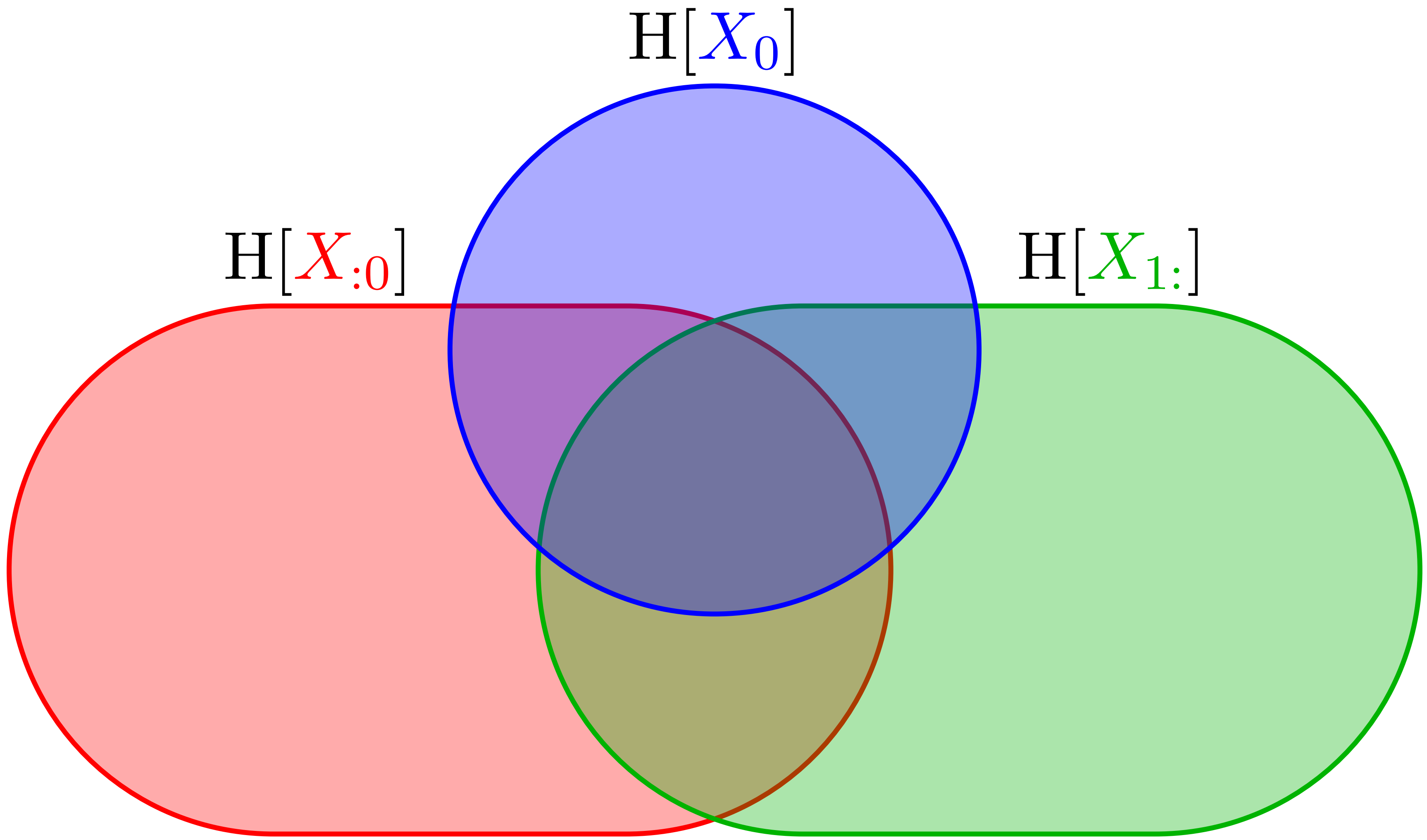
Past, Present & Future



Past, Present & Future

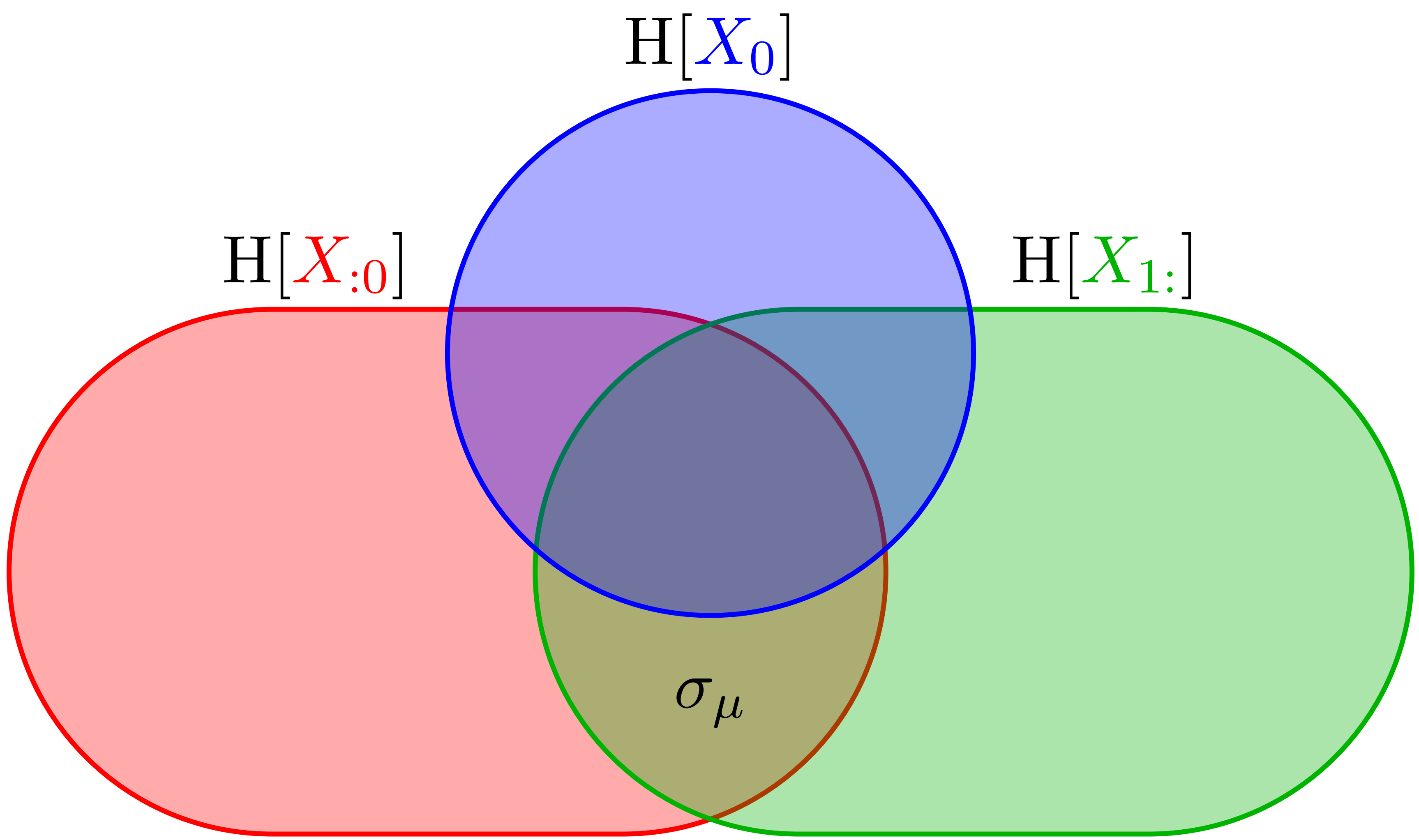


Past, Present & Future



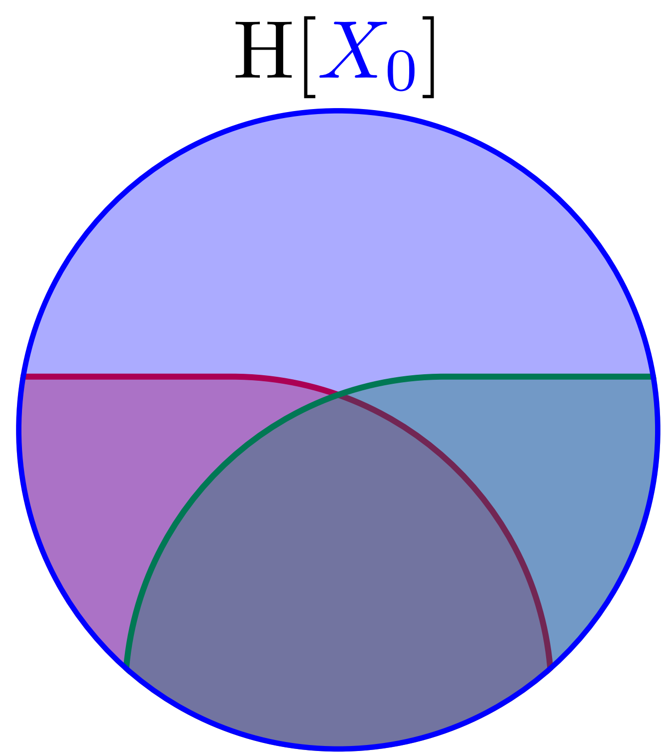
- $H[X_0]$ is *partitioned* by the past and the future

Past, Present & Future

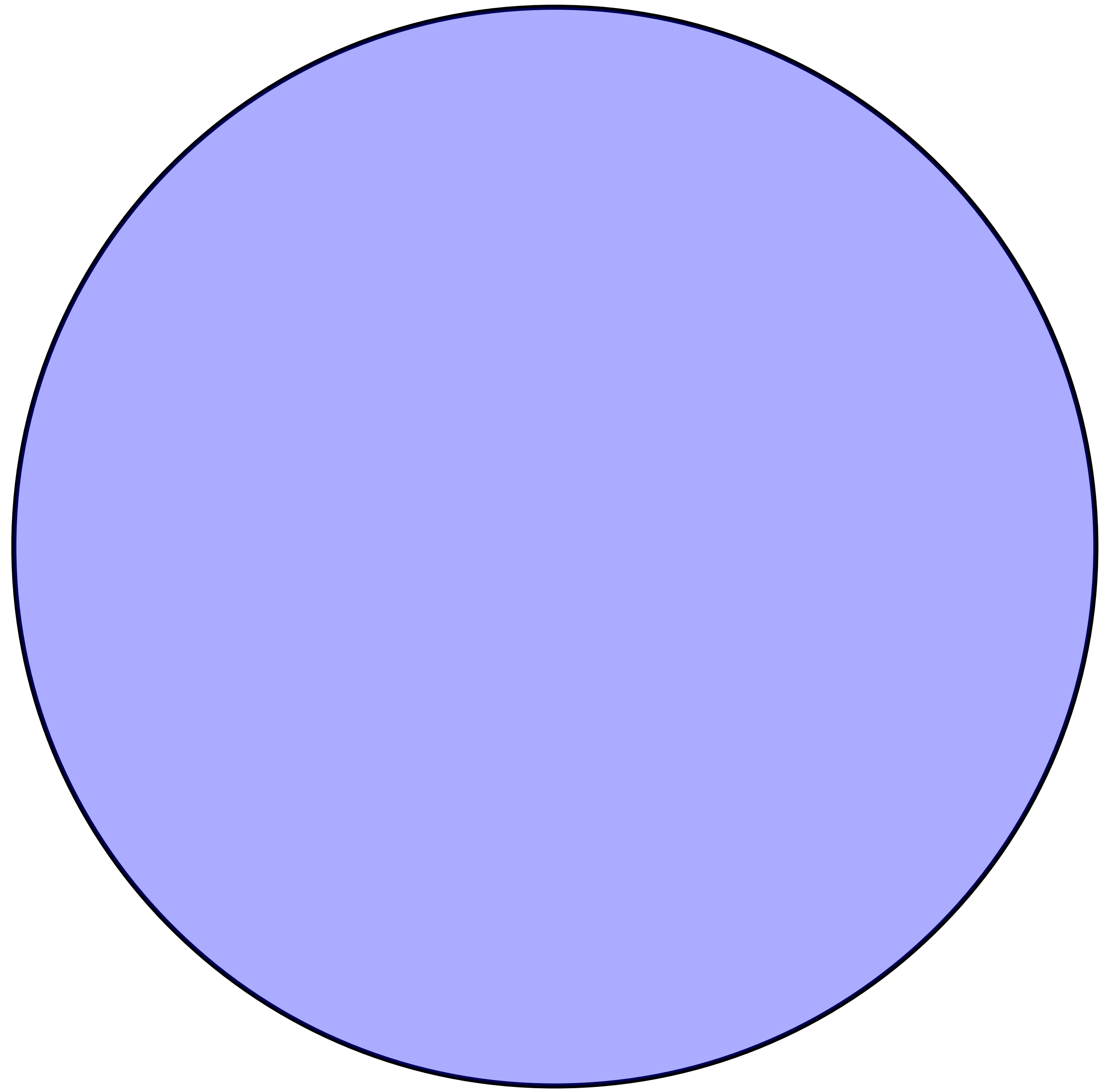


- $H[X_0]$ is *partitioned* by the past and the future
- $\sigma_\mu = I[X_{:0}; X_{1:} | X_0]$: evidence of internal states

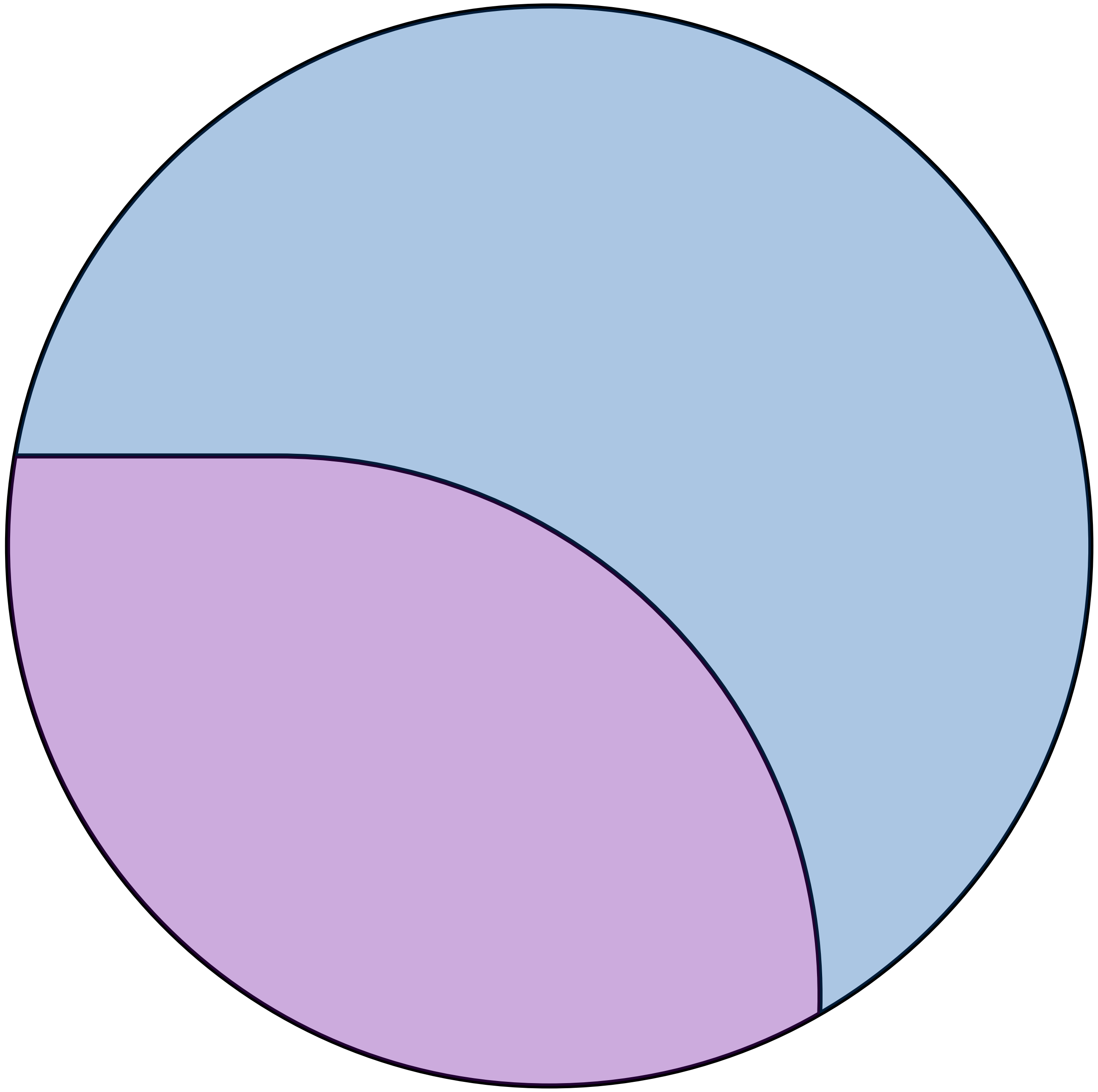
Past, Present & Future



Decompositions of $H[X_0]$

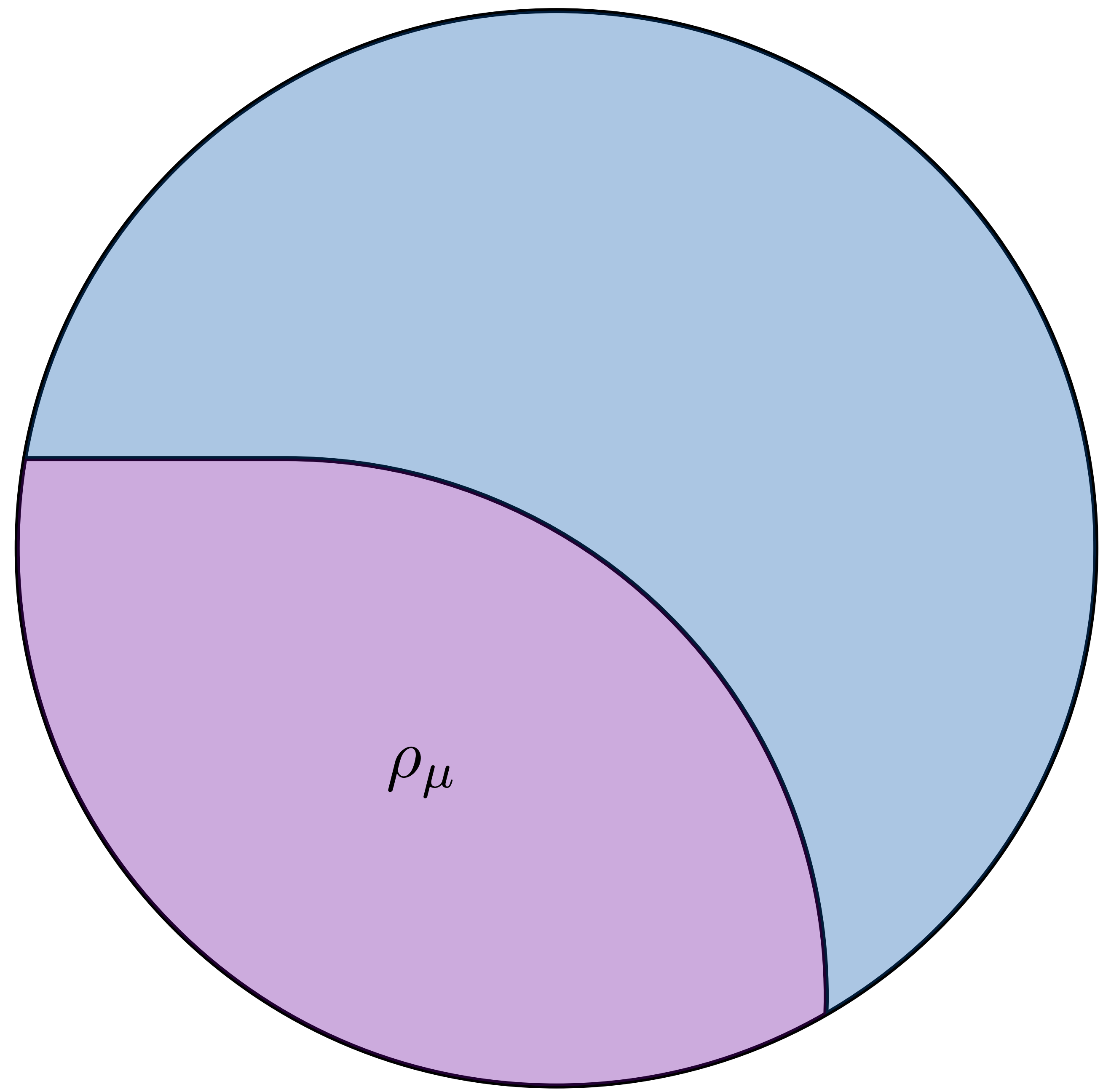


The Human Decomposition of $H[X_0]$



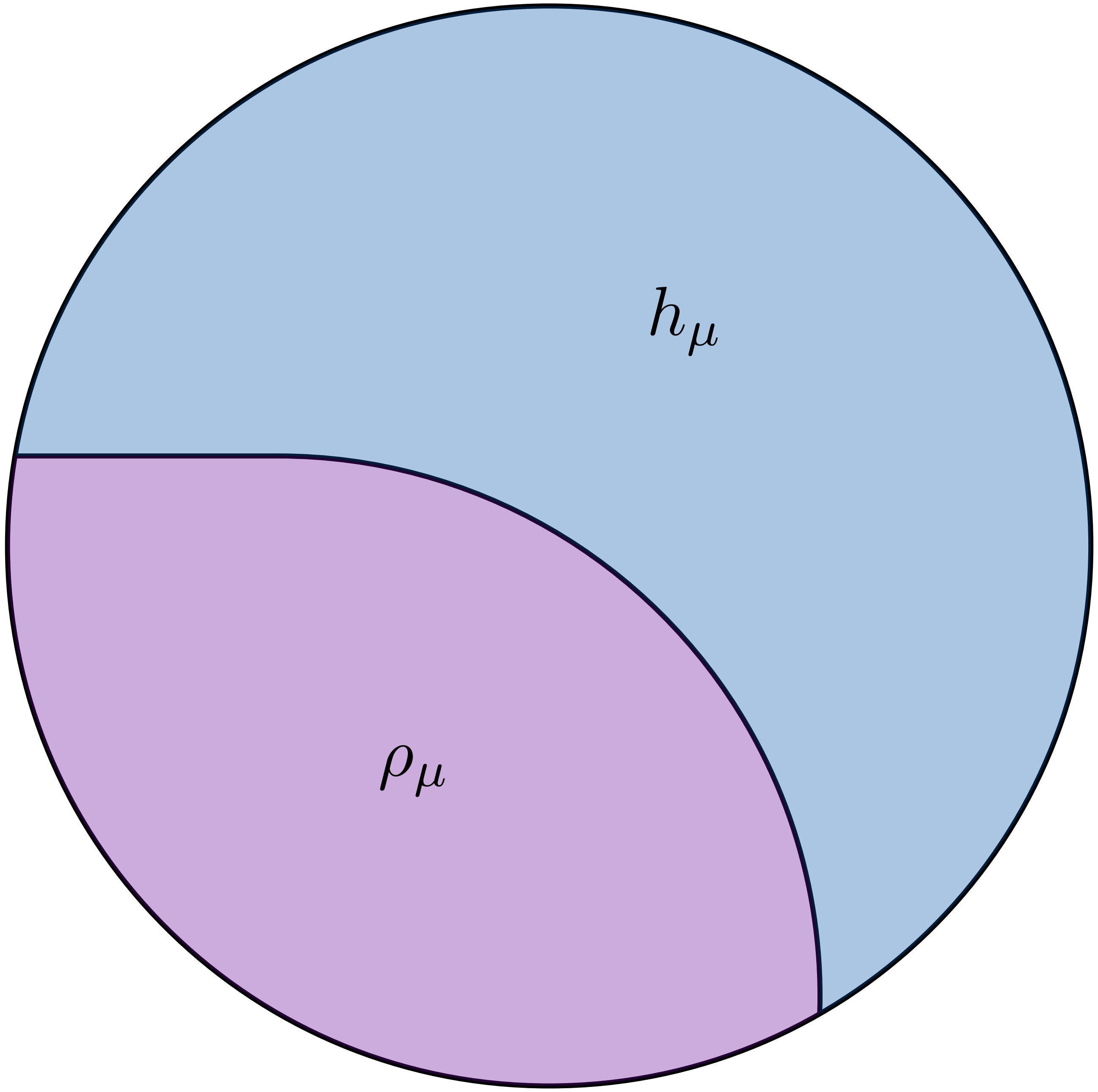
The Human Decomposition of $\mathrm{H}[X_0]$

- $\rho_\mu = \mathrm{I}[X_{:0}; X_0]$: anticipated information



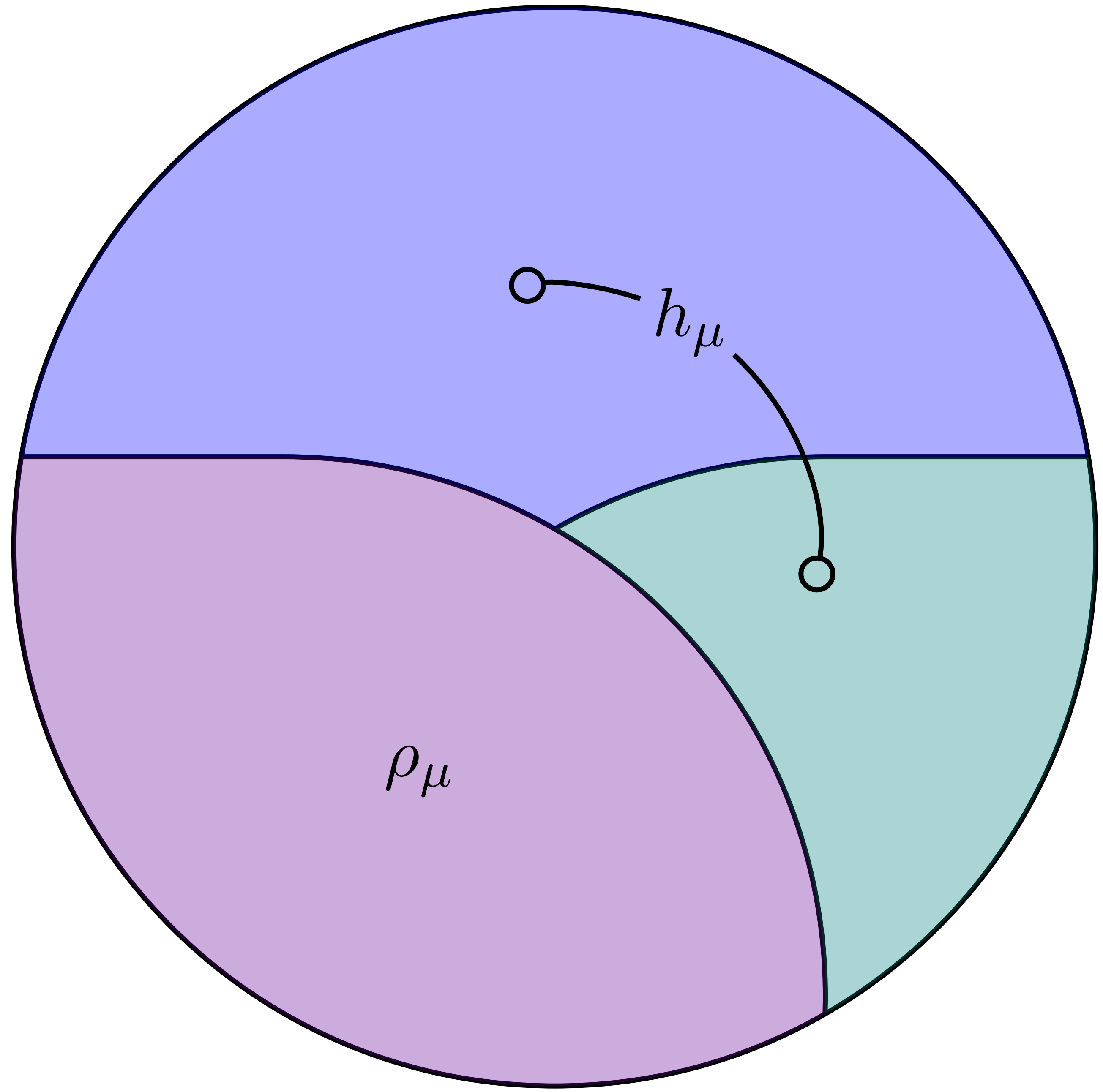
The Human Decomposition of $H[X_0]$

- $\rho_\mu = I[X_{:0}; X_0]$: anticipated information
- $h_\mu = H[X_0 | X_{:0}]$: (*Shannon entropy rate, metric entropy, Kolmogorov-Sinai entropy*) unanticipated information



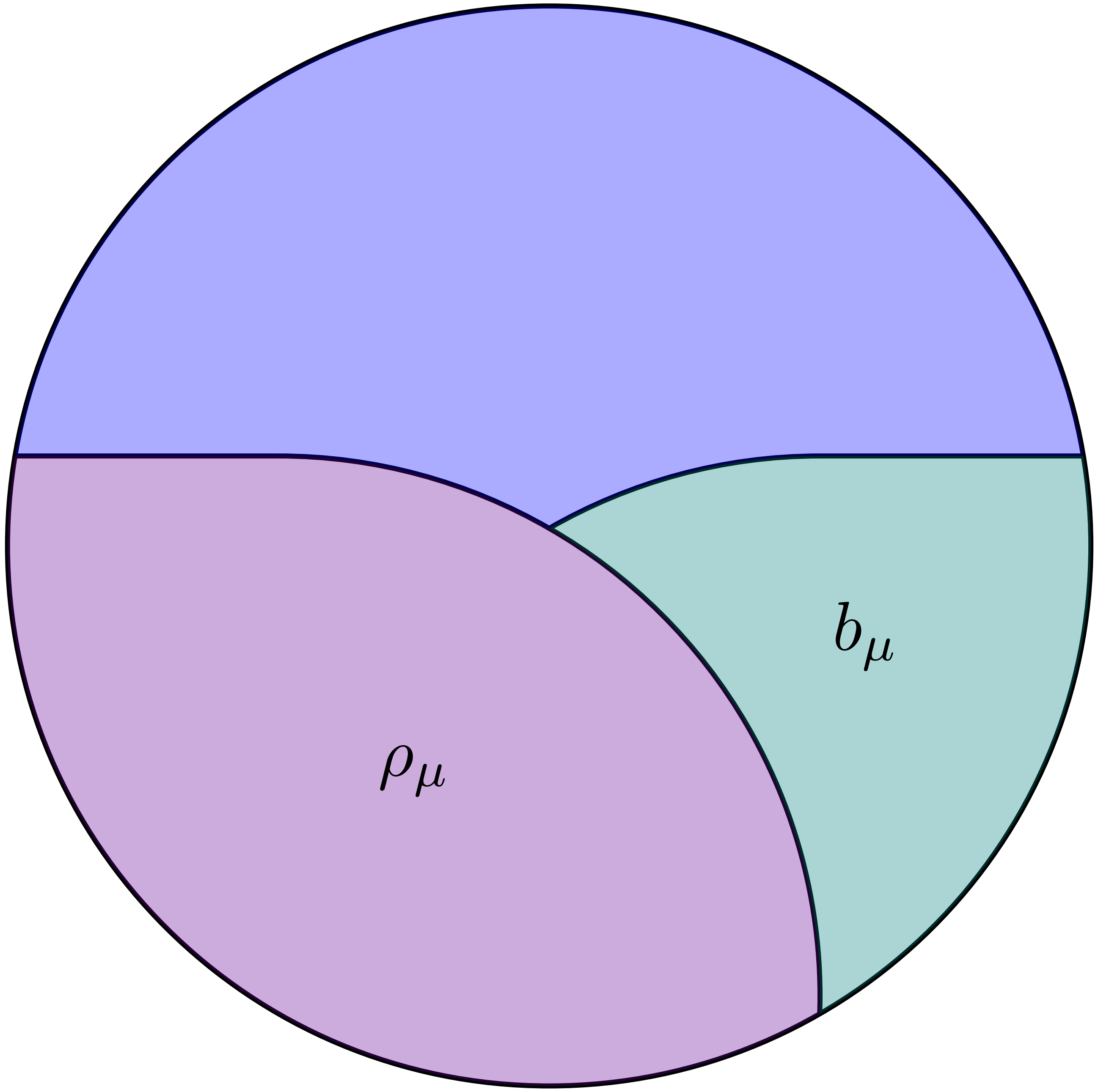
A More Refined Decomposition of $H[X_0]$

- $\rho_\mu = I[X_{:0}; X_0]$: anticipated information



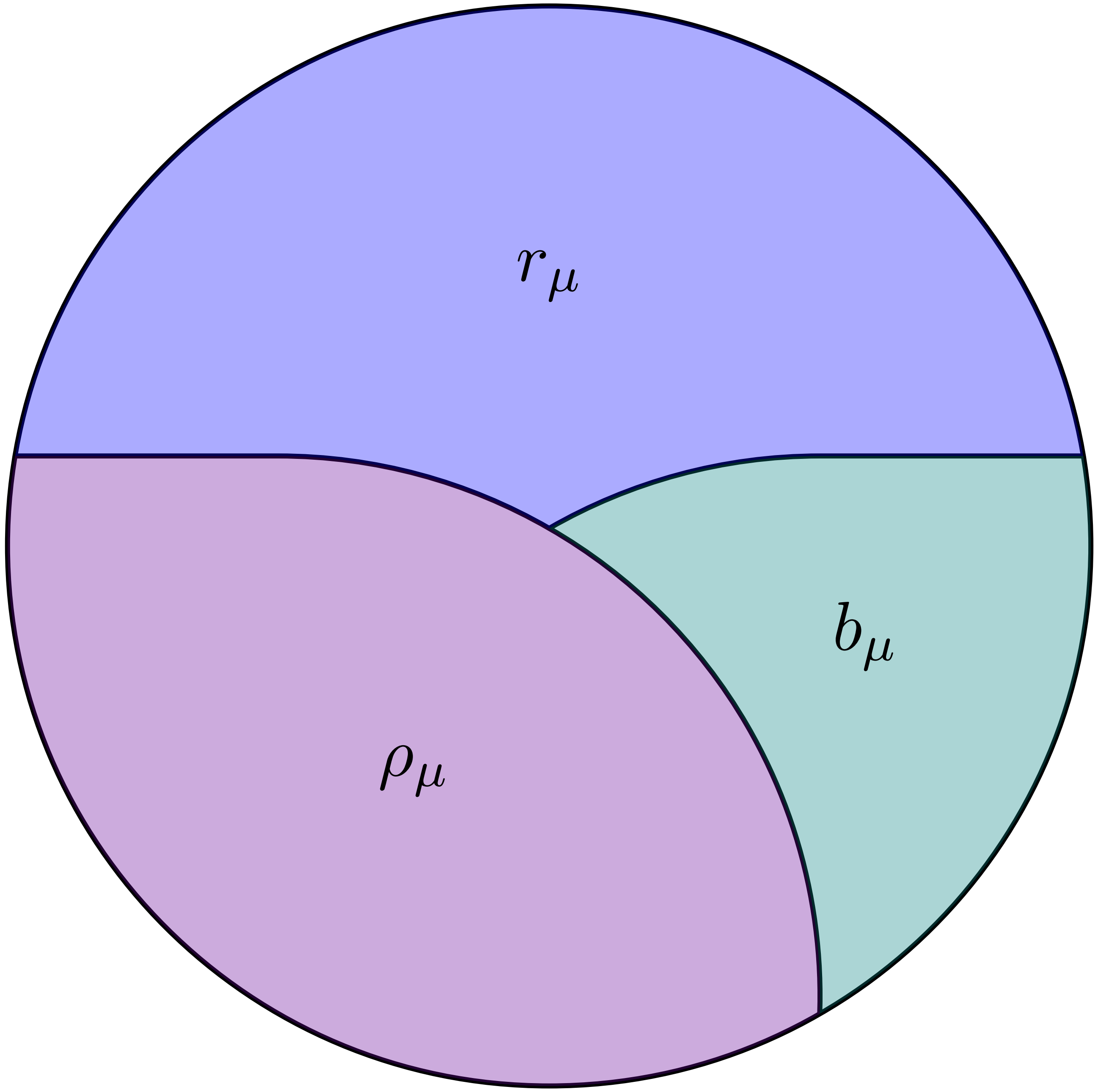
A More Refined Decomposition of $\mathrm{H}[X_0]$

- $\rho_\mu = \mathrm{I}[X_{:0}; X_0]$: anticipated information
- $b_\mu = \mathrm{I}[X_0; X_{1:} | X_{:0}]$: unanticipated and relevant



A More Refined Decomposition of $\mathrm{H}[X_0]$

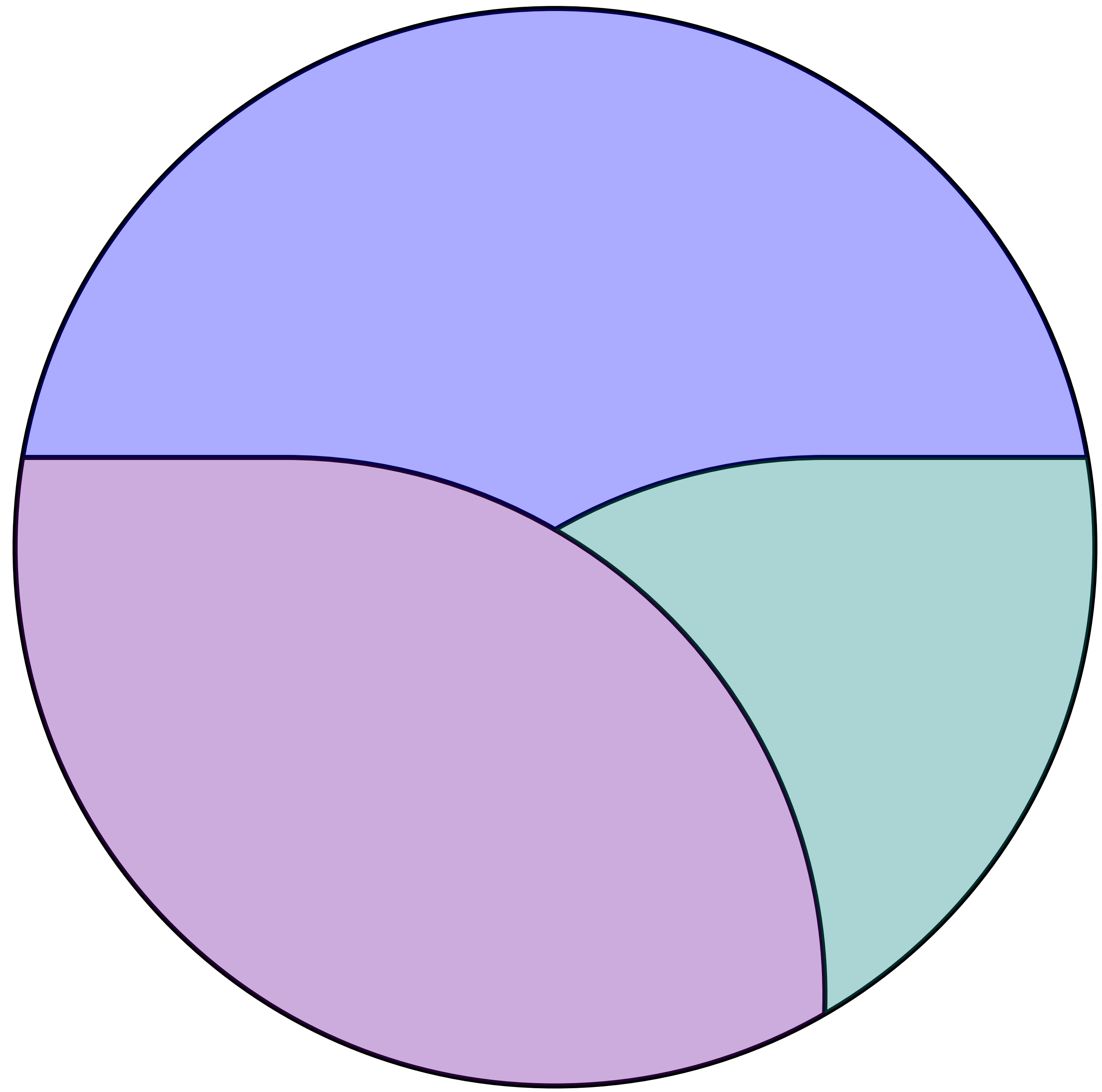
- $\rho_\mu = \mathrm{I}[X_{:0}; X_0]$: anticipated information
- $b_\mu = \mathrm{I}[X_0; X_{1:} | X_{:0}]$: unanticipated and relevant
- $r_\mu = \mathrm{H}[X_0 | X_{:0}; X_{1:}]$: unanticipated and irrelevant



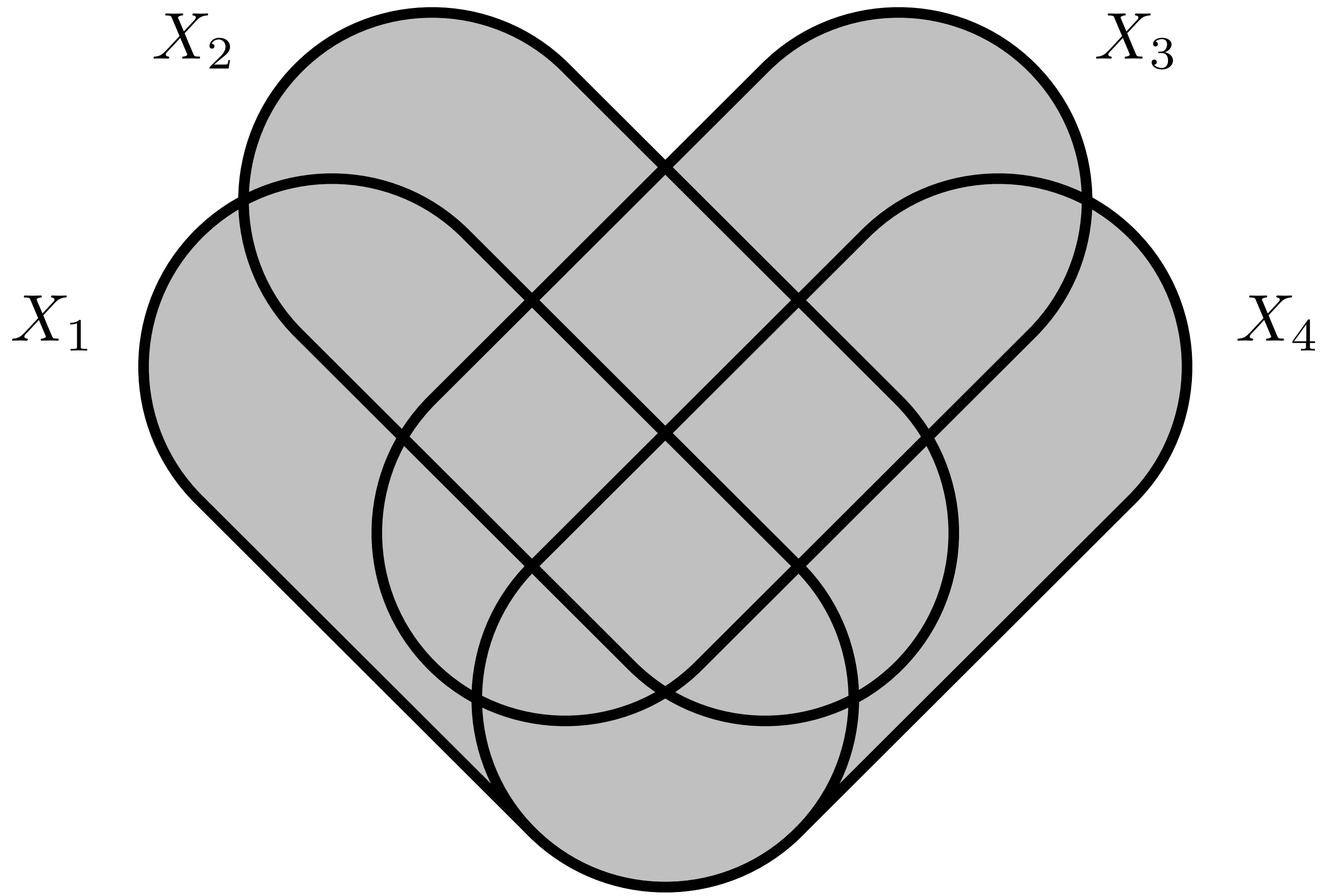
. . . and you know my Achilles tendon is my one Achilles' heal

Like Humans Do

"How do I measure these?"



Entropy



$$H[X_{1:\ell}] = - \sum_{w \in \mathcal{A}^\ell} p_w \log(p_w)$$

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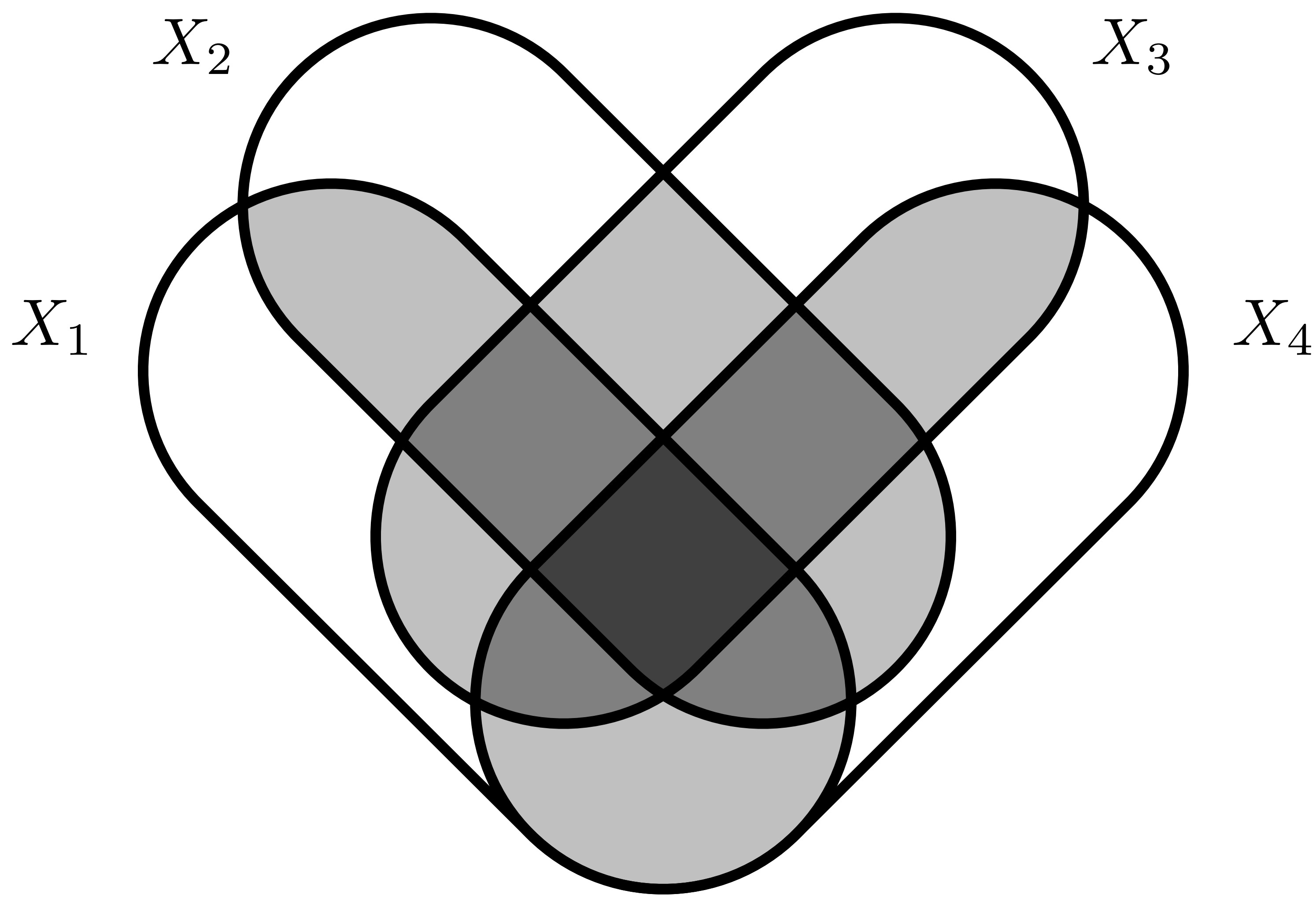
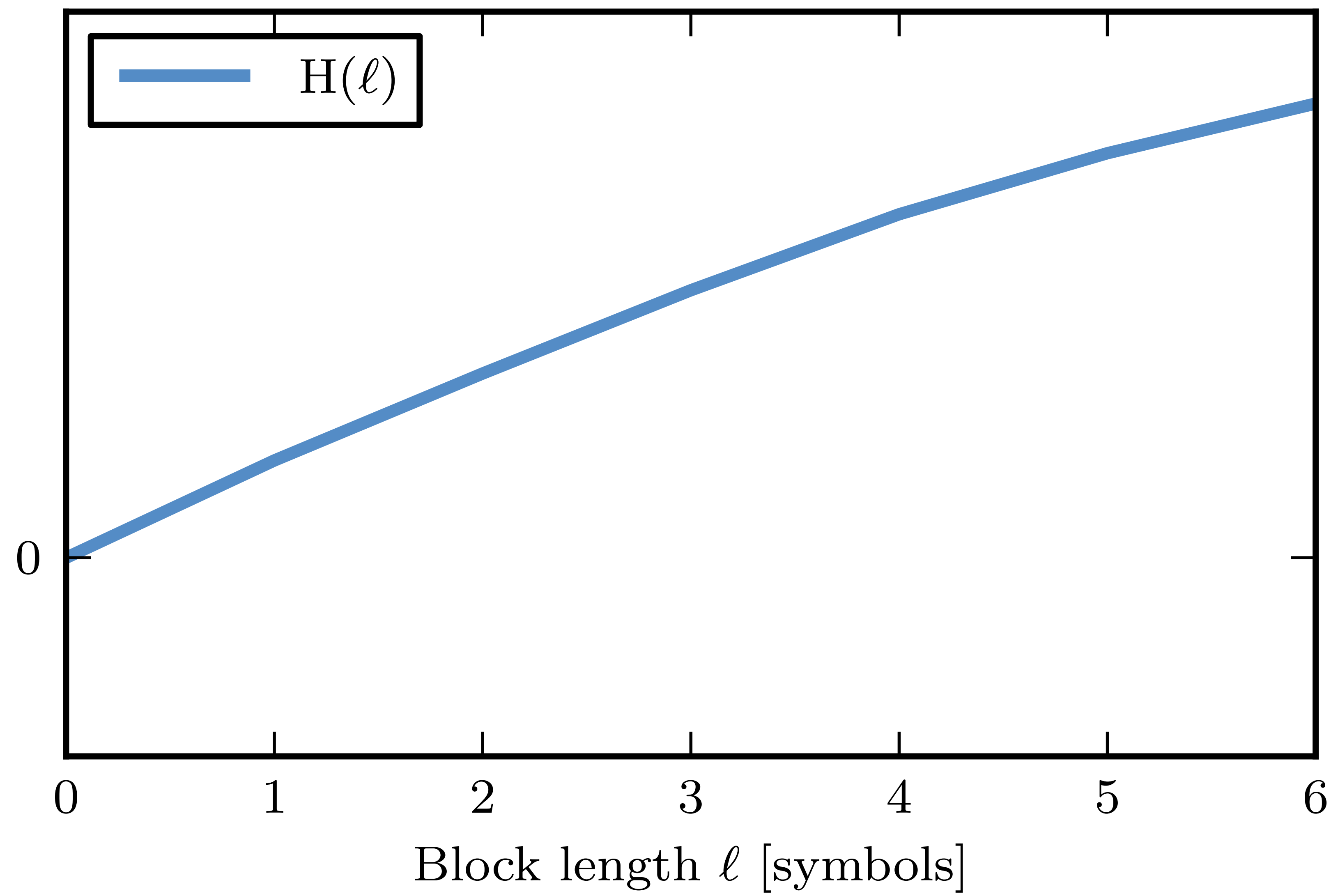
Total Correlation

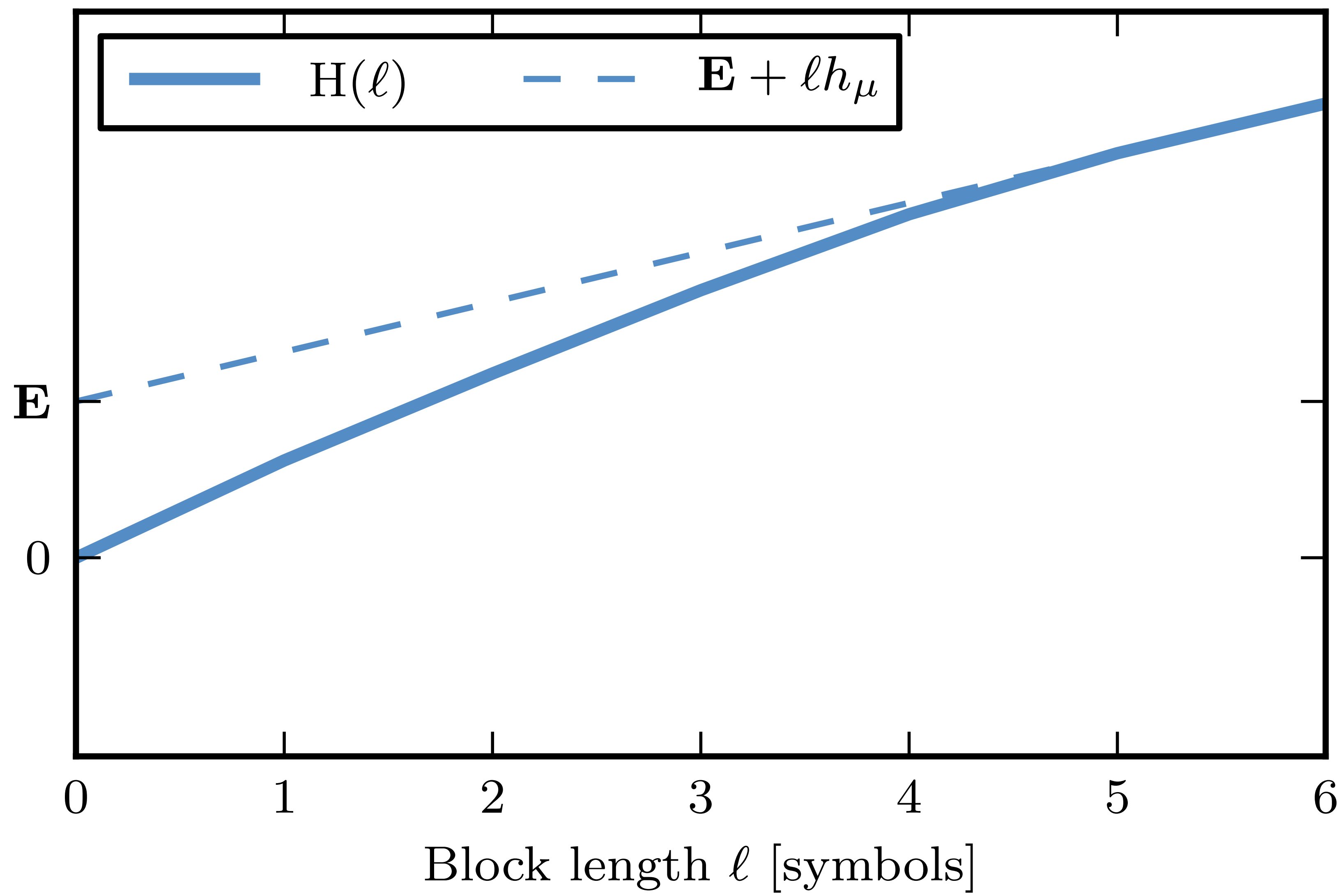


$$\mathrm{T}[X_{1:\ell}] = \sum_{i \in \{1 \ldots \ell\}} \mathrm{H}[X_i] - \mathrm{H}[X_{1:\ell}]$$

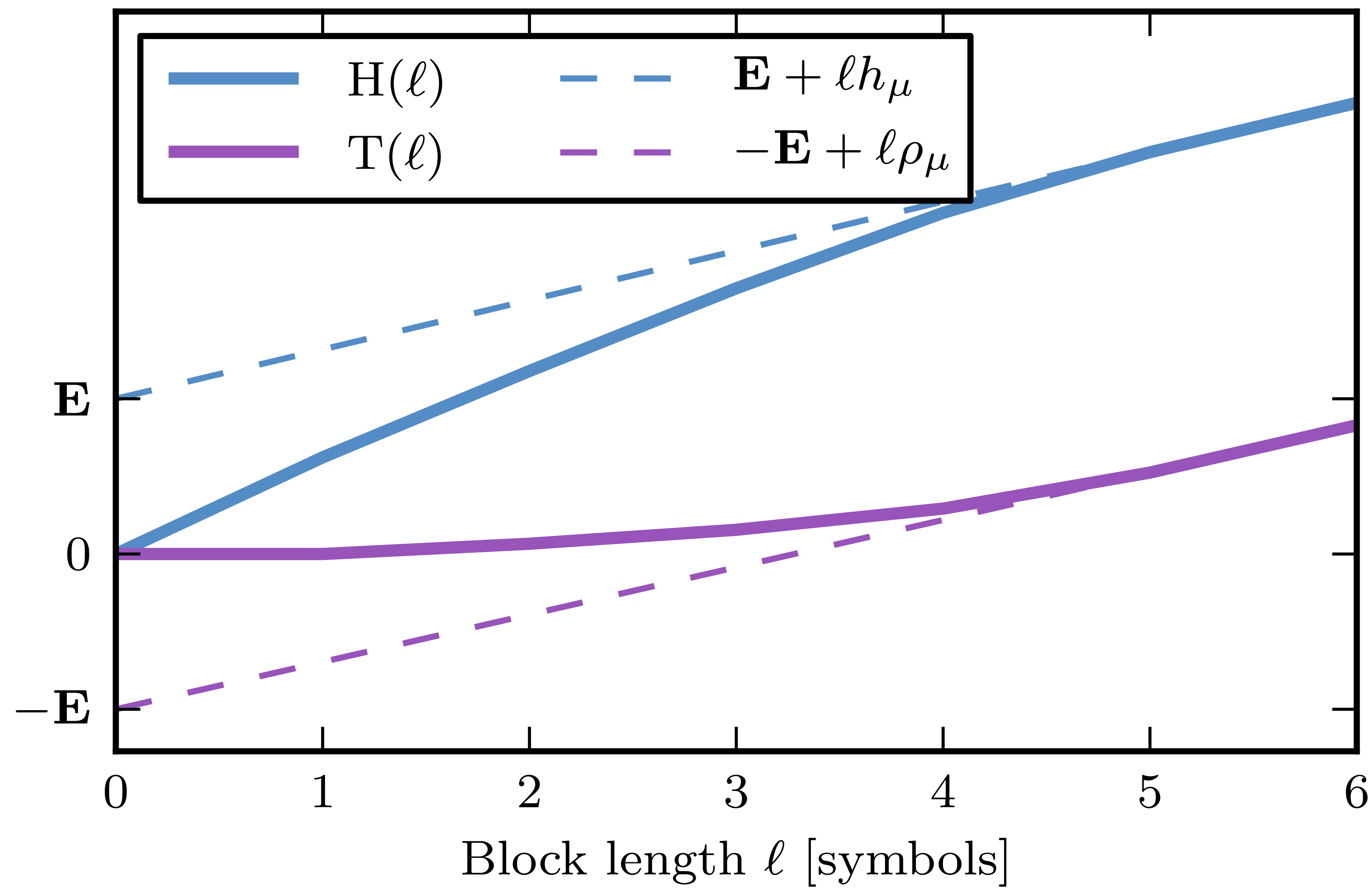
Asymptotic Rates: h_μ & ρ_μ



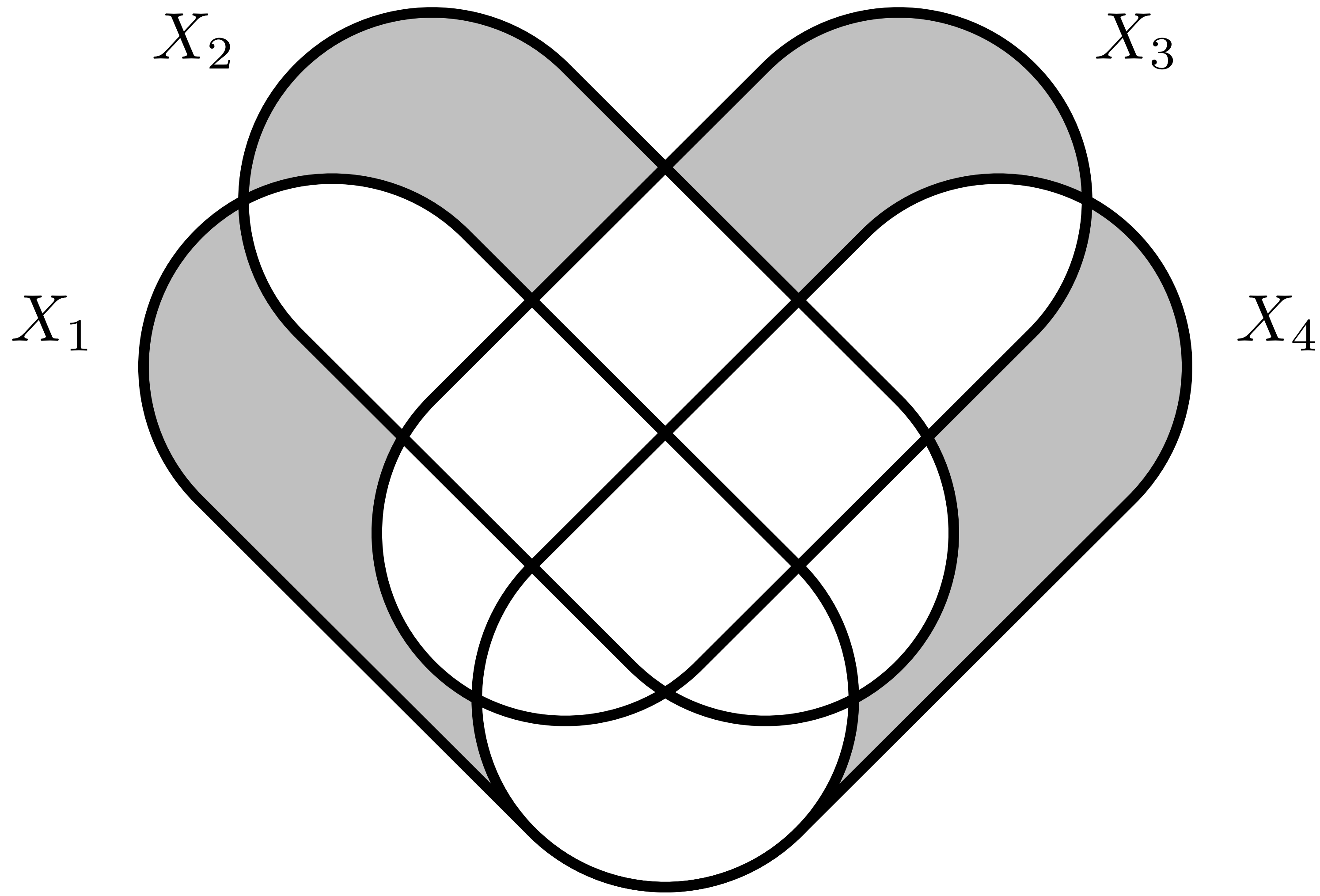
Asymptotic Rates: h_μ & ρ_μ



Asymptotic Rates: h_μ & ρ_μ



Residual Entropy



$$\mathrm{R}[X_{1:\ell}] = \sum_{i \in \{1 \ldots \ell\}} \mathrm{H}[X_i | X_{\{1 \ldots \ell\} \setminus i}]$$

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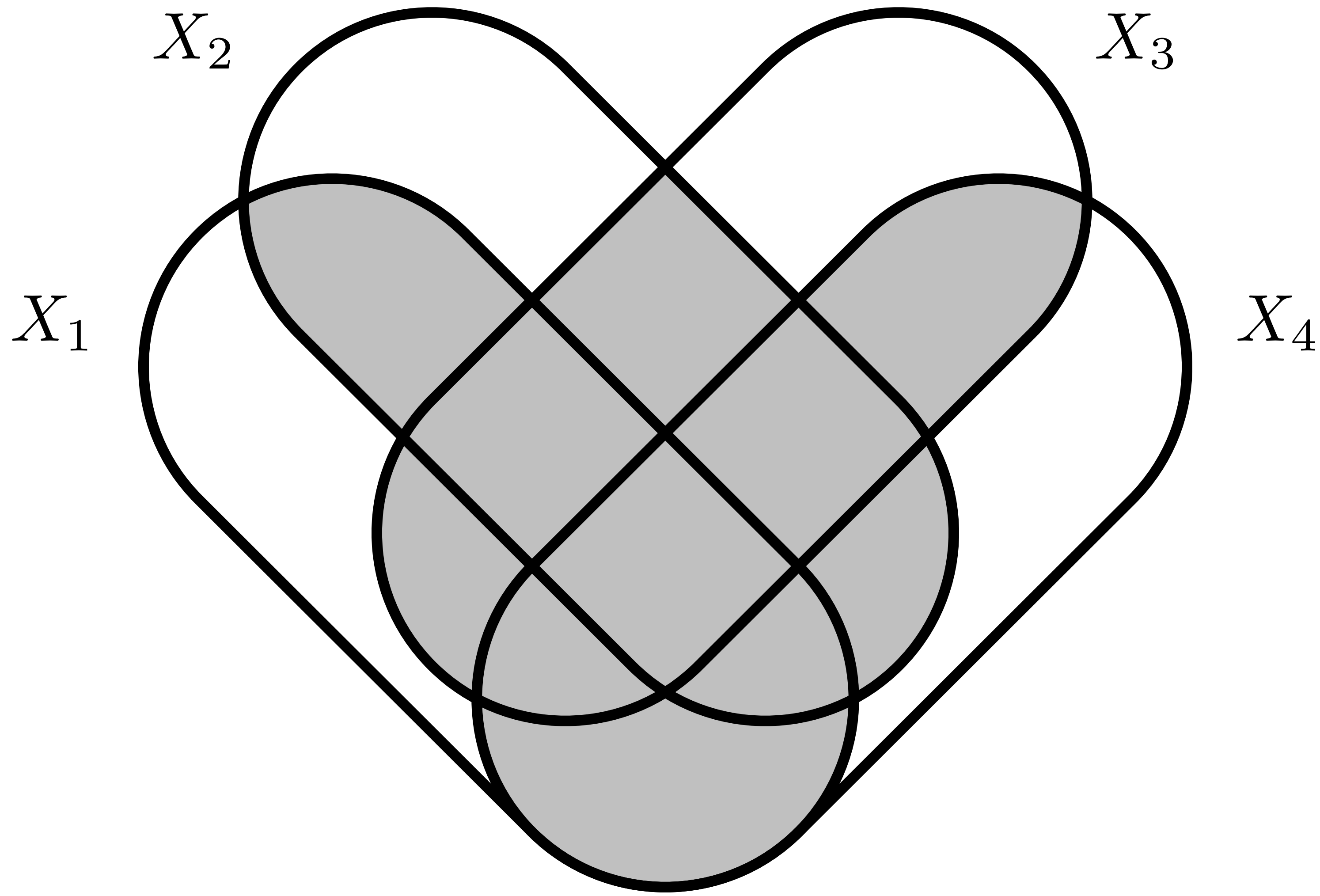
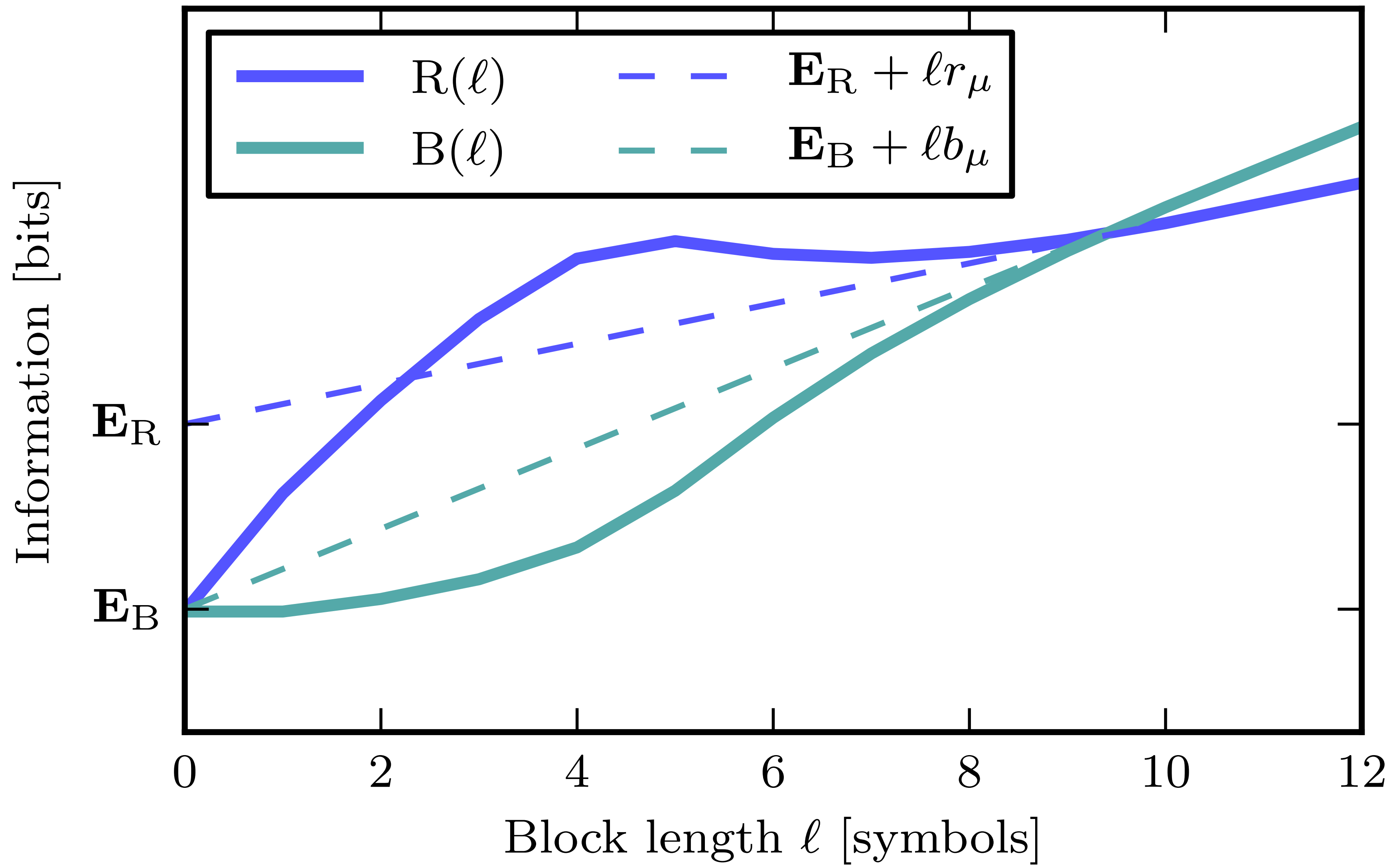
Binding Information



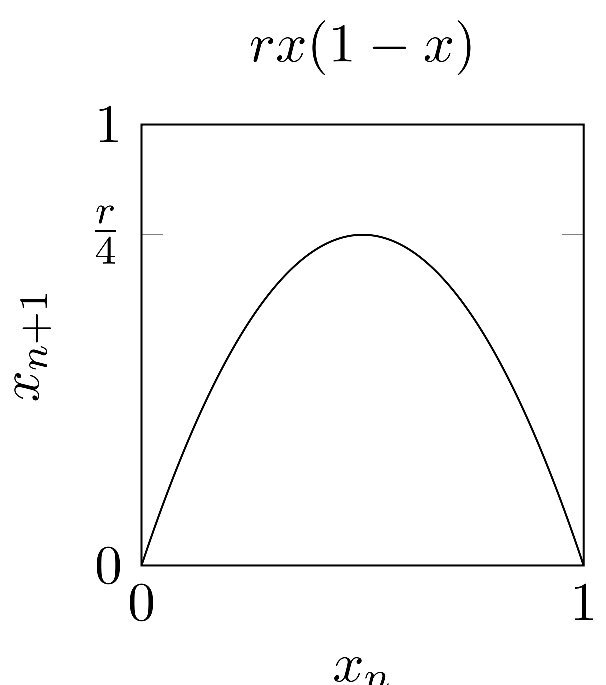
$$\mathrm{B}[X_{1:\ell}] = \mathrm{H}[X_{1:\ell}] - \mathrm{R}[X_{1:\ell}]$$

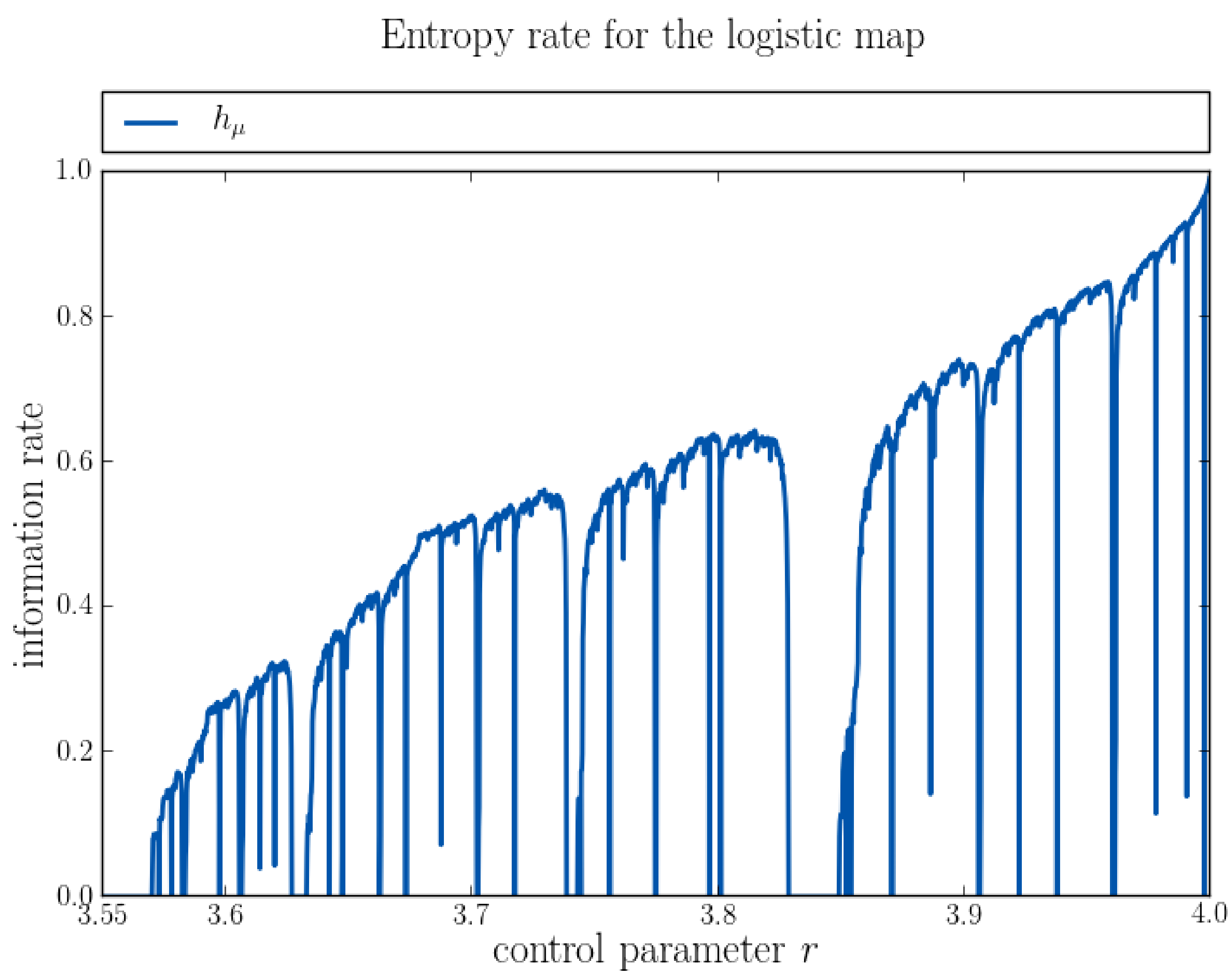
Asymptotic Rates: r_μ & b_μ



Canonical System

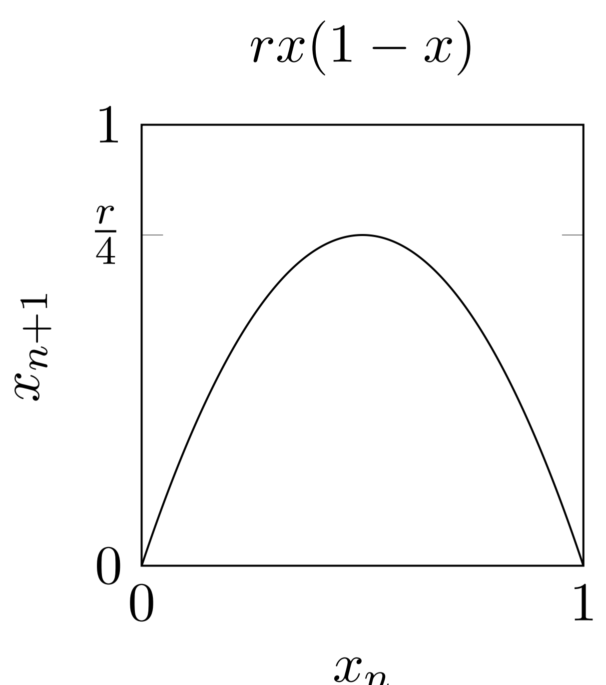
- Pesin's Theorem:
  $h_\mu = \max(0, \lambda)$

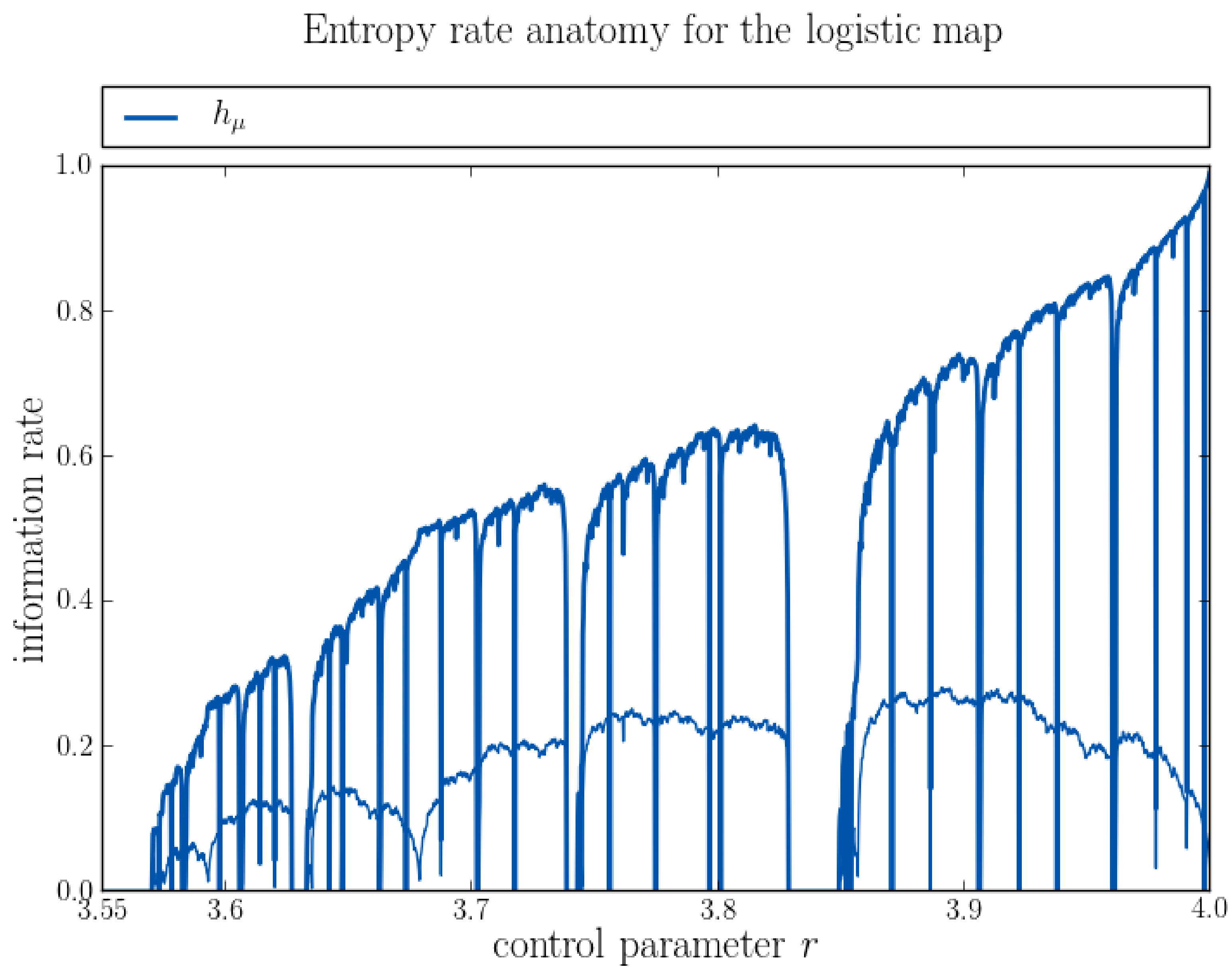




Canonical System

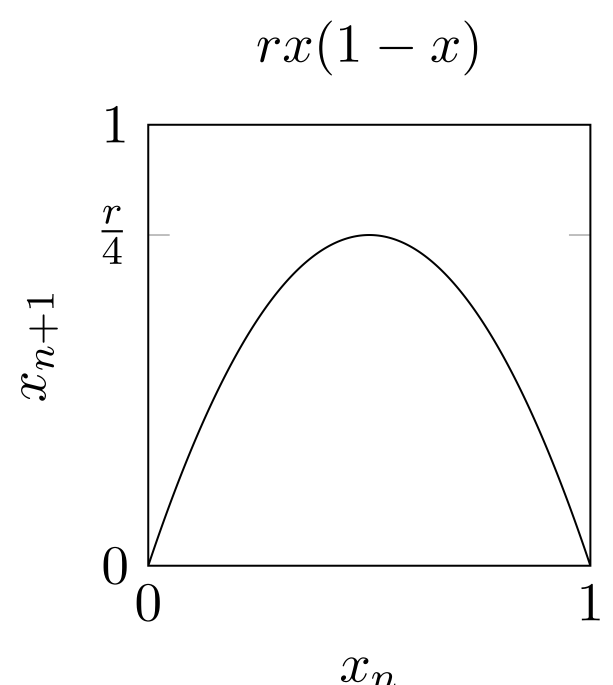
- Pesin's Theorem: $h_\mu = \max(0, \lambda)$
- $h_\mu = \quad +$

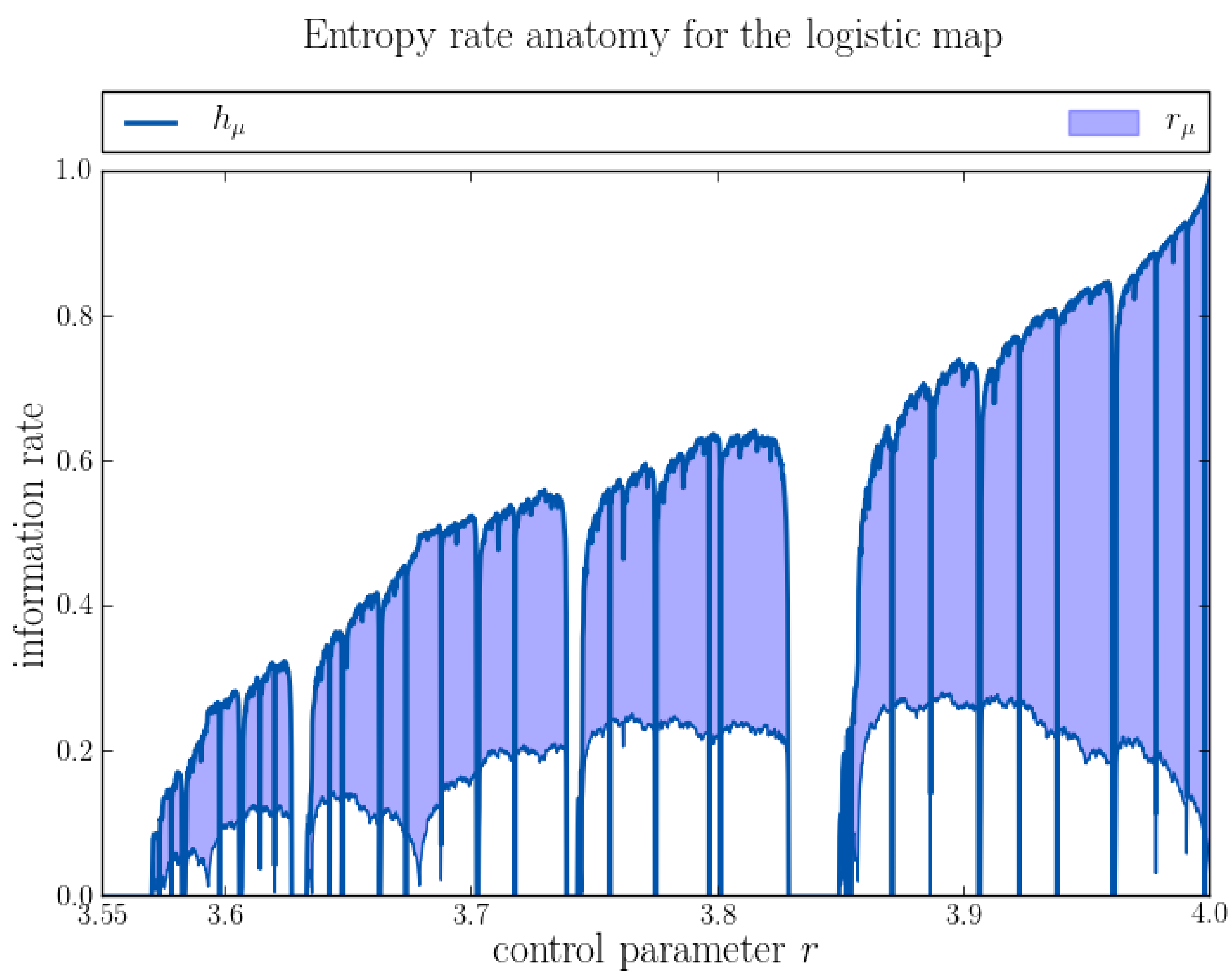




Canonical System

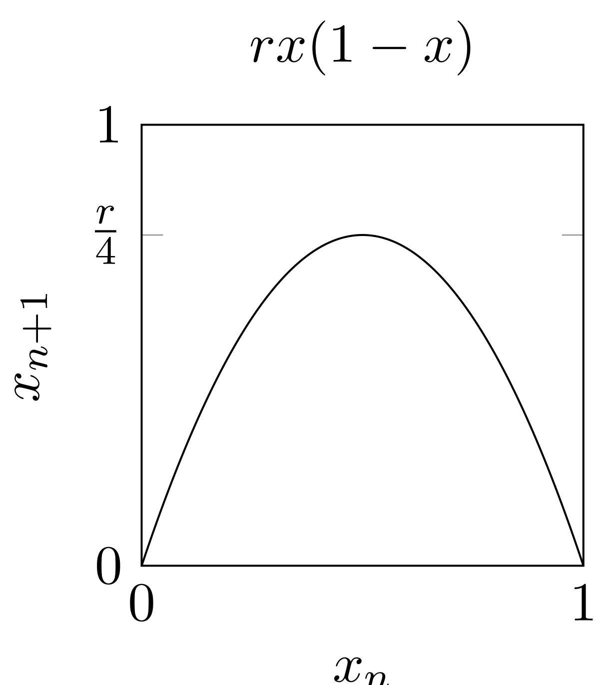
- Pesin's Theorem: $h_\mu = \max(0, \lambda)$
- $h_\mu = r_\mu +$

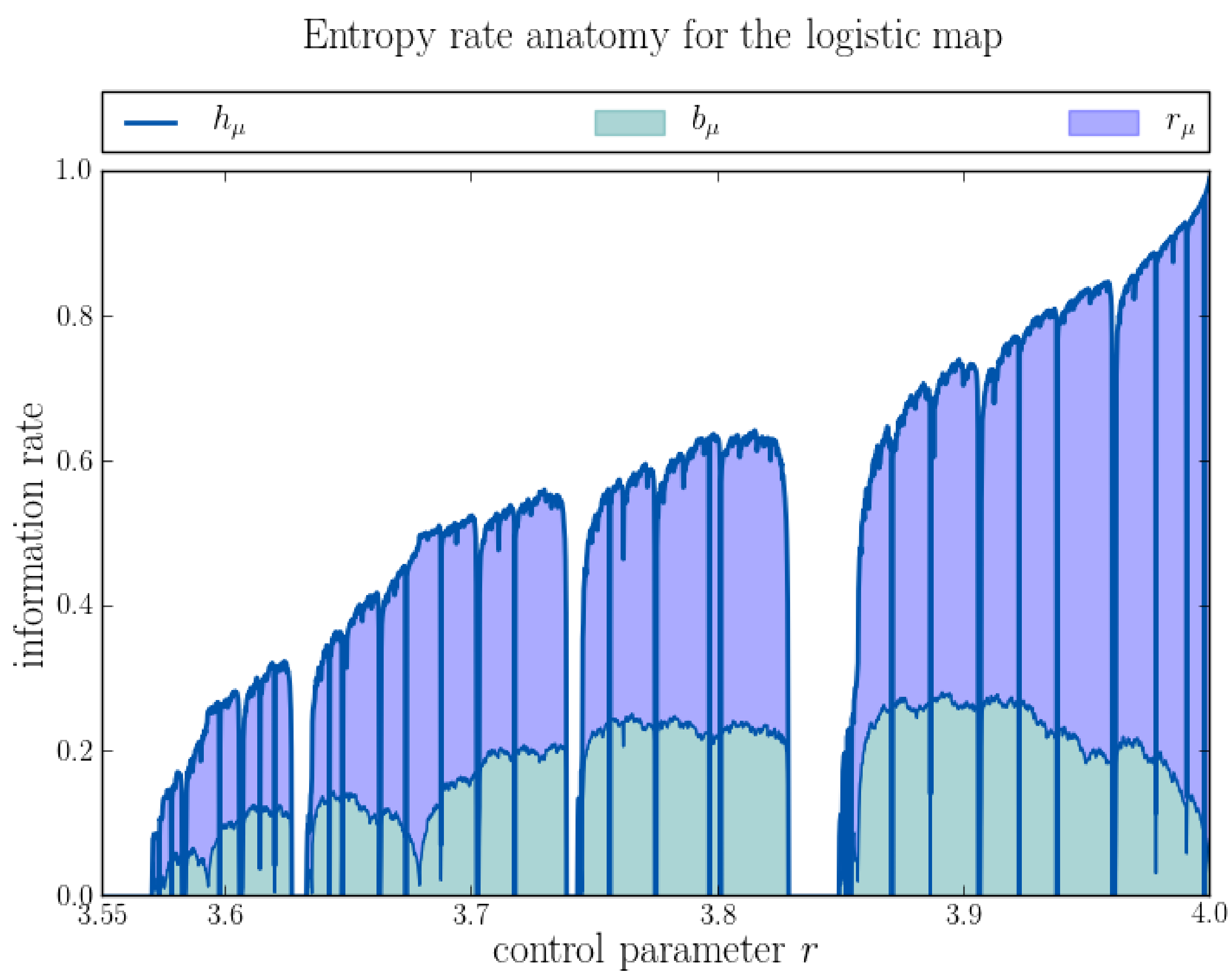




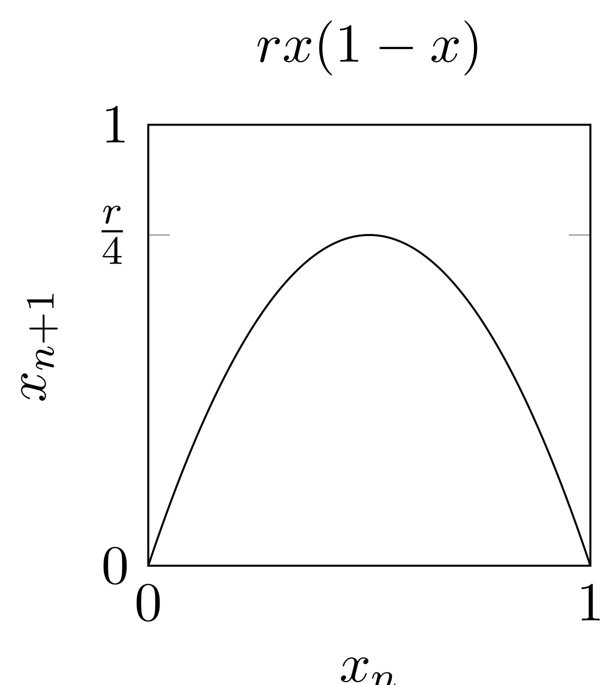
Canonical System

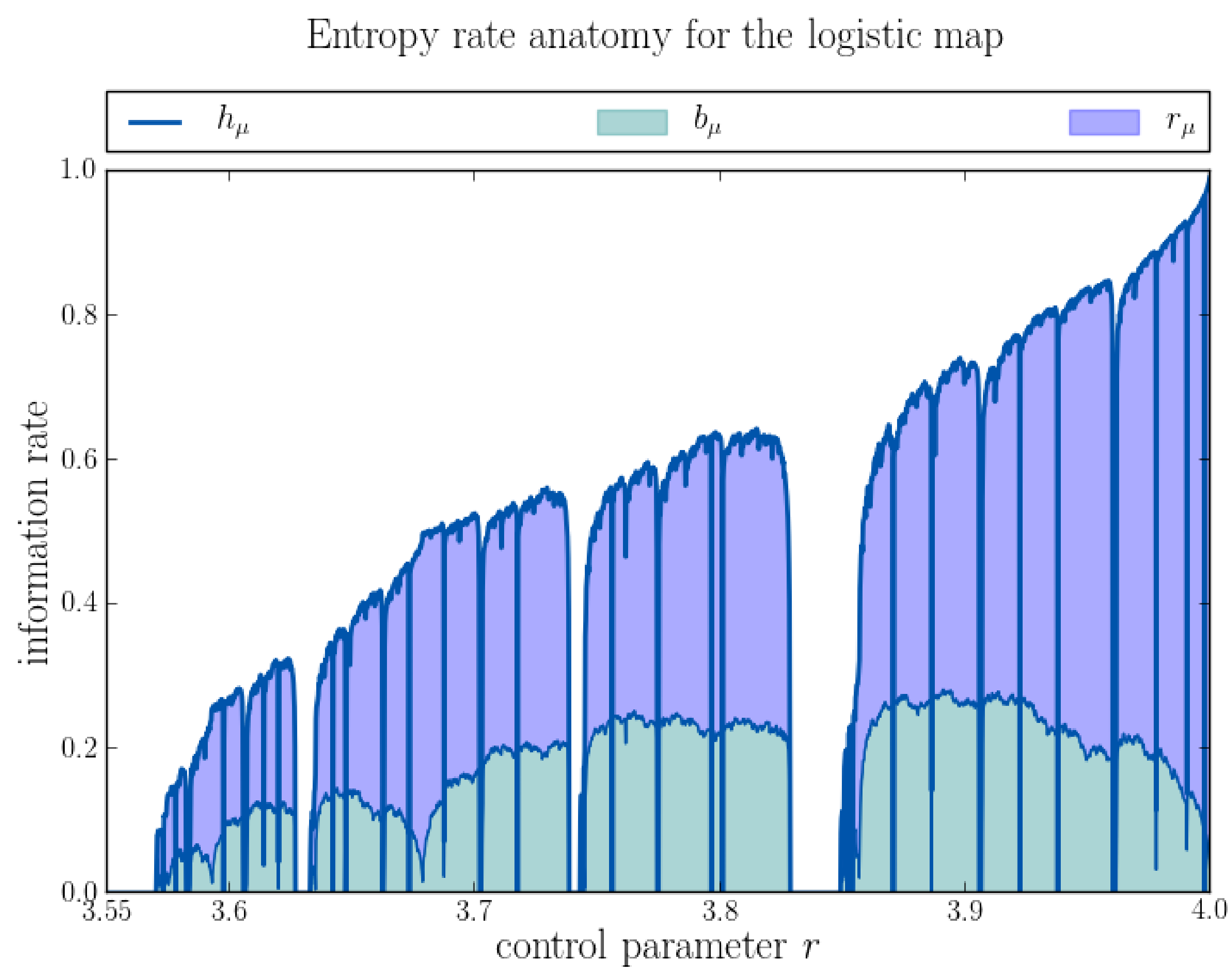
- Pesin's Theorem: $h_\mu = \max(0, \lambda)$
- $h_\mu = r_\mu + b_\mu$





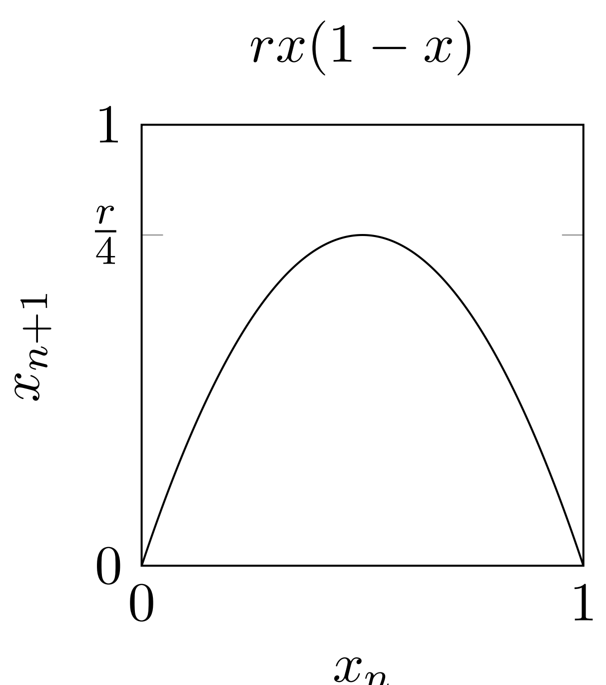
Features

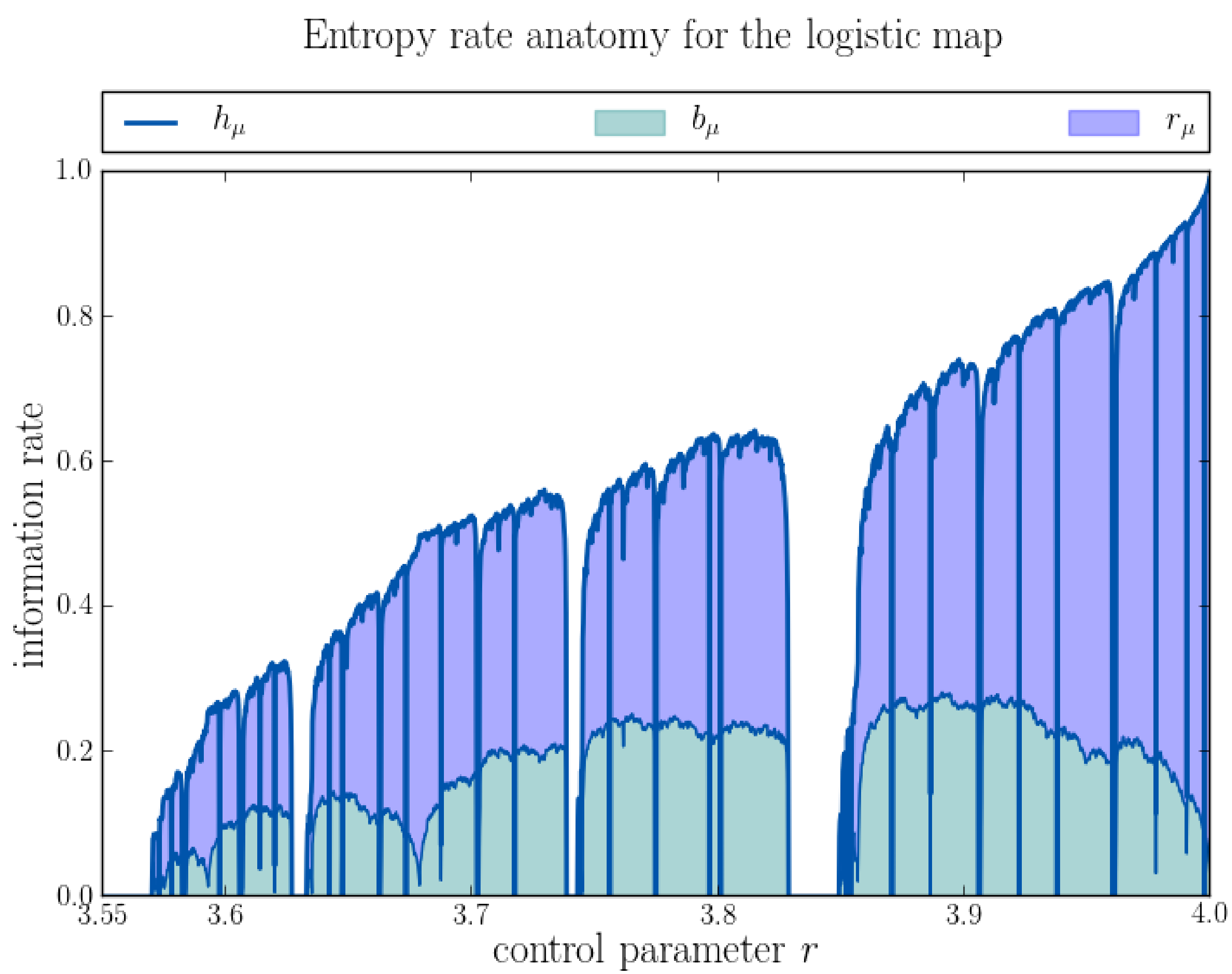

$rx(1-x)$
1
$\frac{r}{4}$
0
0
1
x_{n+1}
x_n


Entropy rate anatomy for the logistic map
h_μ
b_μ
r_μ
1.0
0.8
0.6
0.4
0.2
0.0
information rate
3.55
3.6
3.7
3.8
3.9
4.0
control parameter r

Features

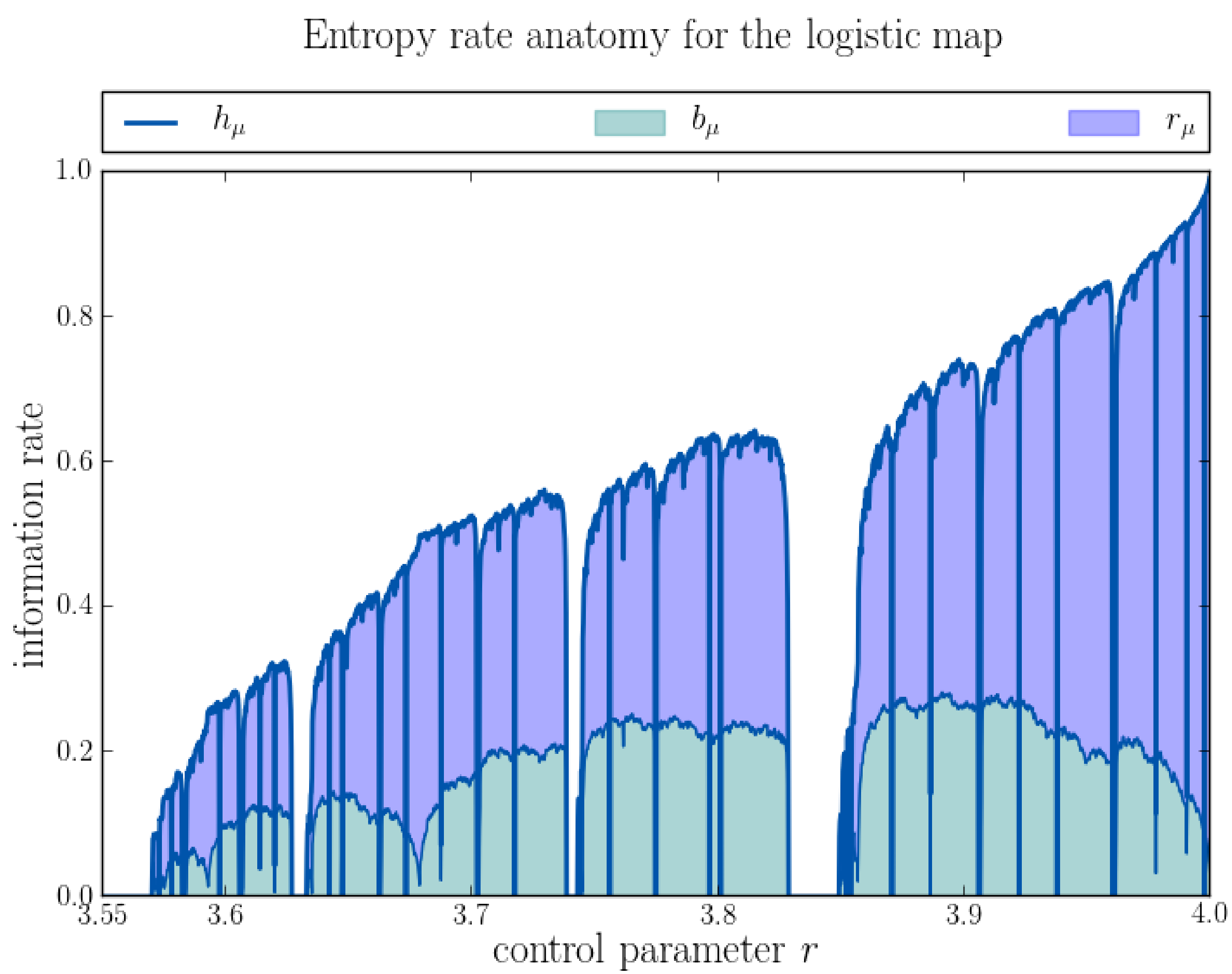
- Nontrivial r_μ, b_μ decomposition

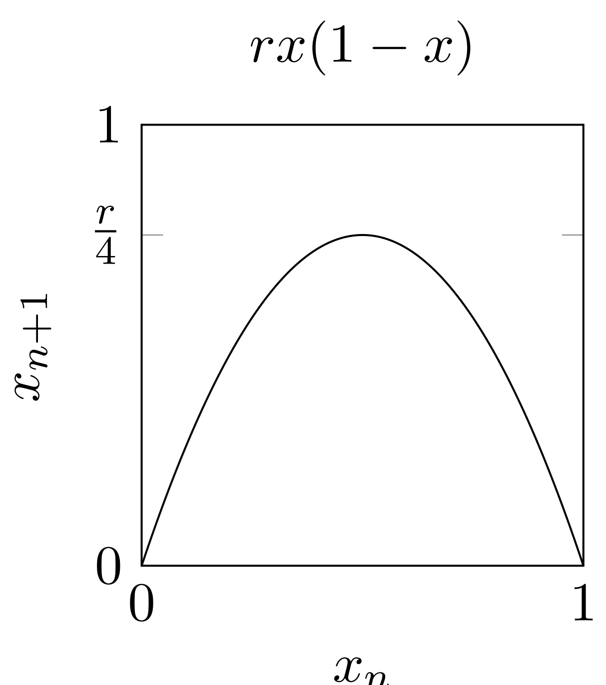




Features

- Nontrivial r_μ, b_μ decomposition
- Self-similarity





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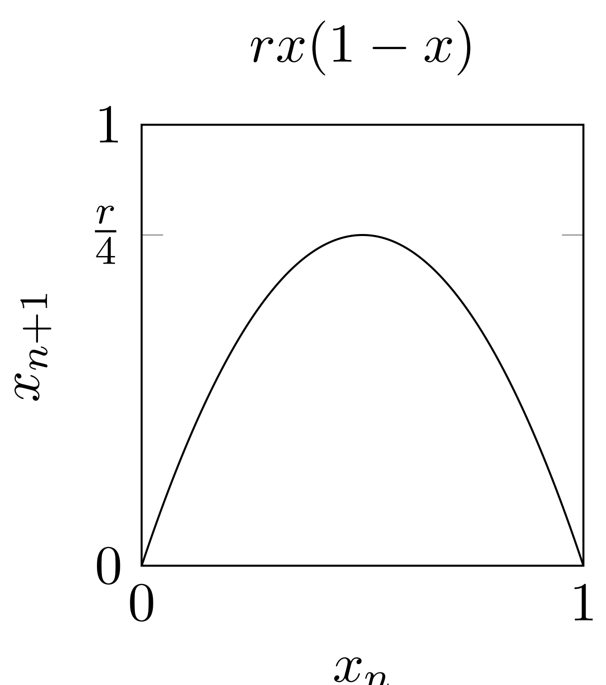
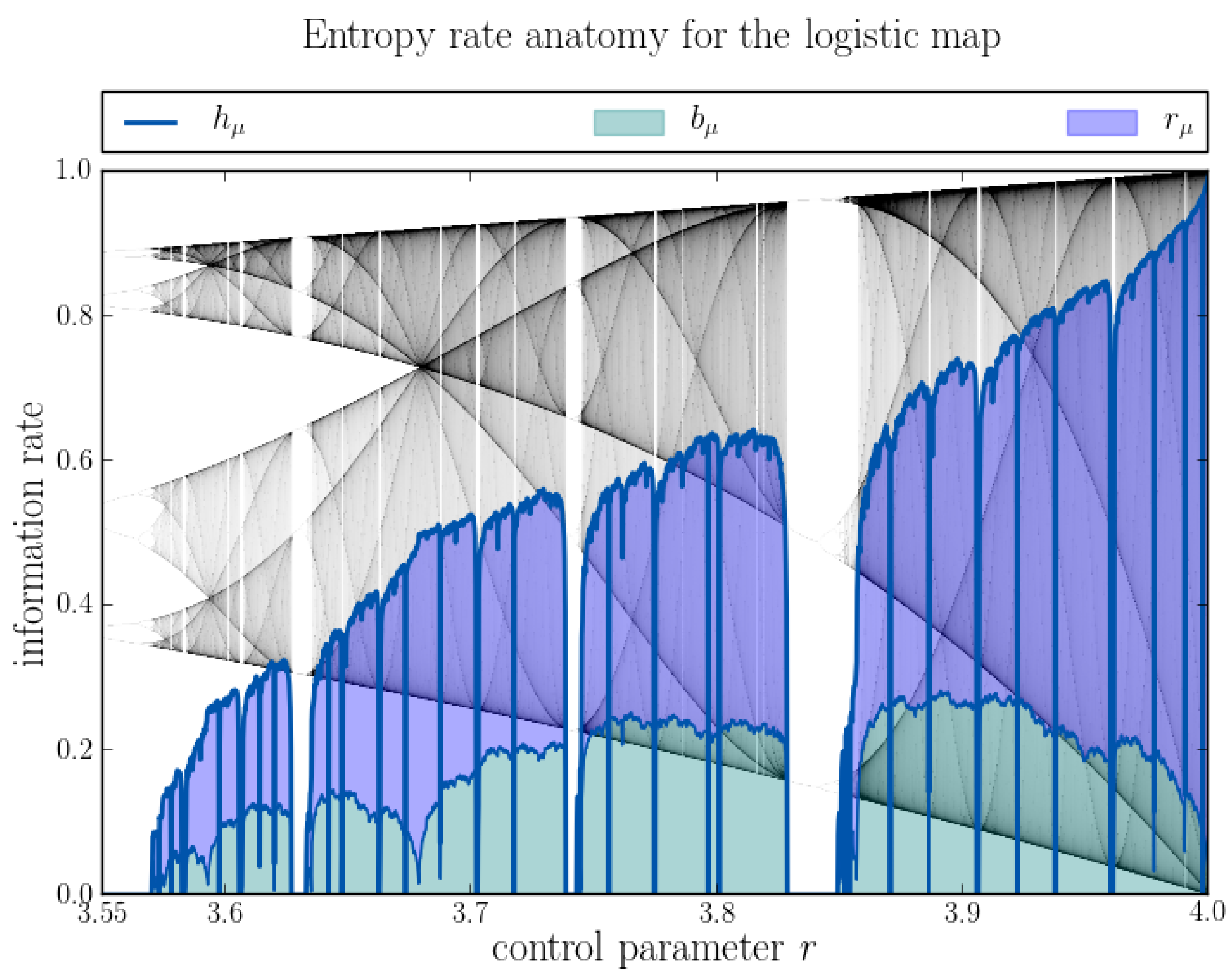
Features

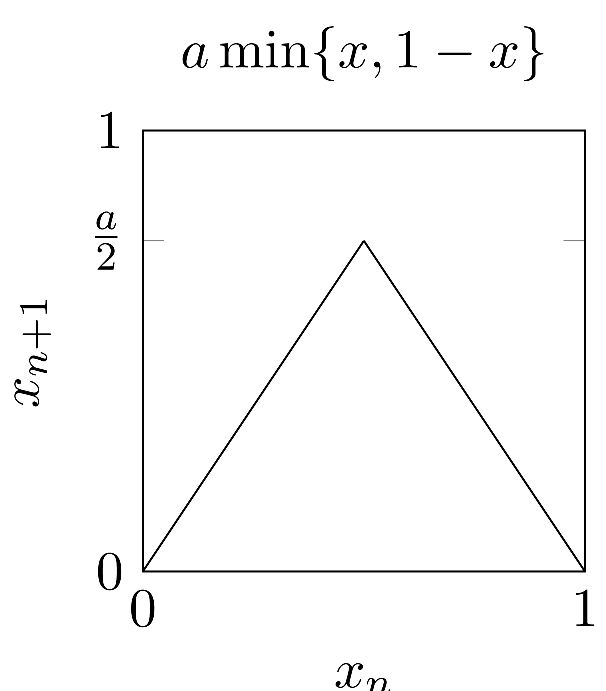
- Nontrivial r_μ, b_μ decomposition
- Self-similarity
- $b_\mu = 0$ at band mergings

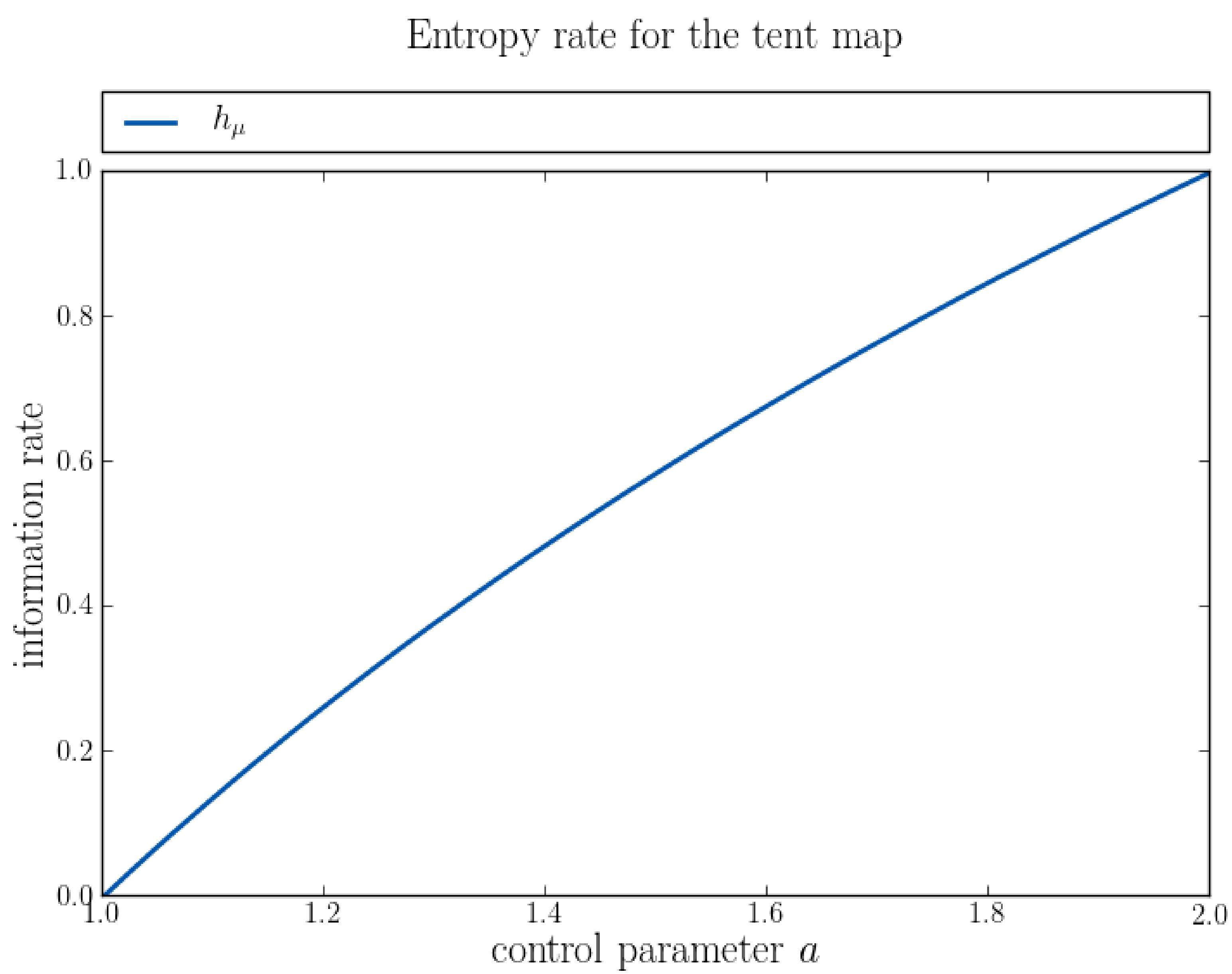




Clean Example

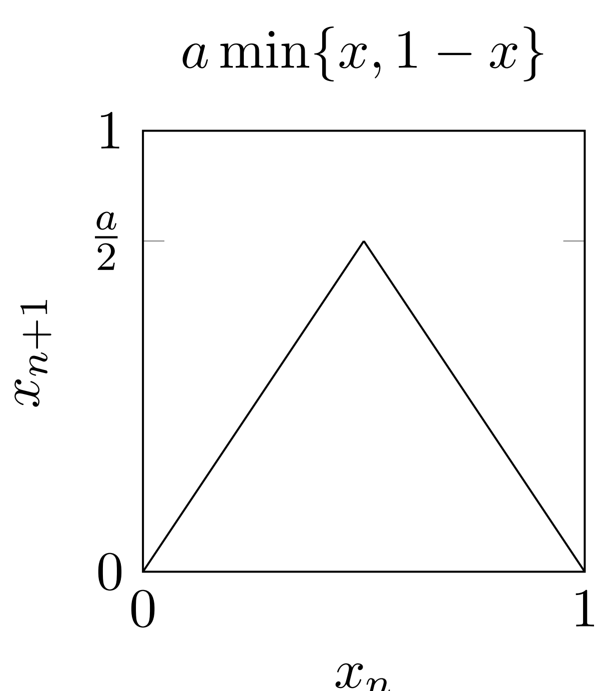
- $h_\mu = \log(a)$

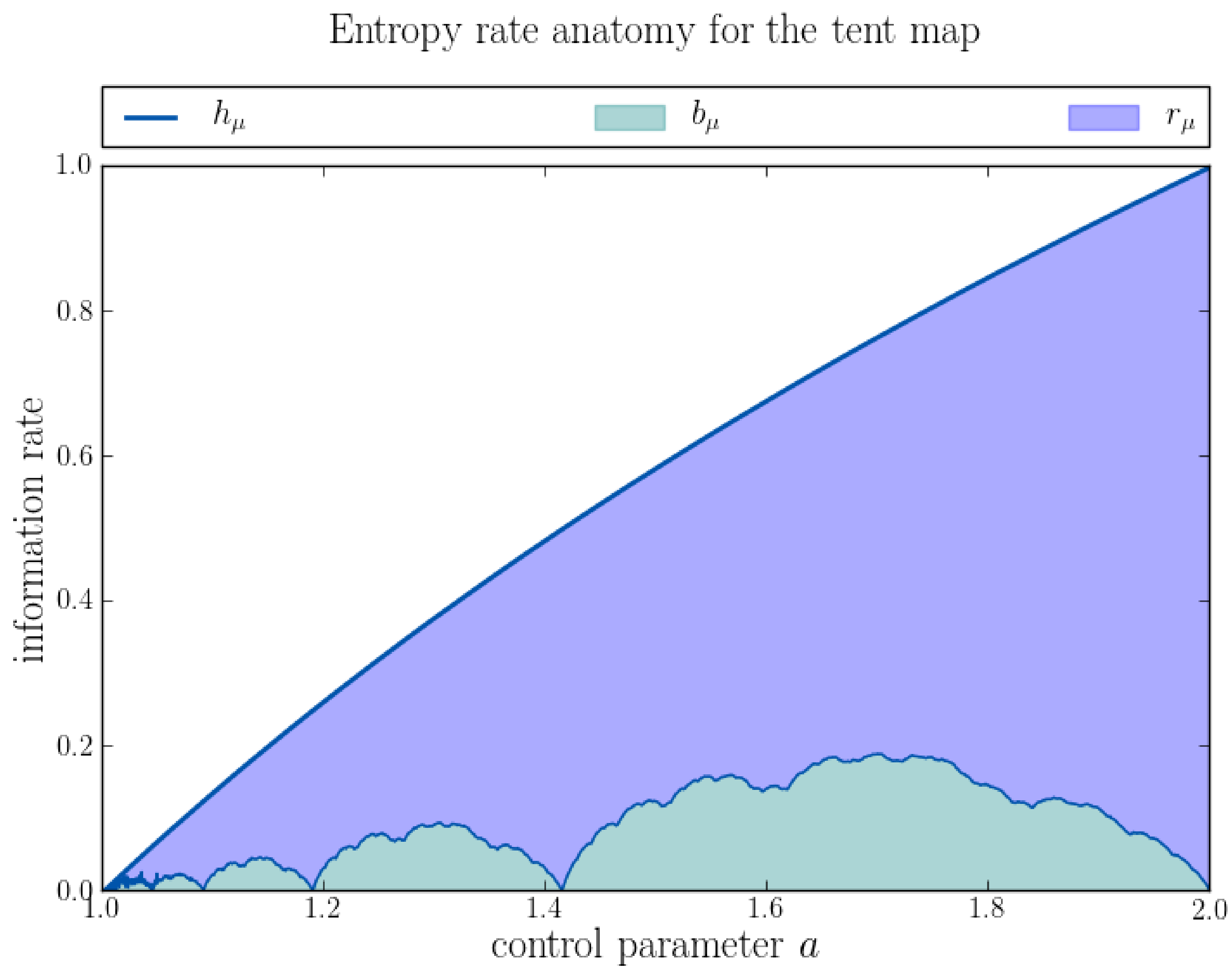




Clean Example

- $h_\mu = \log(a)$
- $h_\mu = r_\mu + b_\mu$





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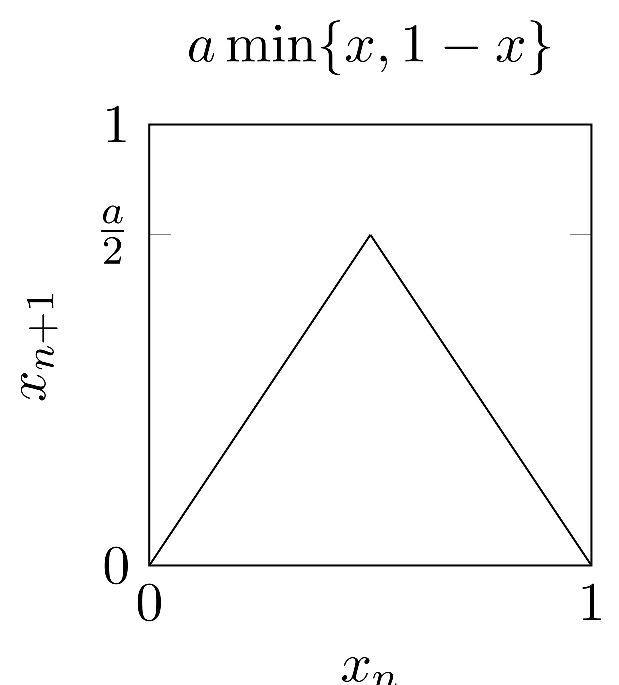
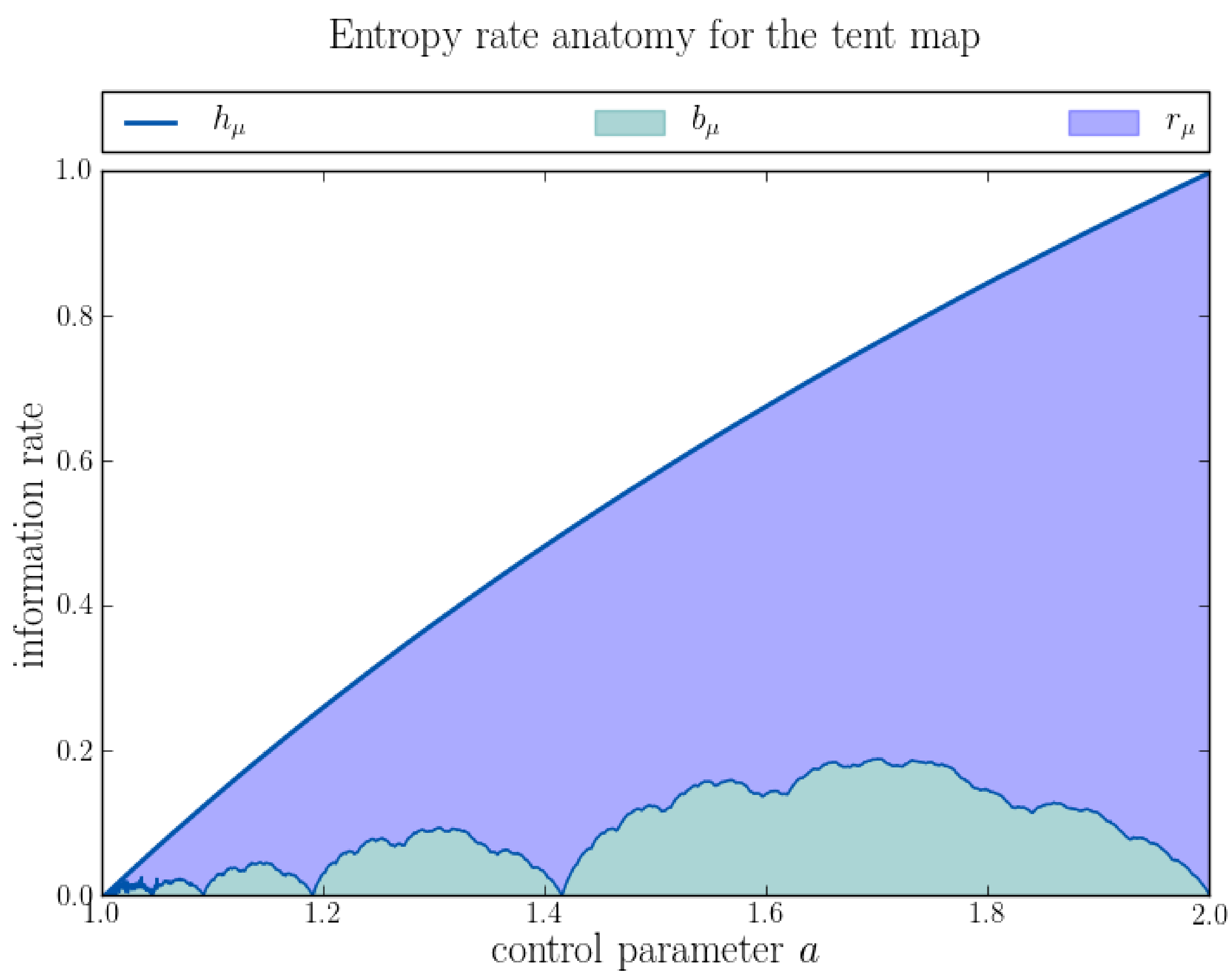
Clean Example

- $h_\mu = \log(a)$
- $h_\mu = r_\mu + b_\mu$
- r_μ, b_μ decomposition still complex





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Clean Example

- $h_\mu = \log(a)$
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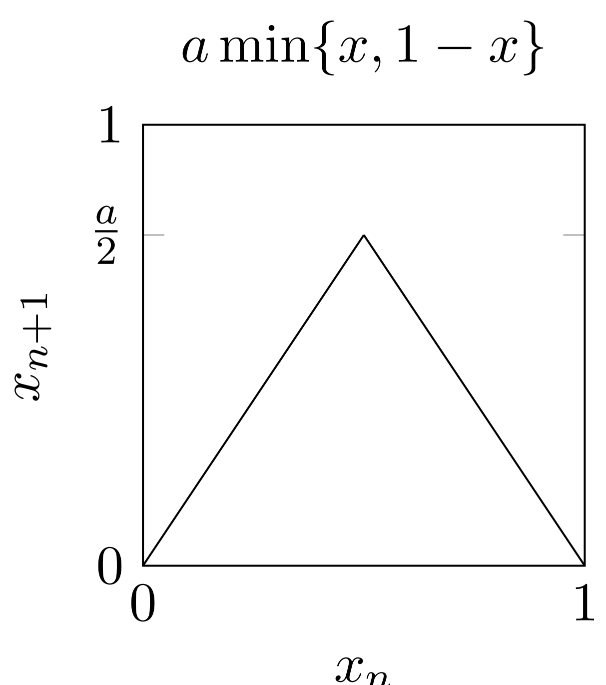
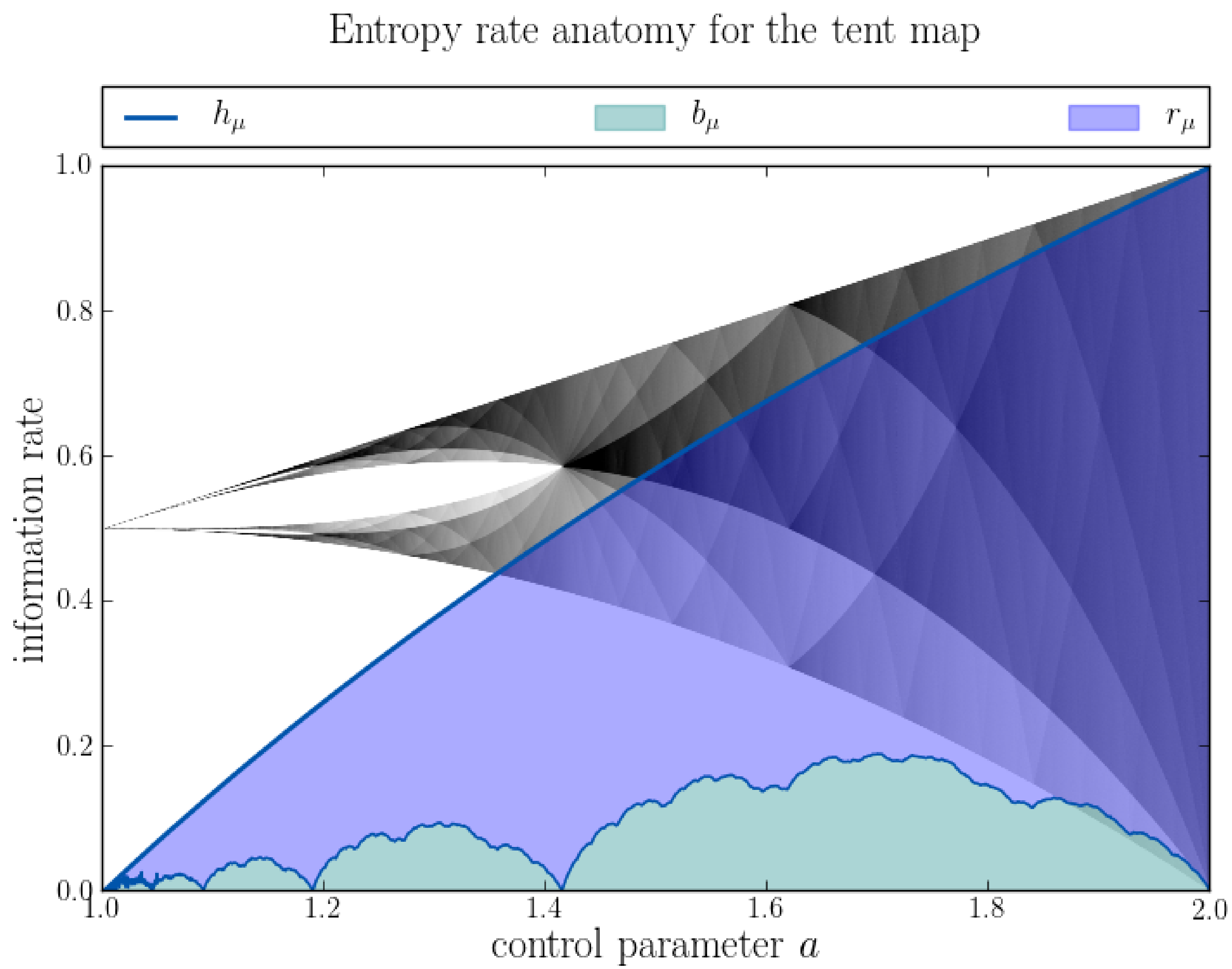
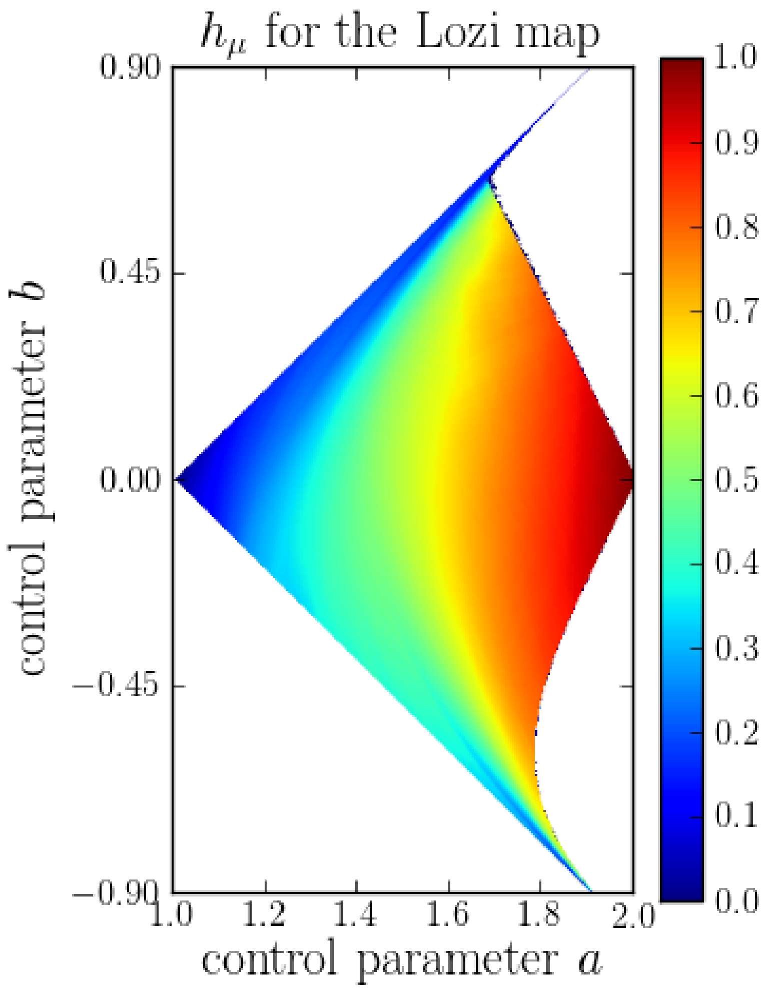


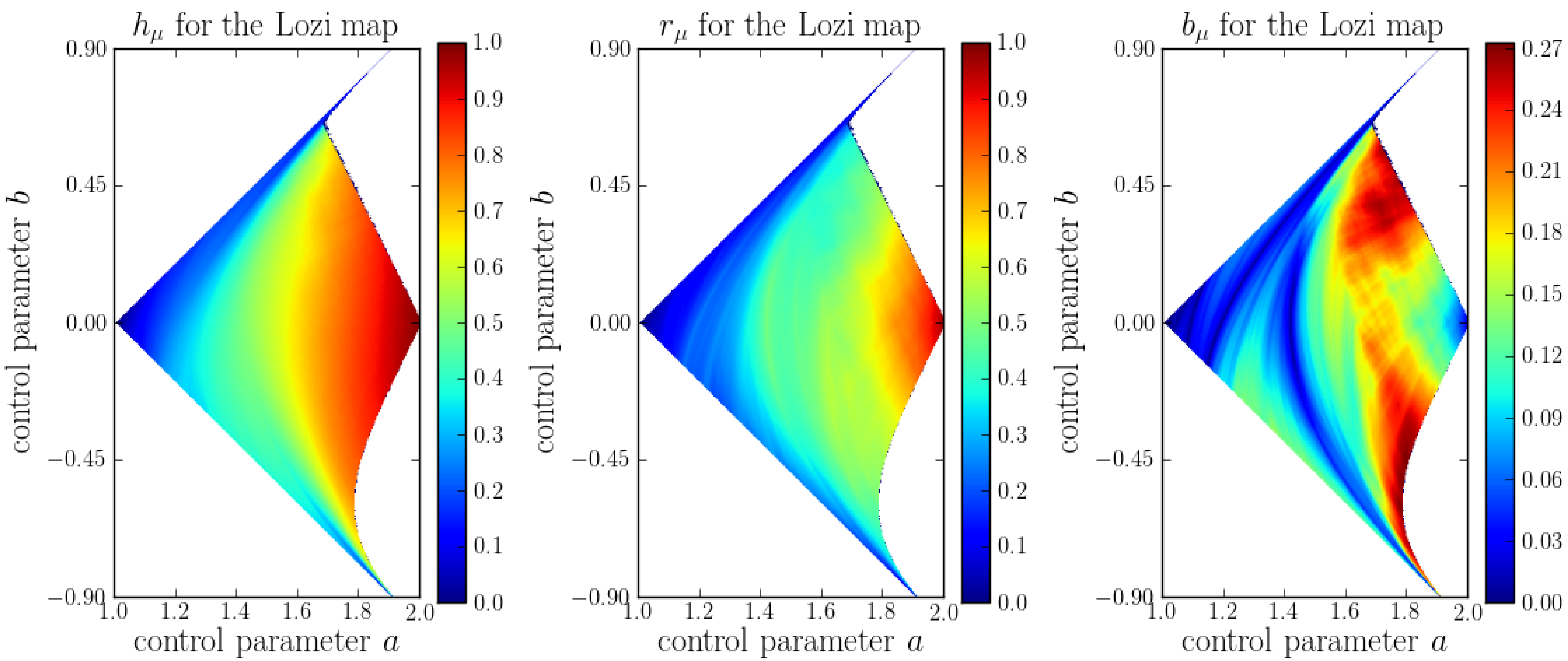


Moving to 2D



$$x_{n+1} = 1 - a\,|x_n| + y_n \qquad y_{n+1} = b x_n$$

Moving to 2D



$$x_{n+1} = 1 - a\,|x_n| + y_n \qquad y_{n+1} = b x_n$$

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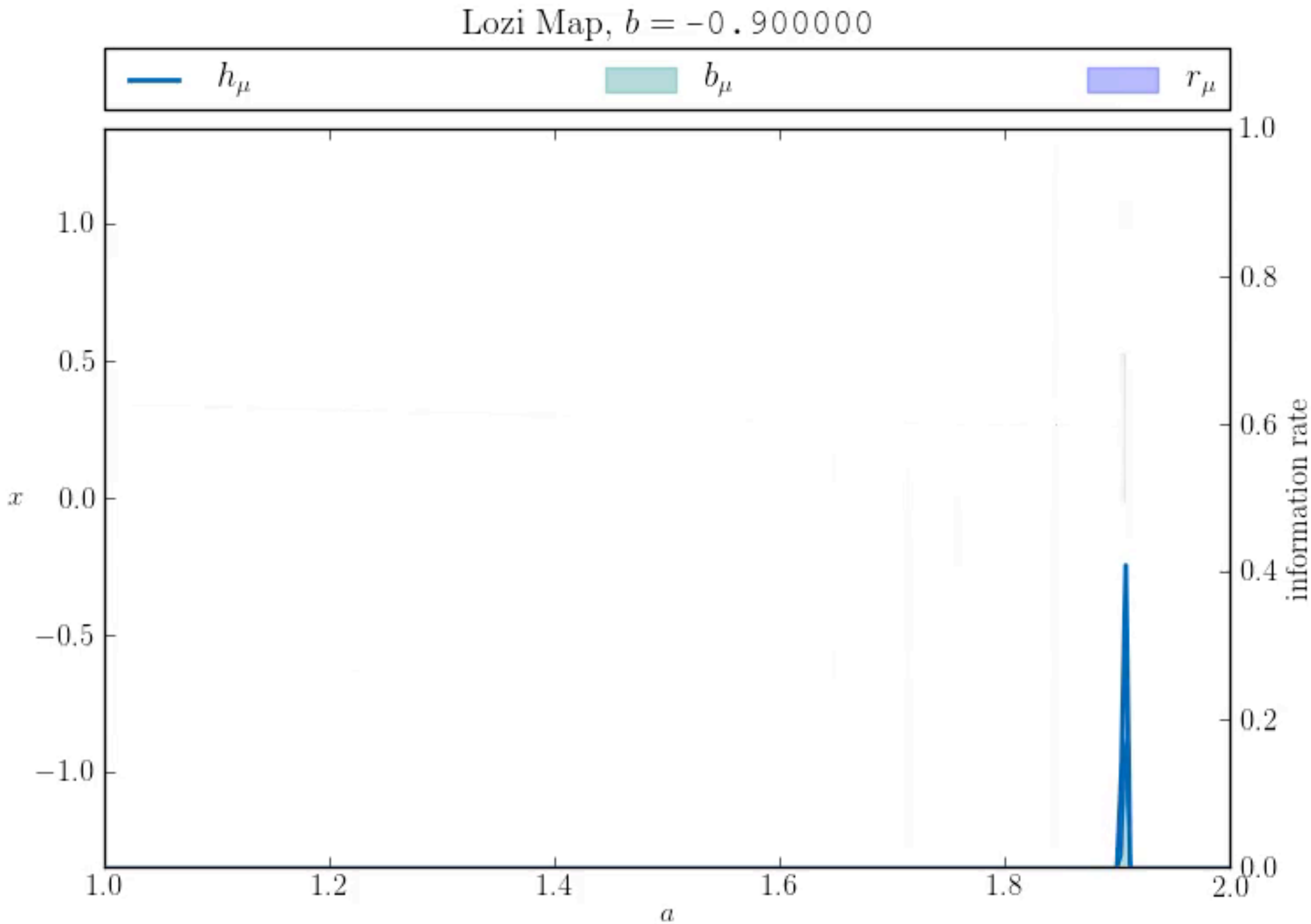
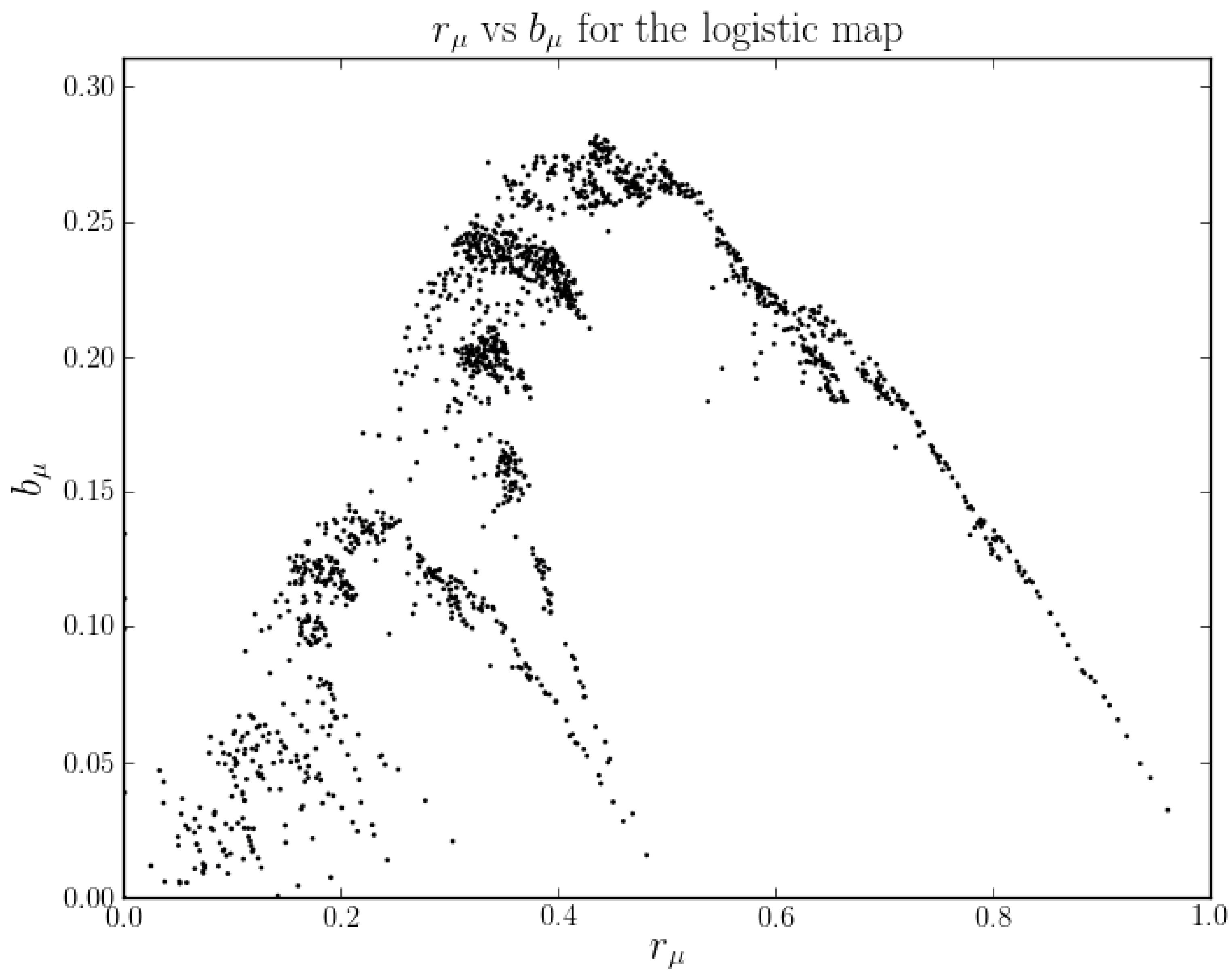
Movie


Lozi Map, $b = -0.900000$
h_μ
b_μ
r_μ
1.0
0.5
0.0
−0.5
−1.0
x
1.0
0.8
0.6
0.4
0.2
0.0
information rate
1.0
1.2
1.4
1.6
1.8
2.0
a

Logistic Map

- Trade-off between noise and structure



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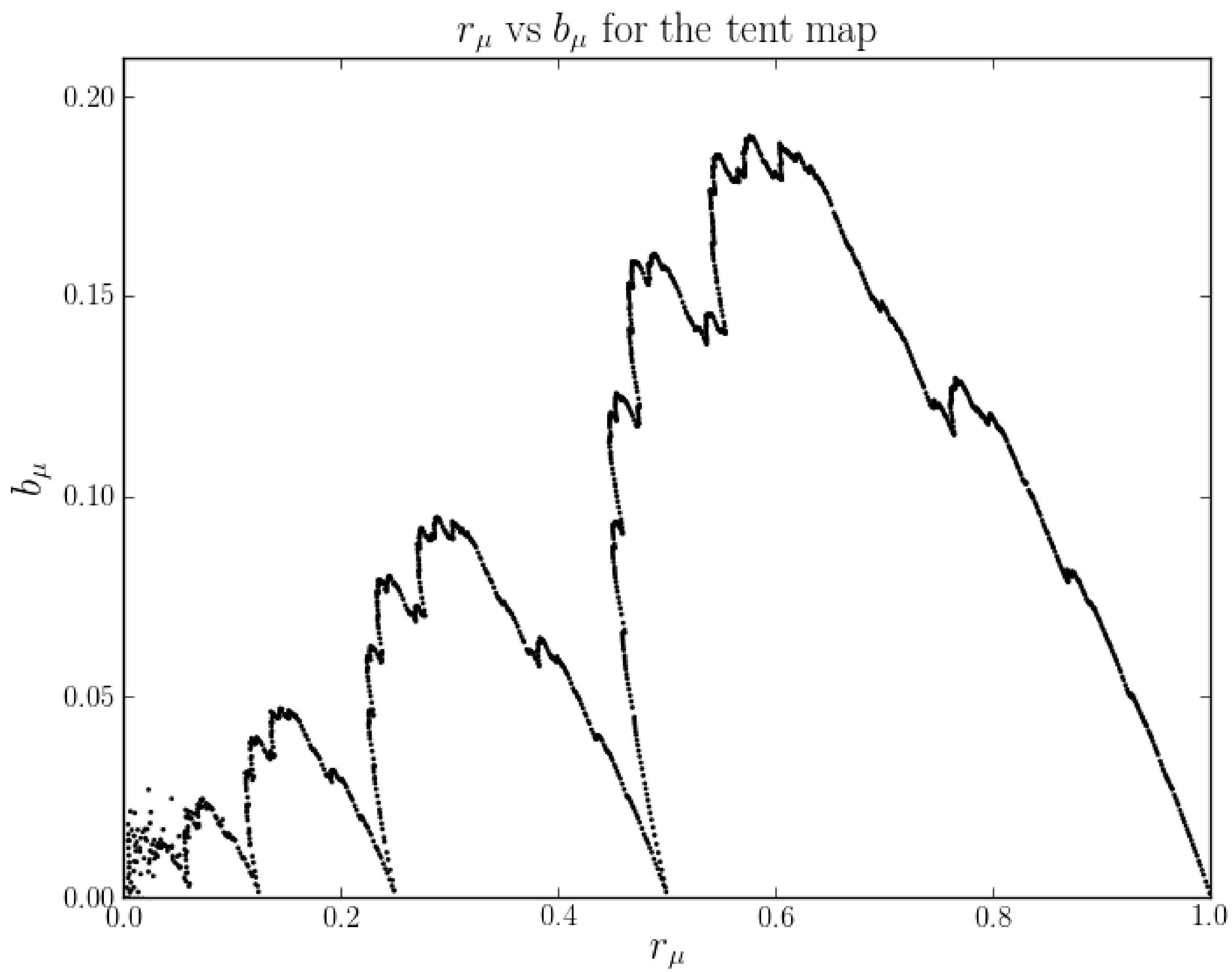
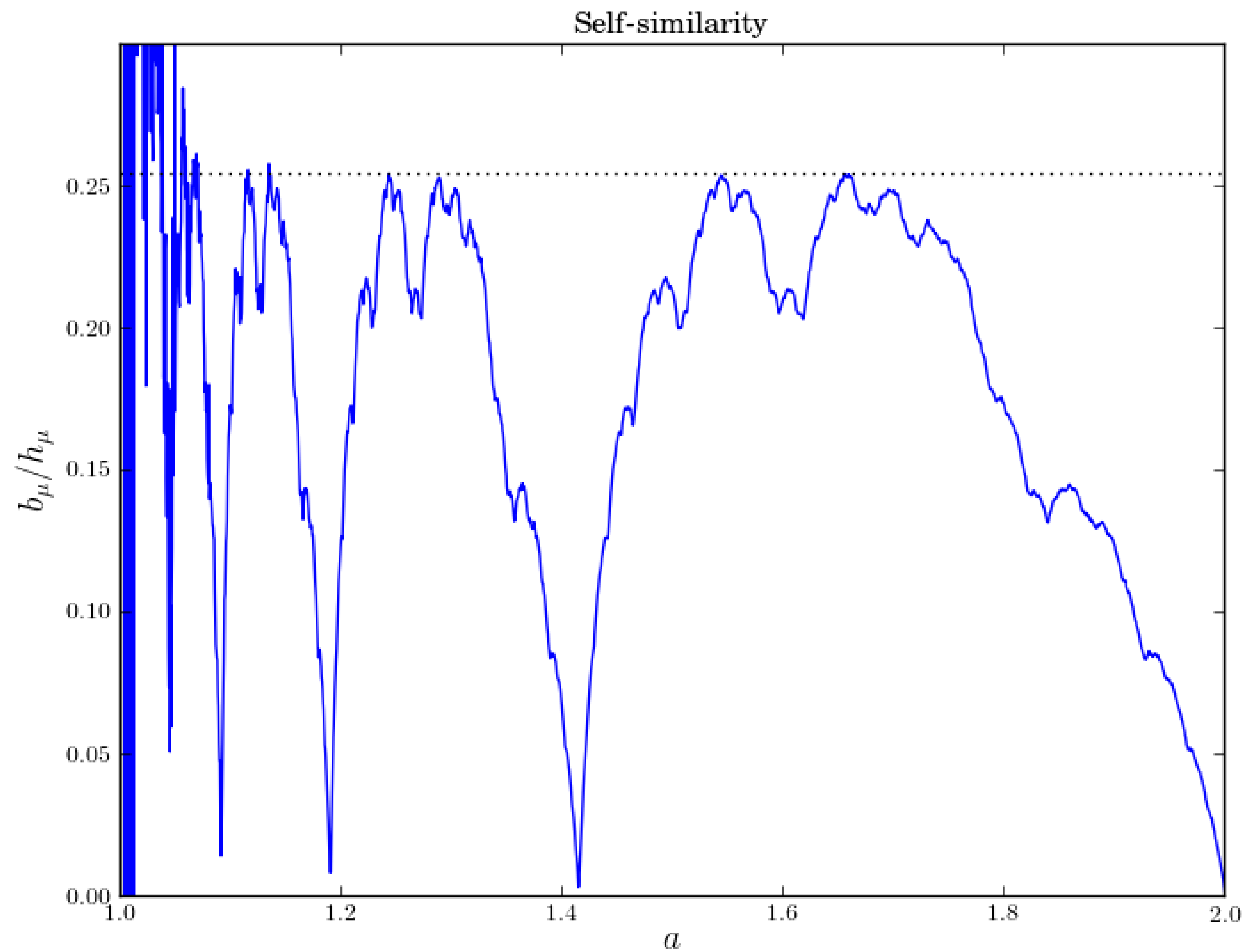
Tent Map

- Trade-off between noise and structure
- Very structured



Tent Map

- Scaling between band mergings is self-similar



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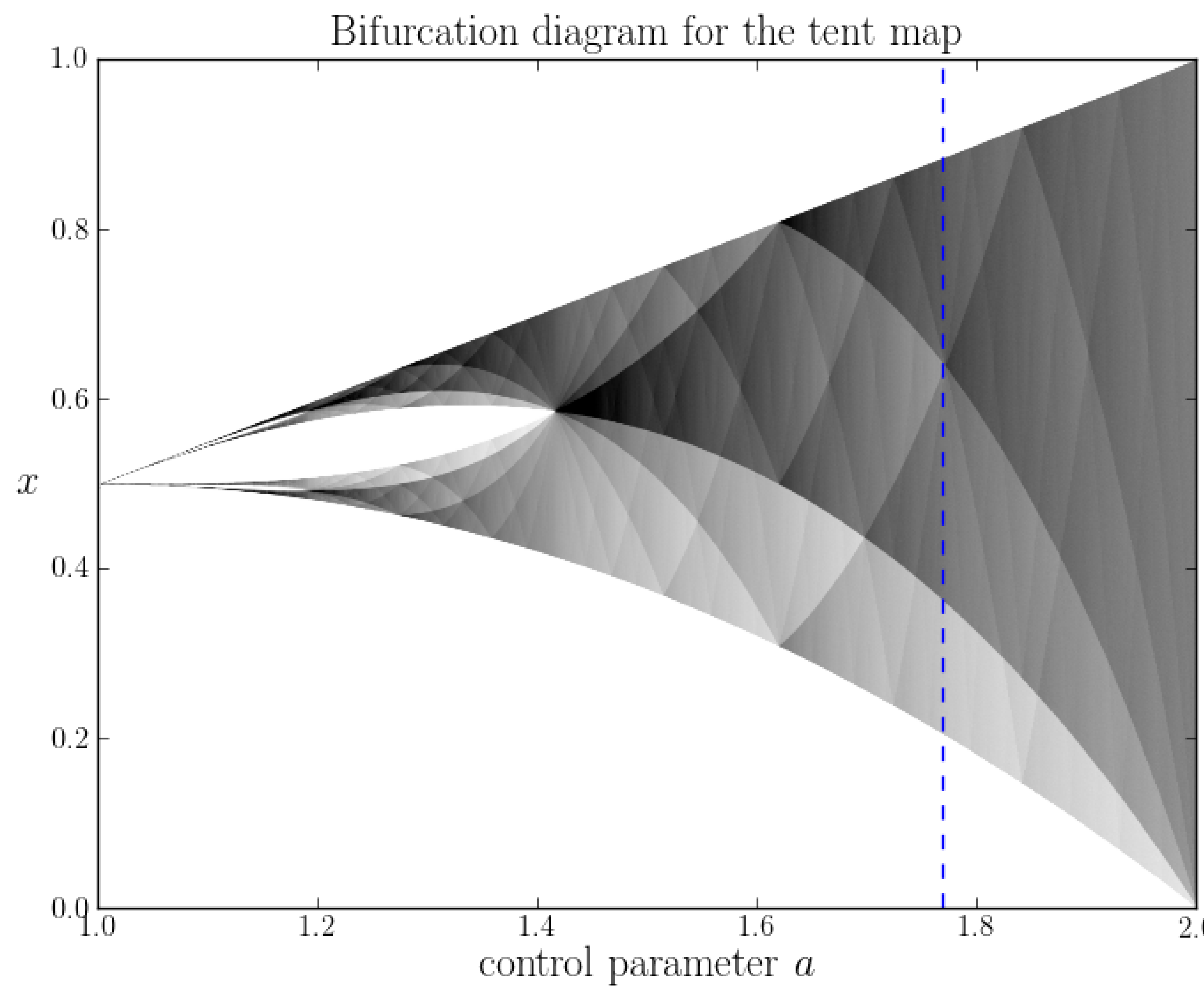
Compute a Misiurewicz Point

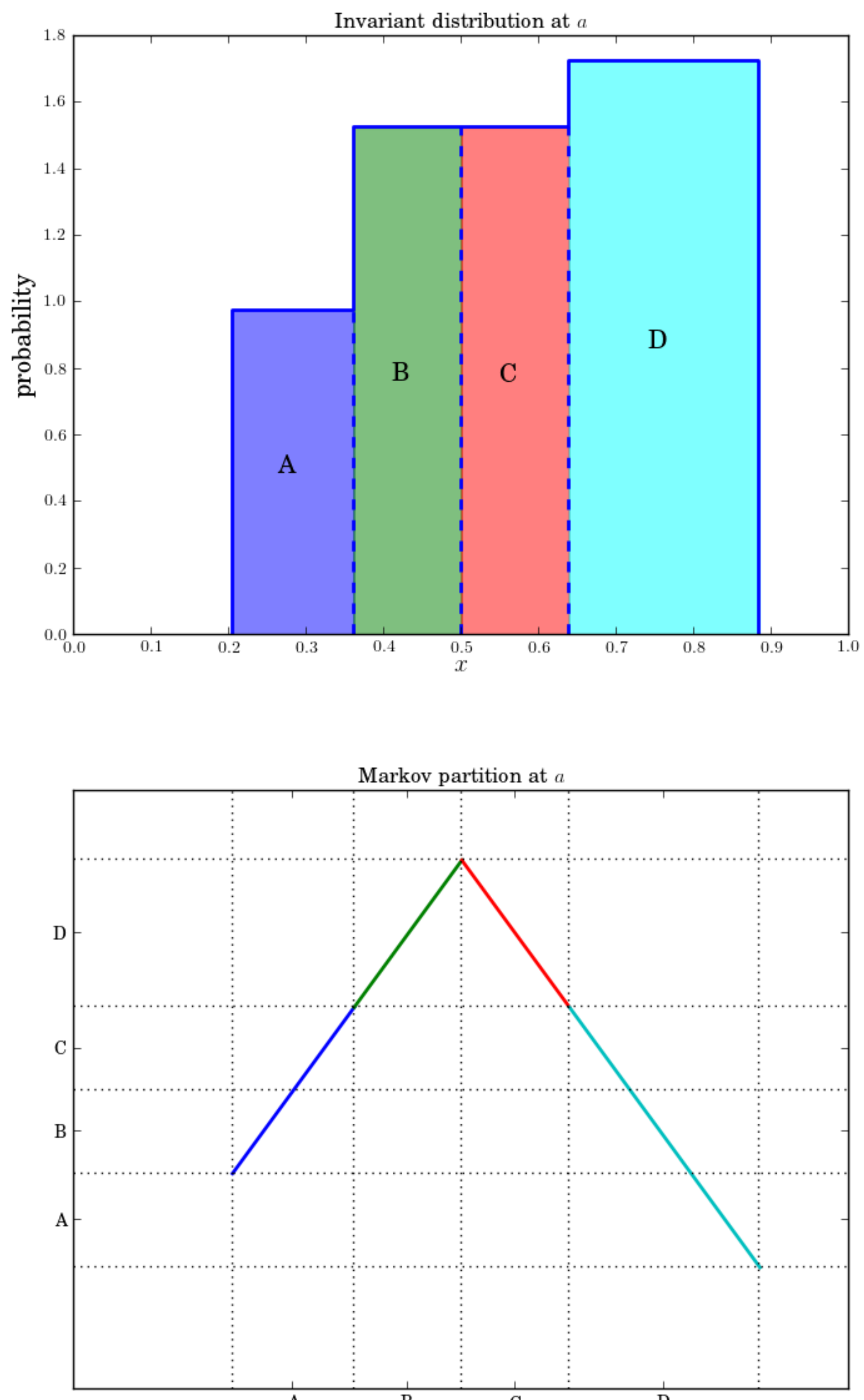
$$\alpha = \sqrt[3]{\sqrt{\frac{19}{27}} + 1}$$

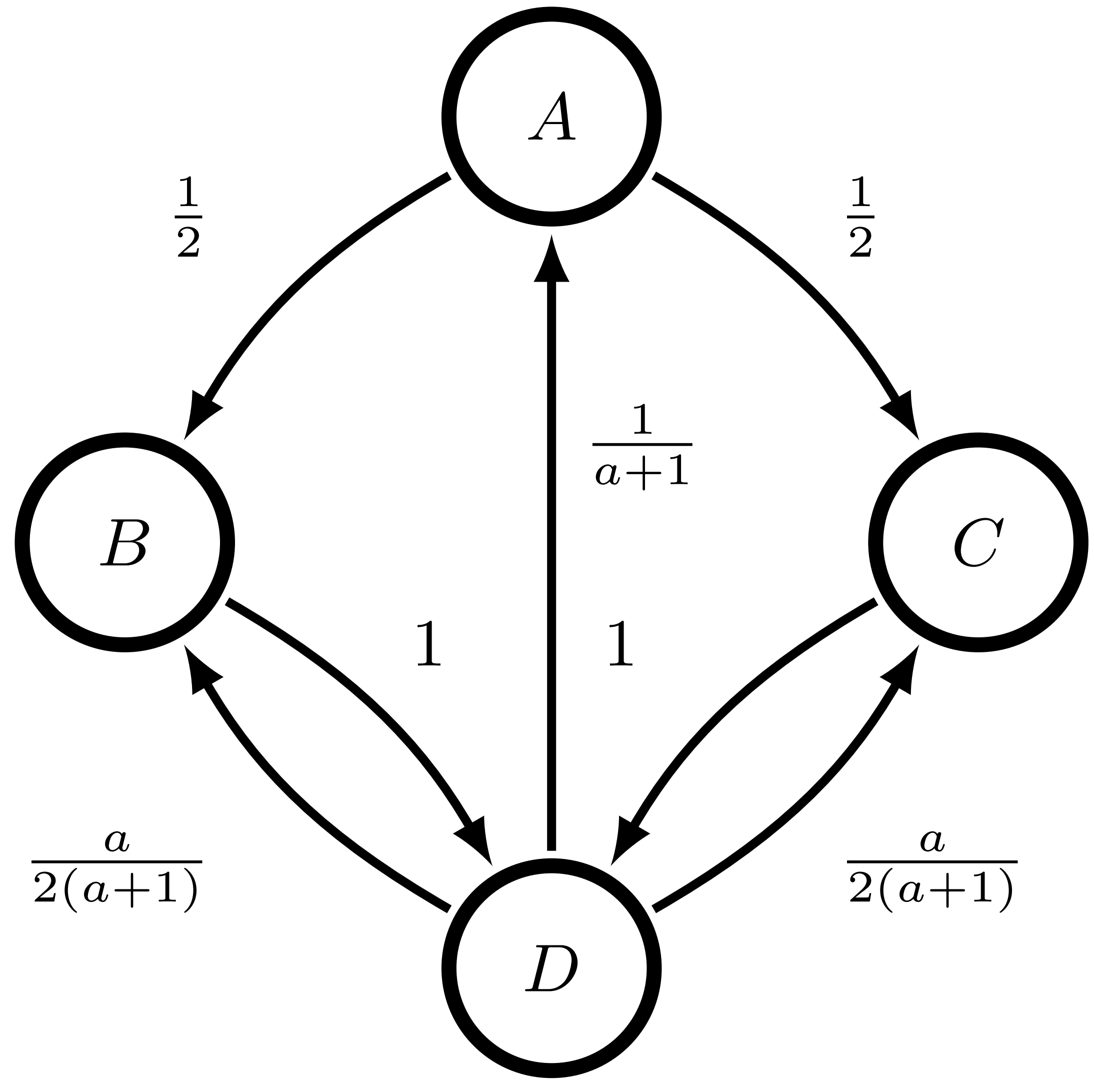
$$a = \alpha + \frac{2}{3\alpha}$$

$$= 1.76929235\ldots$$



The Misiurewicz Point Admits a Markov Partition





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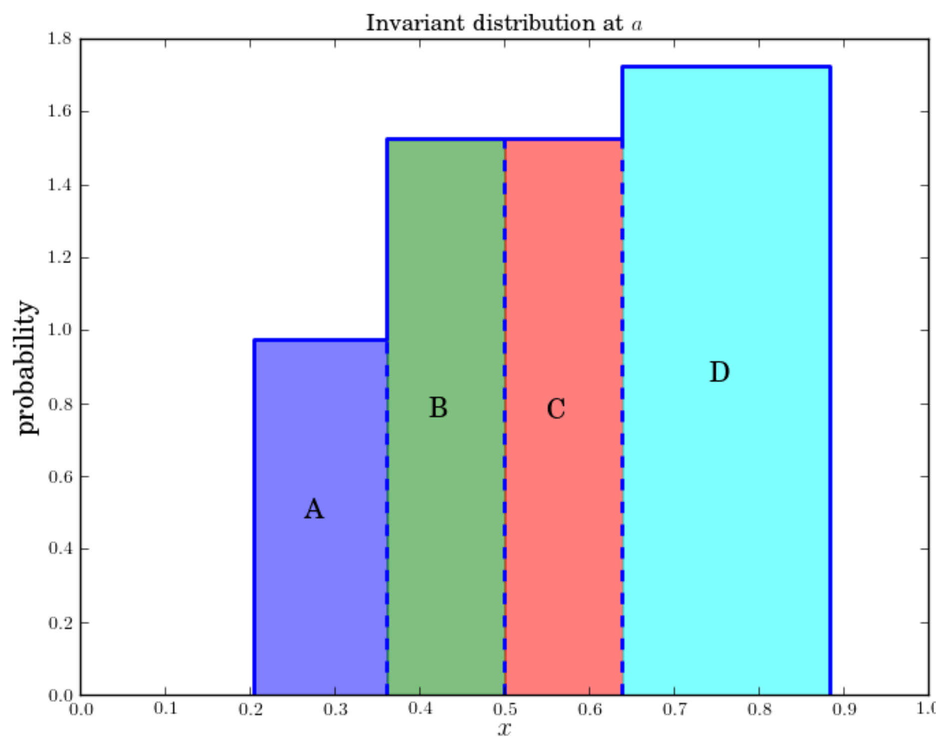

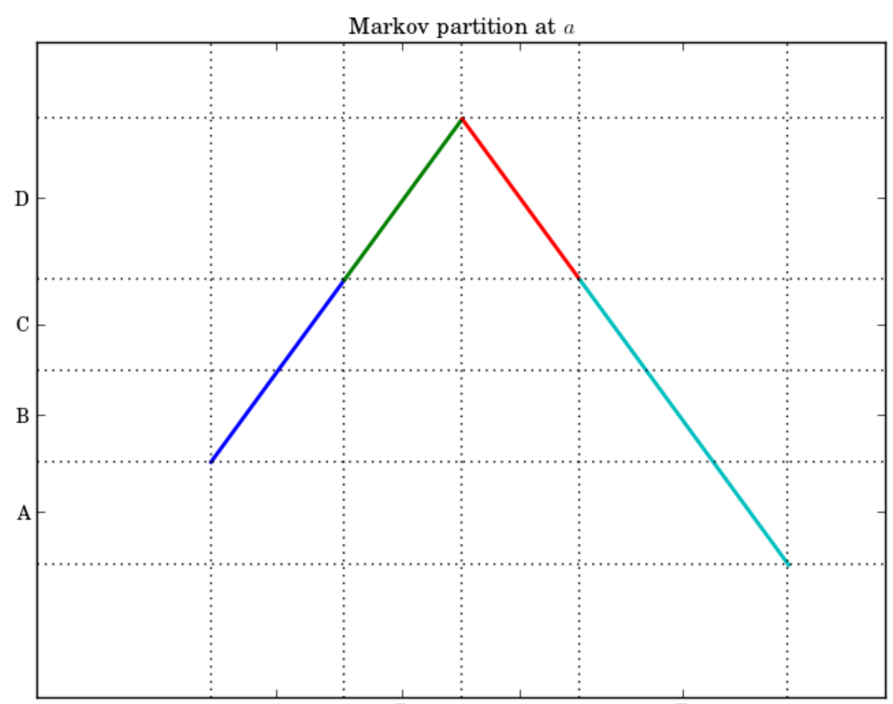

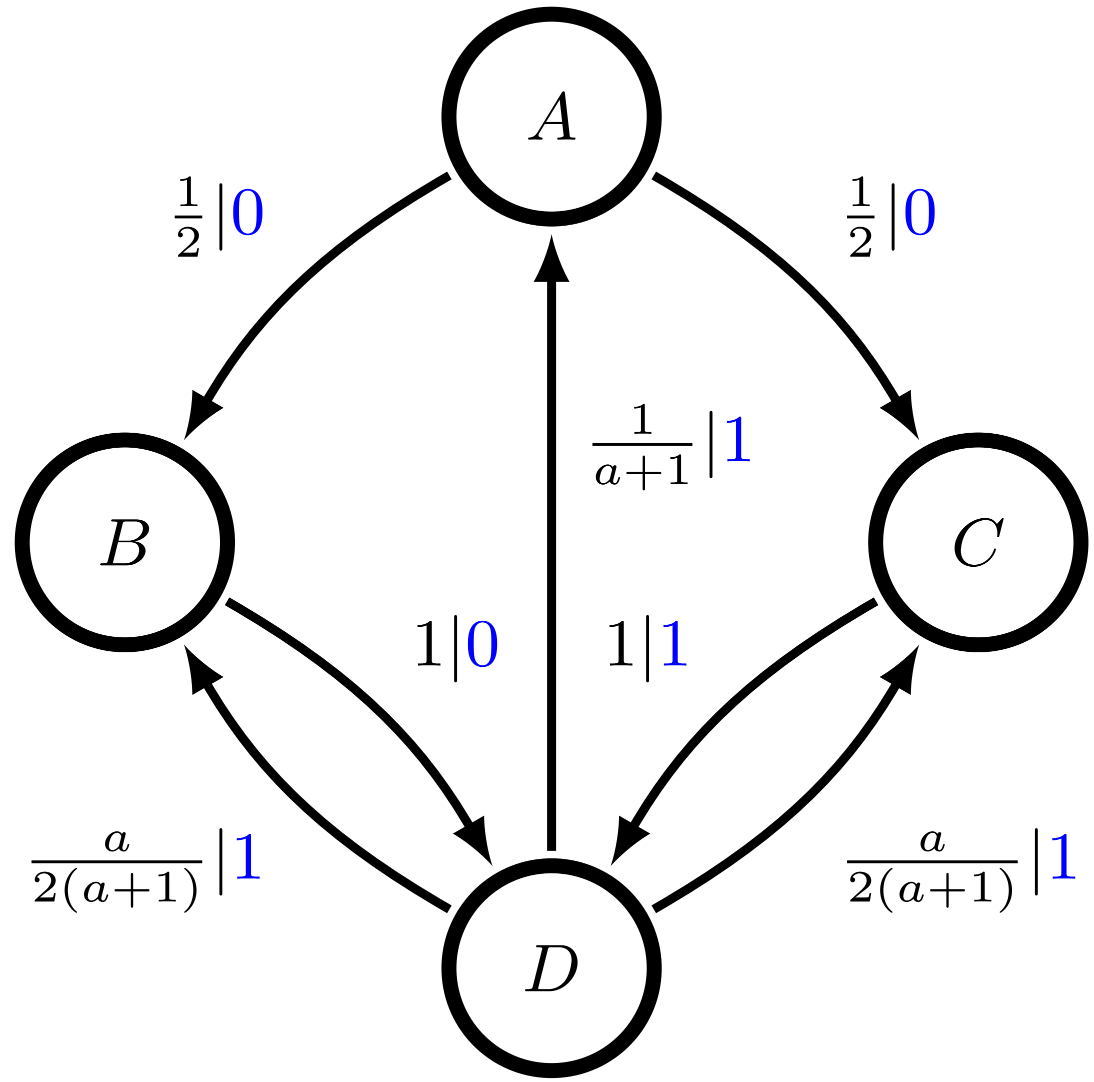

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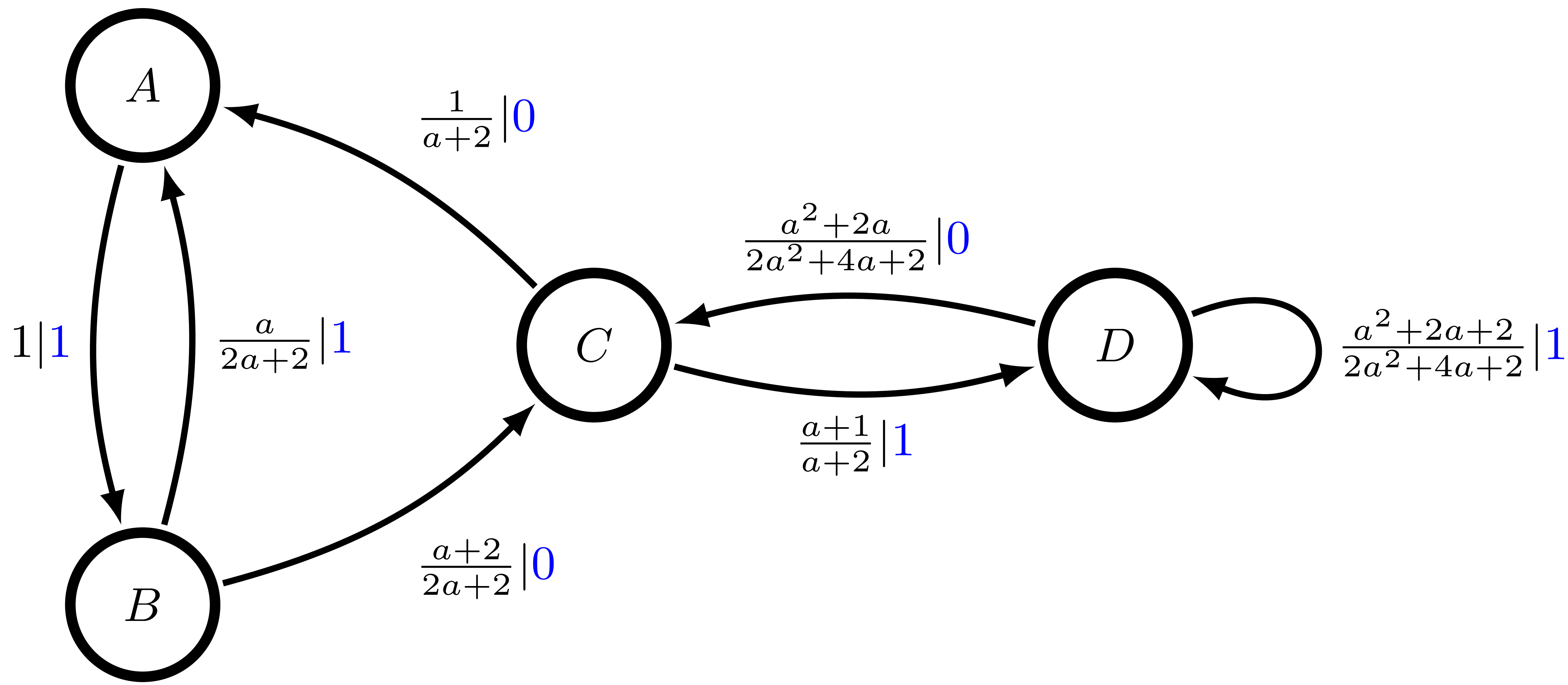
Construct the ϵ-machine



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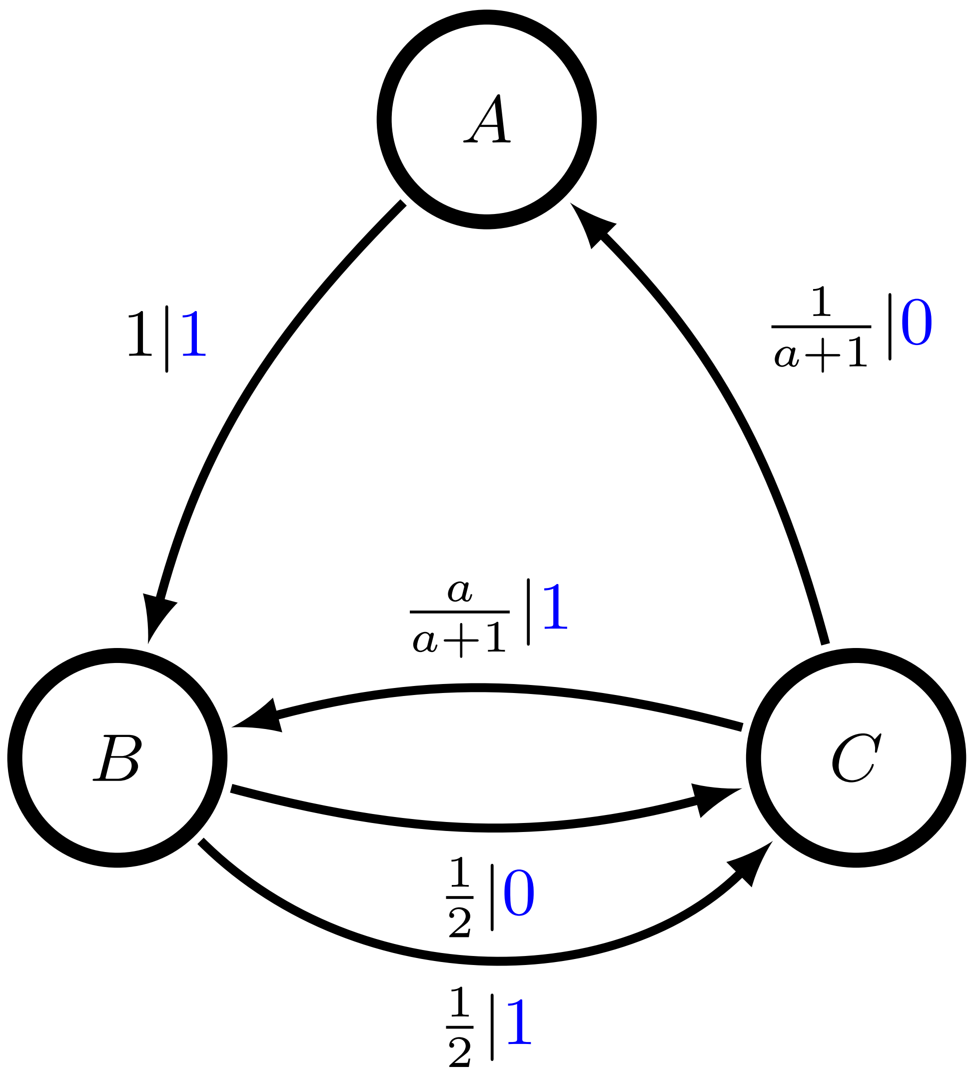
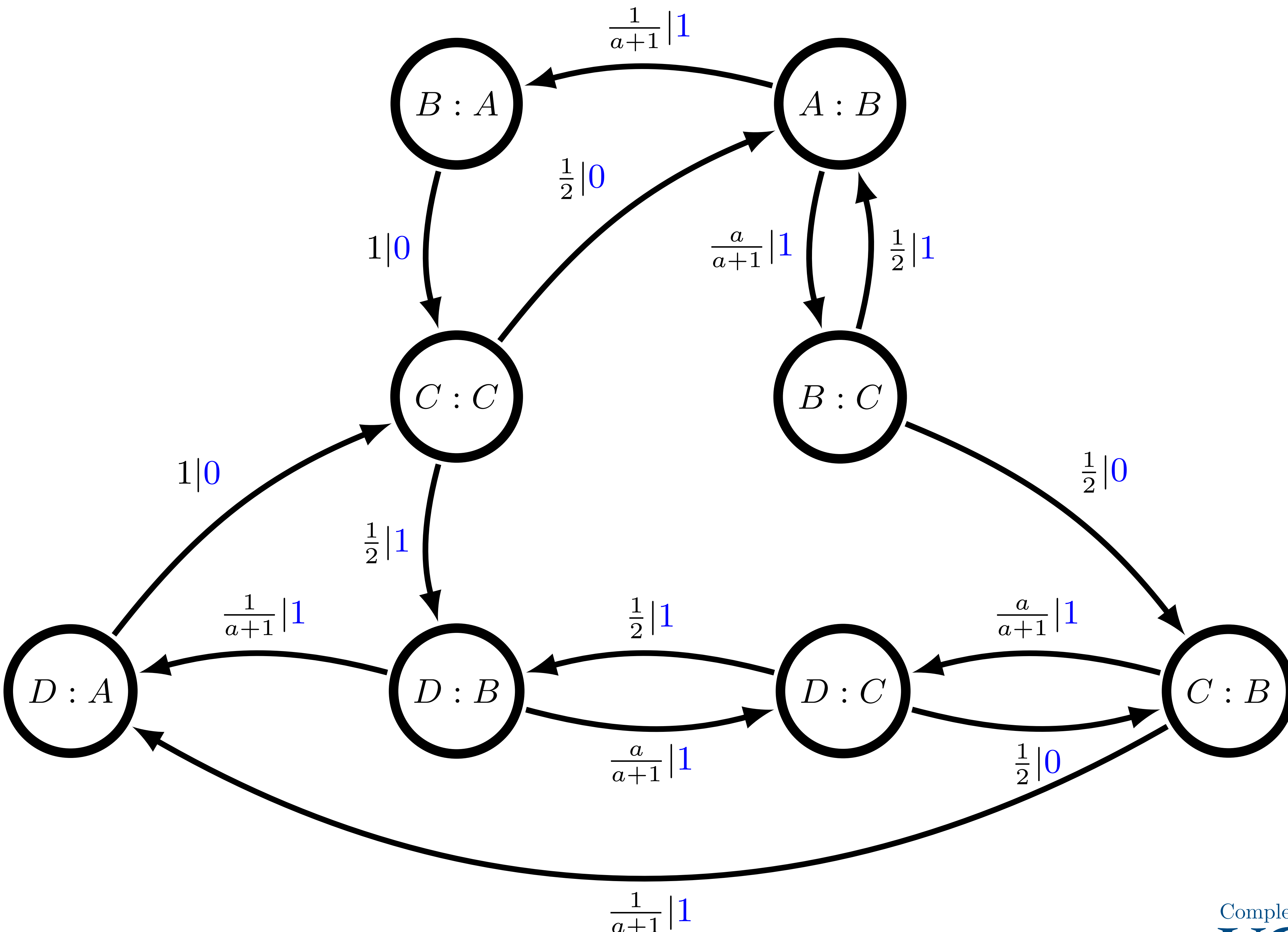


Construct the Bidirectional ϵ-machine



Calculate the Quantities

The bidirectional machine provides $p\left(\mathcal{S}_0^+, \mathcal{S}_0^-, X_0, \mathcal{S}_1^+, \mathcal{S}_1^-\right)$.
This can be marginalized to $p\left(\mathcal{S}_0^+, X_0, \mathcal{S}_1^-\right)$.
$r_\mu = \mathrm{H}\left[X_0 | \mathcal{S}_0^+, \mathcal{S}_1^-\right]$.

$$h_\mu = \log_2 a = \log_2 \left(\frac{\sqrt[3]{9+\sqrt{57}} + \sqrt[3]{9-\sqrt{57}}}{3^{\frac{2}{3}}} \right) = 0.823172\ldots$$

$$r_\mu = \frac{1}{4}\left(3 - \frac{2}{a+1} - \frac{4}{a+2} + \frac{9}{2a+3}\right)$$

$$= \frac{1}{9}\left(\frac{\sqrt[3]{207\sqrt{57}-1349}}{19^{2/3}} - \frac{32}{\sqrt[3]{19\left(207\sqrt{57}-1349\right)}} + 7\right) = 0.648258\ldots$$

$$b_\mu = h_\mu - r_\mu = 0.174915\ldots$$

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It happens sometimes. People just explode. Natural causes.

Thank You

h_μ is two semantically meaningful components: r_μ & b_μ

Interwebs

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Reference

Ryan G. James, Christopher J. Ellison, and James P. Crutchfield
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Chaos 21, 037109 (2011)