Bidirectional Computational Mechanics II

Reading for this lecture: CMR articles
TBA
PRATISP
IACP
IACPLCOCS

Lecture 30: Natural Computation & Self-Organization, Physics 256B (Spring 2014); Jim Crutchfield

Bidirectional Computational Mechanics

Bidirectionality
Excess entropy from ε-machine
Information diagrams for processes
Bidirectional machines
Bidirectional complexities

Bidirectionality

Bidirectionality

Causal shielding for forward and reverse states:

$$\Pr(\overleftarrow{X}, \overrightarrow{X} | \mathcal{S}^+) = \Pr(\overleftarrow{X} | \mathcal{S}^+) \Pr(\overrightarrow{X} | \mathcal{S}^+)$$

$$\Pr(\overleftarrow{Y}, \overrightarrow{Y} | \mathcal{S}^-) = \Pr(\overleftarrow{Y} | \mathcal{S}^-) \Pr(\overrightarrow{Y} | \mathcal{S}^-)$$

Both forward and reverse states equally good at shielding
... but for potentially different, though, related processes:

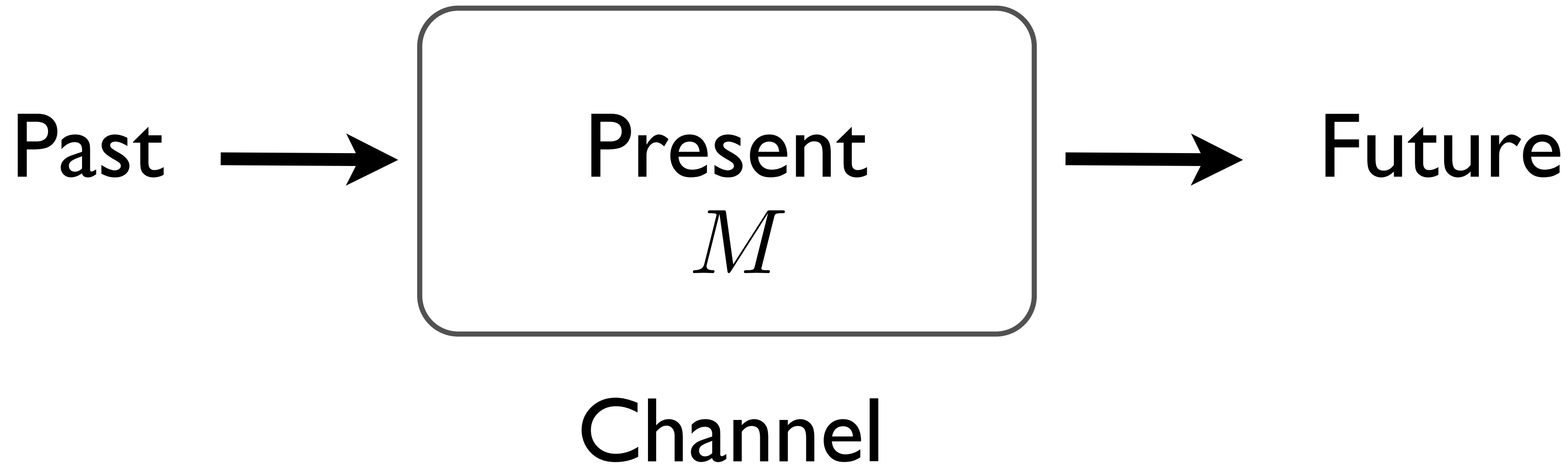
$$\overleftrightarrow{Y} = \widetilde{\overleftrightarrow{X}}$$

Direct relationship between forward and reverse states?

Bidirectionality

Excess entropy from ε-machine:

Process $\Pr(\overleftarrow{X}, \overrightarrow{X})$ is a communication channel from the past $\overleftarrow{X}$ to the future $\overrightarrow{X}$:



Bidirectionality

Excess entropy from ε-machine ...

Mutual information between the past and future

$$\mathbf{E} = I[\overleftarrow{X}; \overrightarrow{X}]$$

A process's channel capacity?

More like the effective channel utilization.

Now, how to get from given ε-machine?

Bidirectionality

Excess entropy from ε-machine ...

Theorem: $\mathbf{E} = I[\mathcal{S}^+; \mathcal{S}^-]$

Proof sketch:
$$\begin{aligned} \mathbf{E} &= I[\overleftarrow{X}; \overrightarrow{X}] \\ &= I[\epsilon^+(\overleftarrow{X}); \epsilon^-(\overrightarrow{X})] \\ &= I[\mathcal{S}^+; \mathcal{S}^-] \end{aligned}$$

⇒ Need $\Pr(\mathcal{S}^+, \mathcal{S}^-)$

New interpretation: Effective transmission capacity of channel between forward and reverse processes.

Bidirectionality

Mixed state presentation of time-reversed ε-machine:

$$\widetilde{M}^{+} = \mathcal{T}(M^{+})$$

$$M^{-} = \mathcal{U}(\widetilde{M}^{+}) \qquad \text{(minimize!)}$$

yields conditional entropy btw forward and reverse states:

$$\Pr(\mathcal{S}^{+}|\mathcal{S}^{-})$$

Bidirectionality

Switching maps between forward and reverse causal states:

Forward-state simplex: $\Delta^m \quad m = |\mathcal{S}^+|$

$$\Pr(\mathcal{S}_0^+ = \sigma_0, \mathcal{S}_1^+ = \sigma_1, \ldots) \in \Delta^m$$

Reverse-state simplex: $\Delta^n \quad n = |\mathcal{S}^-|$

$$\Pr(\mathcal{S}_0^- = \sigma_0, \mathcal{S}_1^- = \sigma_1, \ldots) \in \Delta^n$$

Forward map: $f : \Delta^n \to \Delta^m$

$$f(\sigma^-) = \Pr(\mathcal{S}^+ | \sigma^-)$$

Reverse map: $r : \Delta^m \to \Delta^n$

$$r(\sigma^+) = \Pr(\mathcal{S}^- | \sigma^+)$$

Bidirectionality

Switching maps ...

Uses:

Calculate with conditional & joint state dependencies:

$$\Pr(\mathcal{S}^+ | \mathcal{S}^-)$$
$$\Pr(\mathcal{S}^+, \mathcal{S}^-)$$

Recast quantifiers purely in terms of forward & reverse states.

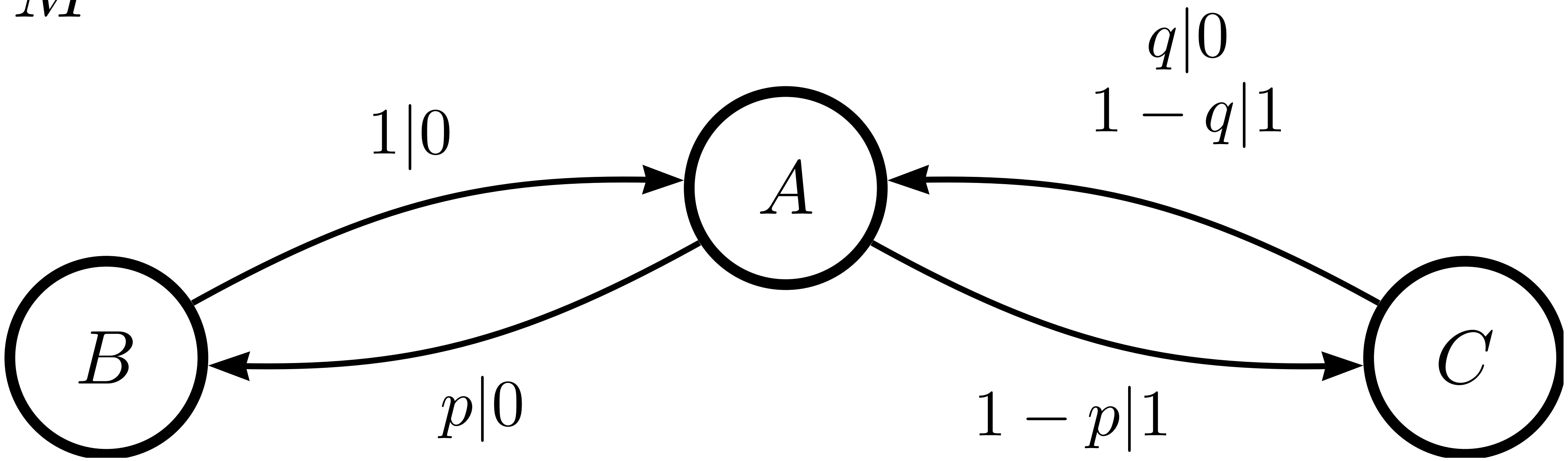
New quantifiers.

A new representation ...

Bidirectionality

Example: Random Noisy Copy (RnC)

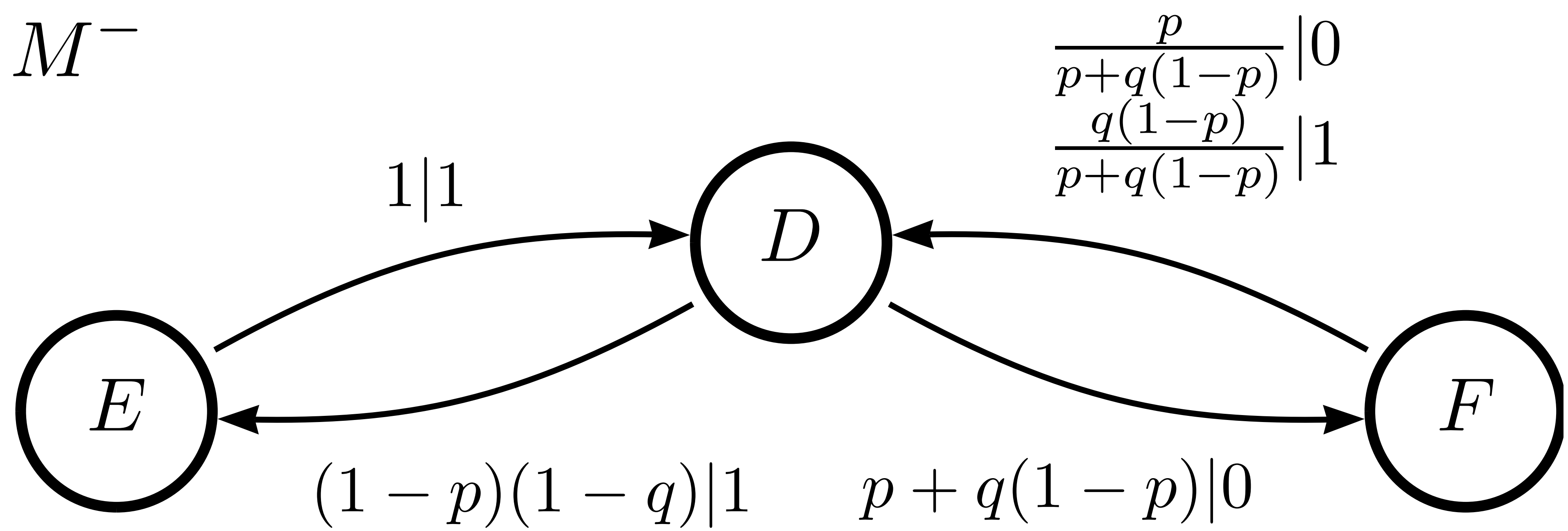
M^+



$$\Pr(\mathcal{S}^+) = \frac{1}{2} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & p & 1-p \end{pmatrix} \end{array}$$

Bidirectionality

Example: Random Noisy Copy (RnC)



$$\Pr(\mathcal{S}^-) = \frac{1}{2} \begin{array}{c} \\ \end{array} \begin{array}{ccc} D & E & F \\ \big(\; 1 & (1-p)(1-q) & p+q(1-p) \; \big) \end{array}$$

Bidirectionality

Example: Random Noisy Copy (RnC)

Forward switching map:

Conditional distribution, from previous MSP calculation

$$
\Pr(\mathcal{S}^+|\mathcal{S}^-) = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} D \\ E \\ F \end{array} & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \frac{p}{p+q(1-p)} & \frac{q(1-p)}{p+q(1-p)} \end{pmatrix} \end{array}
$$

Bidirectionality

Example: Random Noisy Copy (RnC)

Joint distribution:

$$\Pr(\mathcal{S}^+, \mathcal{S}^-) = \frac{1}{2} \times \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} D \\ E \\ F \end{array} & \begin{pmatrix} 0 & 0 & 1 \\ (1-p)(1-q) & 0 & 0 \\ 0 & p & q(1-p) \end{pmatrix} \end{array}$$

Componentwise calculation:

$$\Pr(\mathcal{S}^+ = i, \mathcal{S}^- = j) = \Pr(\mathcal{S}^+ = i | \mathcal{S}^- = j) \Pr(\mathcal{S}^- = j)$$

Bidirectionality

Example: Random Noisy Copy (RnC)

Reverse switching map:

"Transpose" of joint distribution

$$\Pr(\mathcal{S}^-, \mathcal{S}^+) = \frac{1}{2} \times \begin{array}{c c} & \begin{array}{ccc} D & E & F \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0 & (1-p)(1-q) & 0 \\ 0 & 0 & p \\ 1 & 0 & q(1-p) \end{pmatrix} \end{array}$$

Componentwise:

$$\Pr(\mathcal{S}^- = i | \mathcal{S}^+ = j) = \frac{\Pr(\mathcal{S}^- = i, \mathcal{S}^+ = j)}{\Pr(\mathcal{S}^+ = j)}$$

$$\Pr(\mathcal{S}^- | \mathcal{S}^+) = \begin{array}{c c} & \begin{array}{ccc} D & E & F \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0 & (1-p)(1-q) & 0 \\ 0 & 0 & 1 \\ \frac{1}{1-p} & 0 & q \end{pmatrix} \end{array}$$

Bidirectionality

Example: Random Noisy Copy (RnC)

Reverse switching map ...

Normalize rows to get actual transition probabilities:

$$\Pr(\mathcal{S}^-|\mathcal{S}^+) = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} \quad D \quad & \quad E \quad & \quad F \quad \end{array} \\ \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{1+q(1-p)} & 0 & \frac{q(1-p)}{1+q(1-p)} \end{array} \right) \end{array}$$

Bidirectionality

Example: Random Noisy Copy (RnC) ...

Excess entropy:

$$\begin{aligned}
\mathbf{E} &= I[\mathcal{S}^+;\mathcal{S}^-] \\
&= H[\mathcal{S}^+] - H[\mathcal{S}^+|\mathcal{S}^-] \\
&= H\left[\tfrac{1}{2}(1 \;\; p \;\; 1-p)\right] - H[\{B,C\}|F]\ \Pr(F) \\
&= H\left[\tfrac{1}{2}(1 \;\; p \;\; 1-p)\right] - H\left(\tfrac{p}{p+q(1-p)}\right)(p+q(1-p))
\end{aligned}$$

$$\Pr(\mathcal{S}^+) = \frac{1}{2}\ \begin{array}{ccc} A & B & C \\ \hline \end{array}$$

$$\Pr(\mathcal{S}^+) = \frac{1}{2} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & p & 1-p \end{pmatrix} \end{array}$$

$$\Pr(\mathcal{S}^+|\mathcal{S}^-) = \begin{array}{c} \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \frac{p}{p+q(1-p)} & \frac{q(1-p)}{p+q(1-p)} \end{pmatrix} \end{array}$$

$$\Pr(\mathcal{S}^-) = \frac{1}{2} \begin{array}{c} \begin{array}{ccc} D & E & F \end{array} \\ \begin{pmatrix} 1 & (1-p)(1-q) & p+q(1-p) \end{pmatrix} \end{array}$$

Bidirectionality

Crypticities, recast:

Forward:

$$\begin{aligned} \chi^{+} &= H[\mathcal{S}^{+}|\overrightarrow{X}] \\ &= H[\mathcal{S}^{+}|\epsilon^{-}(\overrightarrow{X})] \\ &= H[\mathcal{S}^{+}|\mathcal{S}^{-}] \end{aligned}$$

Reverse:

$$\begin{aligned} \chi^{-} &= H[\mathcal{S}^{-}|\overleftarrow{X}] \\ &= H[\mathcal{S}^{-}|\epsilon^{+}(\overleftarrow{X})] \\ &= H[\mathcal{S}^{-}|\mathcal{S}^{+}] \end{aligned}$$

Bidirectionality

Example: Random Noisy Copy (RnC) ...

$$\begin{aligned}
\chi^{+} &= H[\mathcal{S}^{+}|\mathcal{S}^{-}] \\
&= H\left[\mathcal{S}^{+} = \{B, C\} | \mathcal{S}^{-} = F\right] \\
&= H\left(\tfrac{p}{p+q(1-p)}\right)(p + q(1-p))
\end{aligned}$$

$$\begin{aligned}
\chi^{-} &= H[\mathcal{S}^{-}|\mathcal{S}^{+}] \\
&= H\left[\mathcal{S}^{-} = \{D, F\} | \mathcal{S}^{+} = C\right] \\
&= H\left(\tfrac{1}{1+q(1-p)}\right)\left(\frac{1-p}{2}\right)
\end{aligned}$$

Bidirectionality

Causal irreversibility, recast:

$$
\begin{aligned}
\Xi &\equiv C_\mu^+ - C_\mu^- \\
&= H[\mathcal{S}^+|\mathcal{S}^-] - H[\mathcal{S}^-|\mathcal{S}^+]
\end{aligned}
$$

using

$$
C_\mu^+ = \mathbf{E} + \chi^+
$$

$$
C_\mu^- = \mathbf{E} + \chi^-
$$

Bidirectionality

Example: Random Noisy Copy (RnC) ...

$$
\begin{aligned}
\Xi &= H[\mathcal{S}^+|\mathcal{S}^-] - H[\mathcal{S}^-|\mathcal{S}^+] \\
&= H\left[\tfrac{1}{2}(1 \quad p \quad 1-p)\right] - \\
&\quad\; H\left[\tfrac{1}{2}\left(1 \quad (1-p)(1-q) \quad p + q(1-p)\right)\right]
\end{aligned}
$$

Bidirectionality

Summary:

New meaning for excess entropy:

$$\mathbf{E} = I[\mathcal{S}^+ ; \mathcal{S}^-]$$

$\mathbf{E}$ can be directly (and accurately!) calculated from ε-machine.

New level of analytical calculation possible.

New algorithms for measuring intrinsic computation using bidirectional representations.

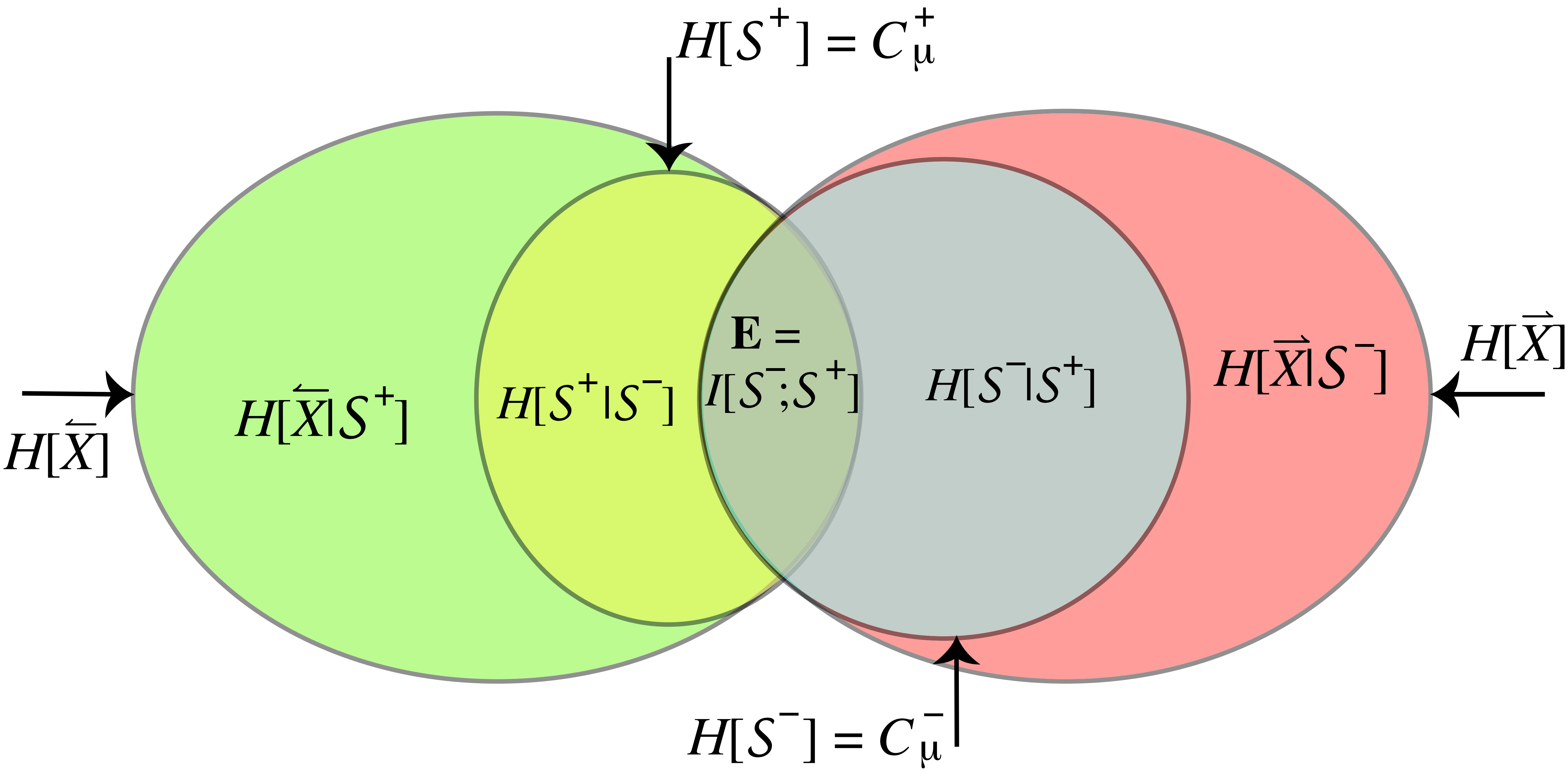
Information Diagrams for Processes

Information Diagrams for Processes

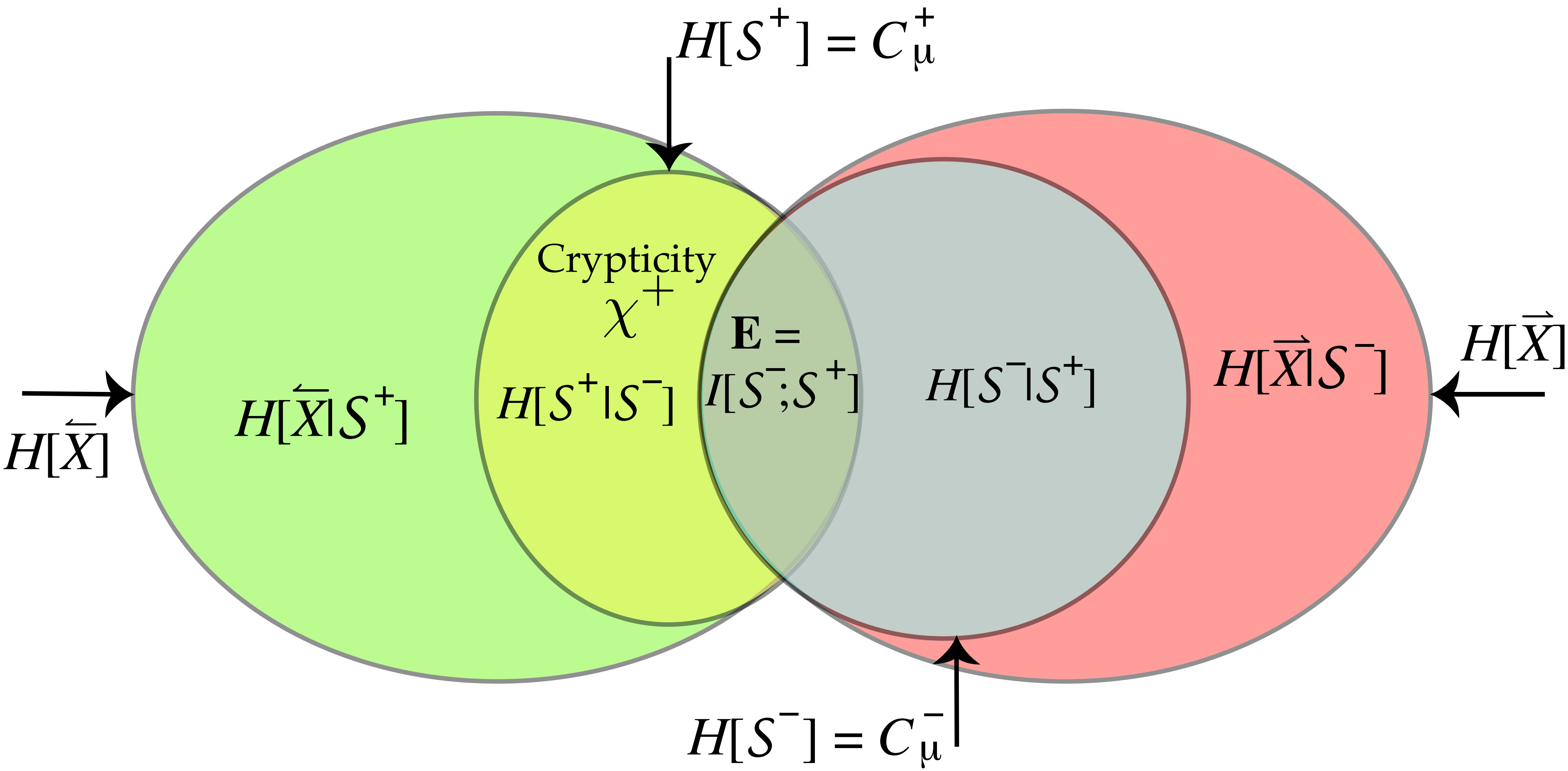
Now, ready for full information diagram of a process.

Requires both forward and reverse scanning of a process.

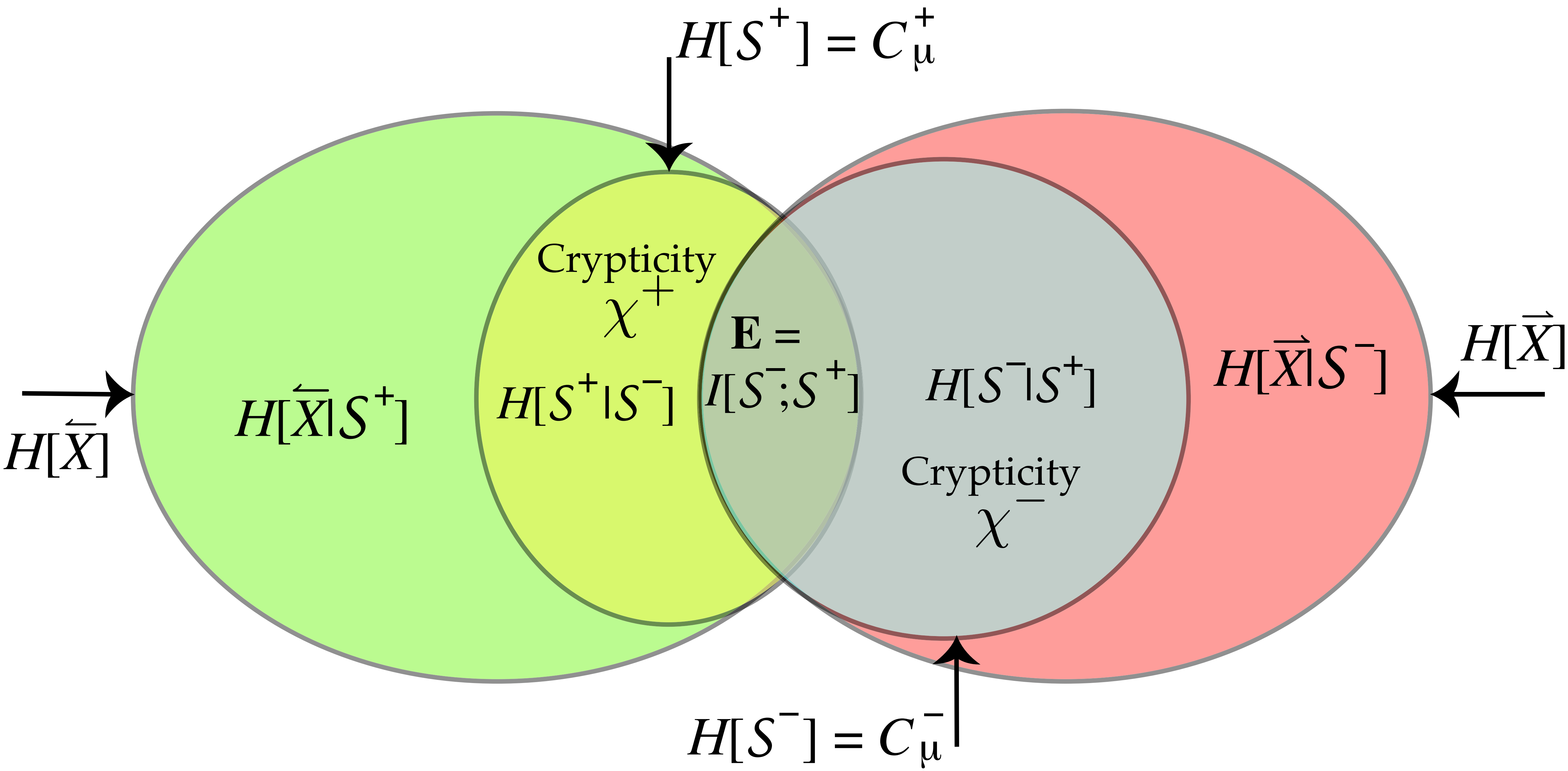
ϵ-machine information Diagram



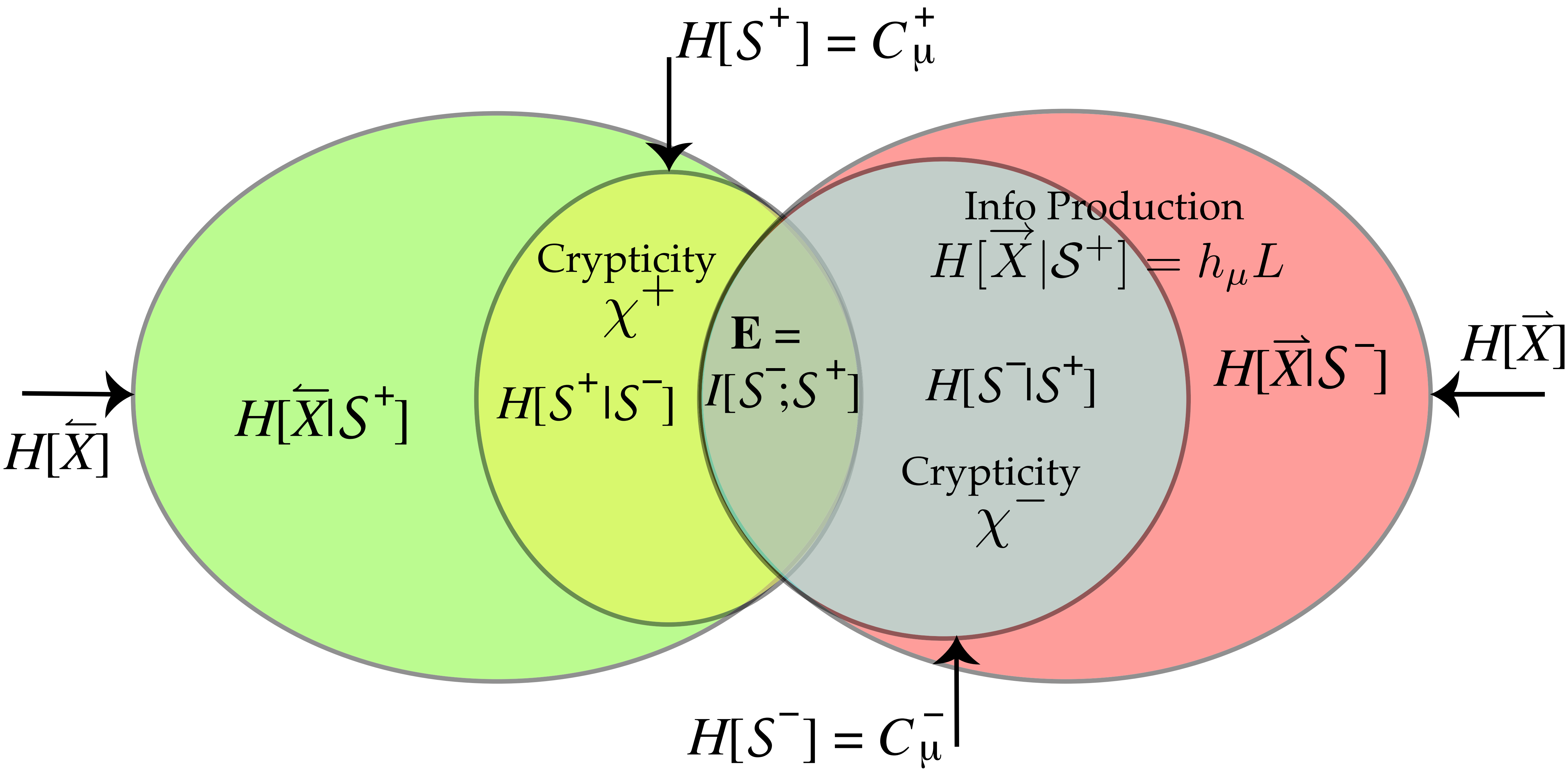
ϵ-machine information Diagram



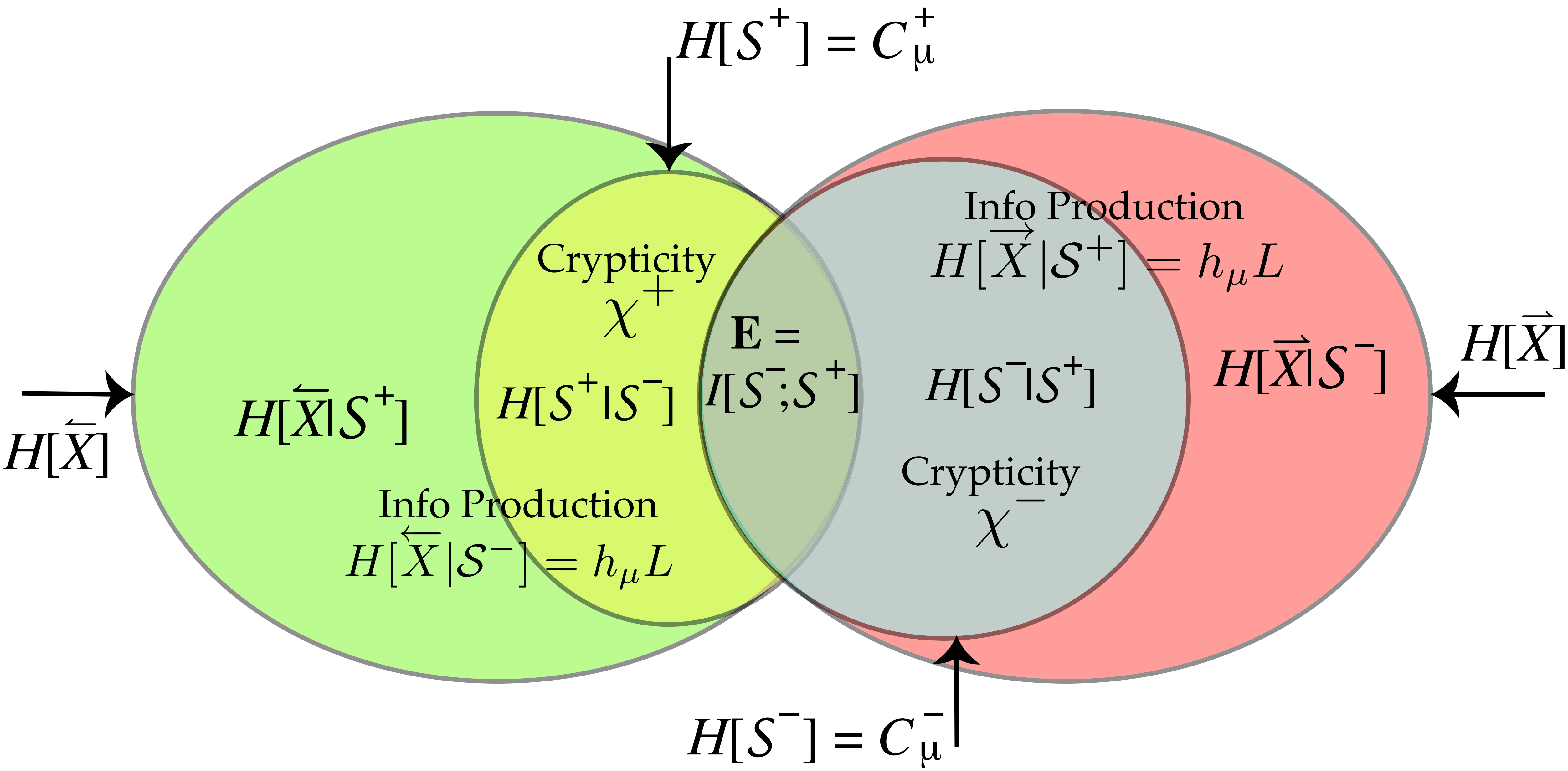
ϵ-machine information Diagram



ϵ-machine Information Diagram



ϵ-machine information Diagram



Bidirectional Machines

Bidirectional Machines

Summary:

So far, re-expressed key measures
in terms of forward and reverse ε-machines

$$\mathbf{E} = I[\mathcal{S}^-;\mathcal{S}^+]$$

$$\chi^+ = H[\mathcal{S}^+|\mathcal{S}^-]$$

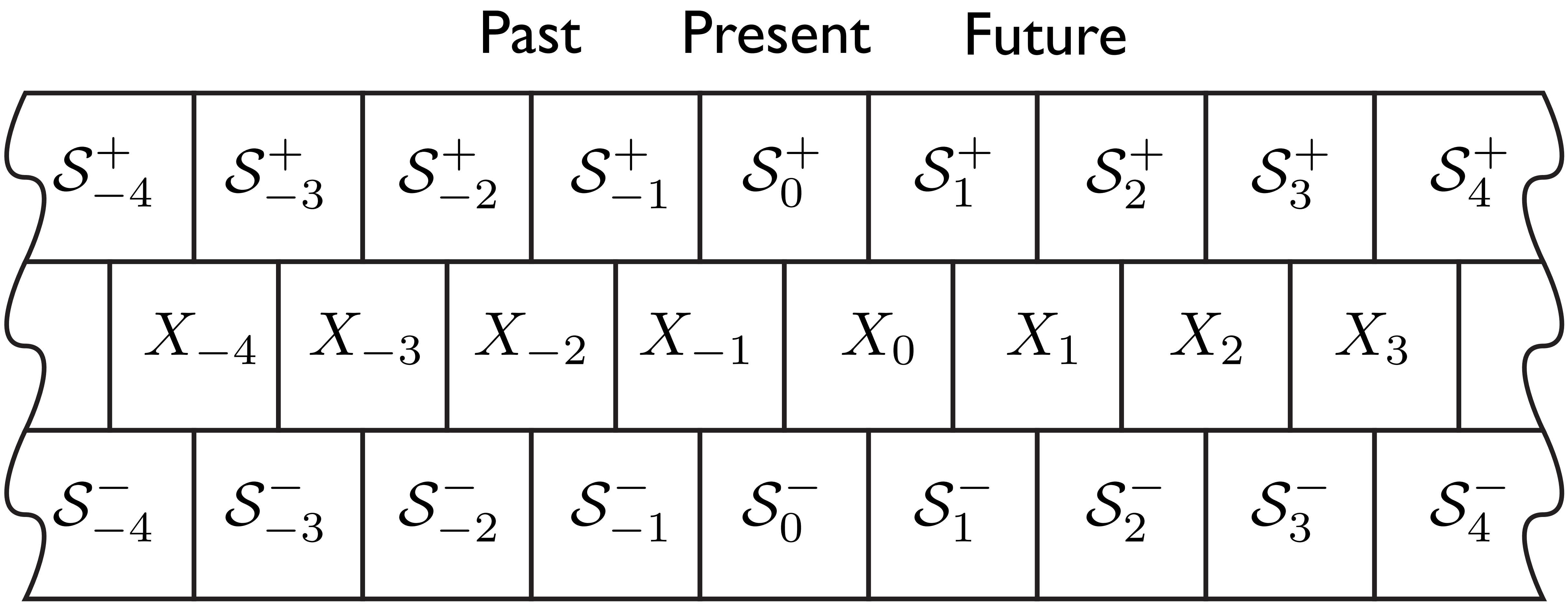
$$\chi^- = H[\mathcal{S}^-|\mathcal{S}^+]$$

$$\Xi = H[\mathcal{S}^+|\mathcal{S}^-] - H[\mathcal{S}^-|\mathcal{S}^+]$$

How to more directly represent the interdependence between forward and reverse ε-machines?

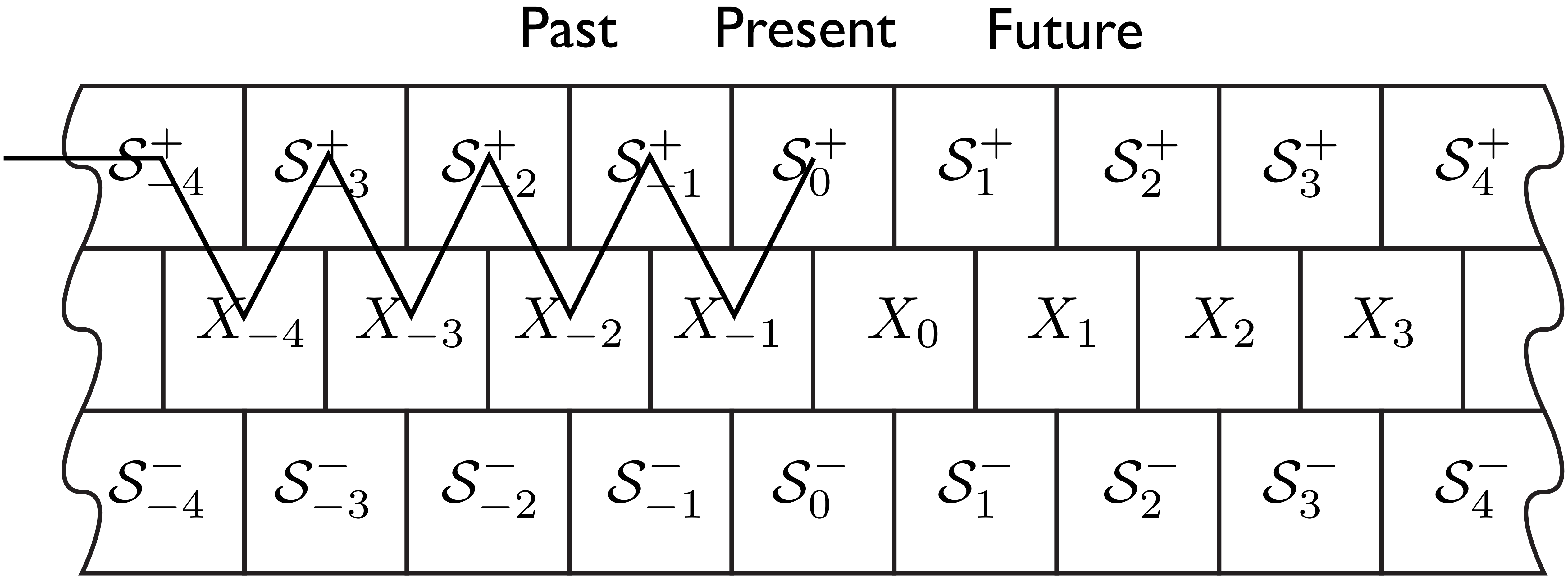
Bidirectional Machines

Bidirectional Process Lattice:



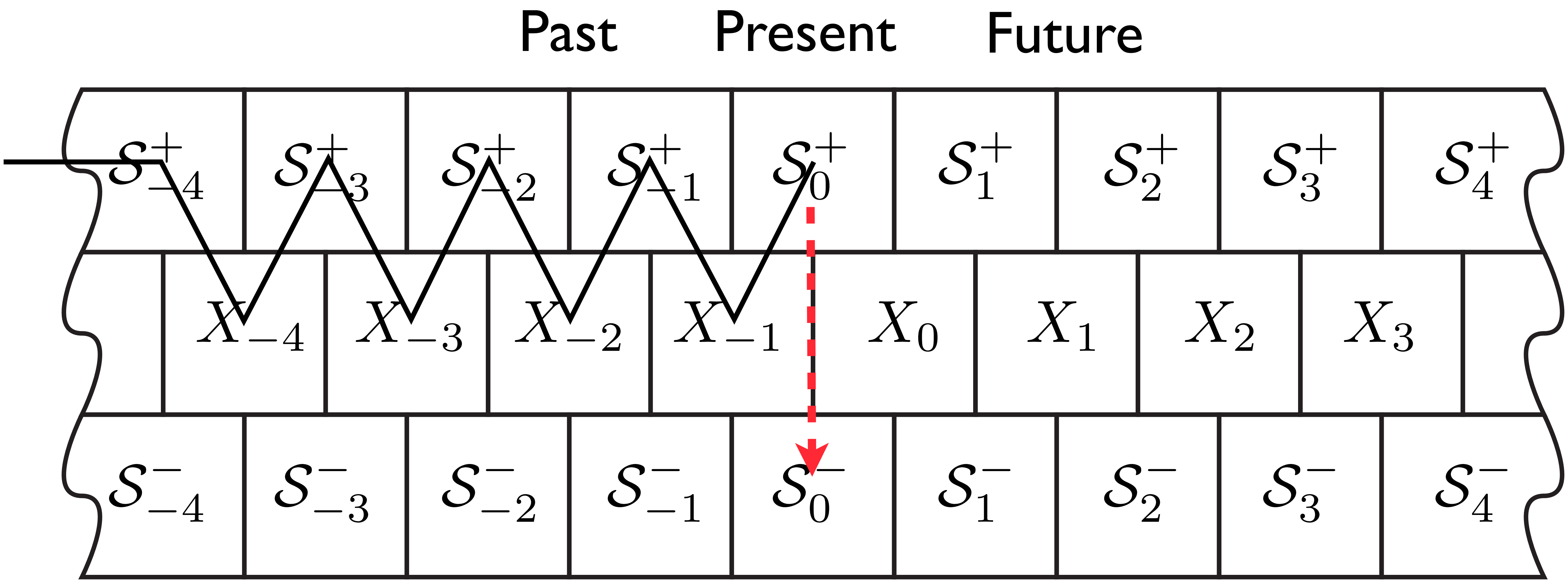
Bidirectional Machines

Bidirectional Process Lattice:



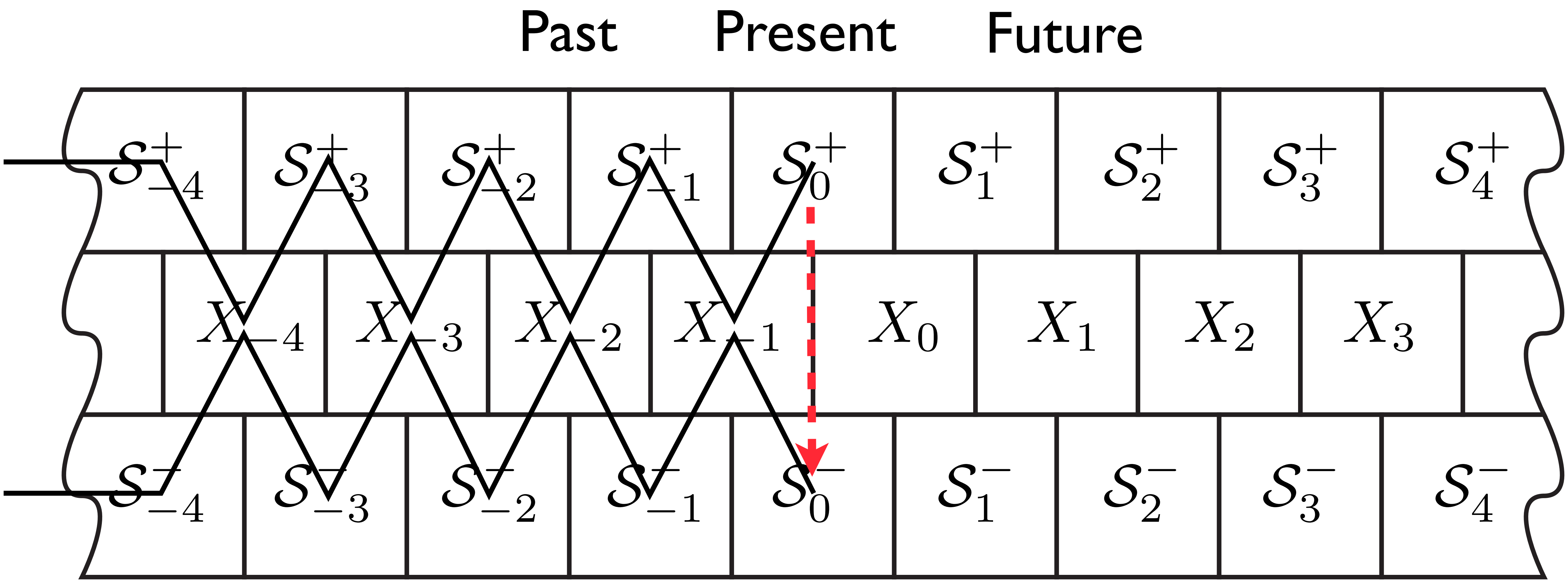
Bidirectional Machines

Bidirectional Process Lattice:



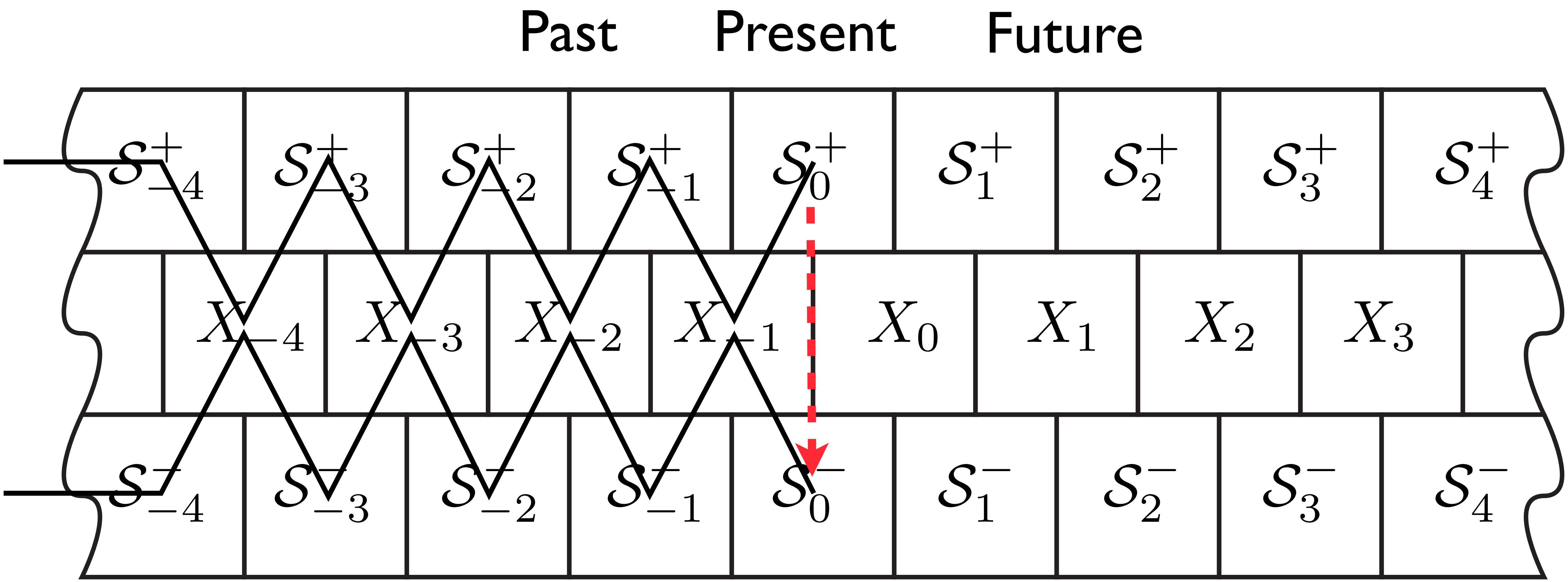
Bidirectional Machines

Bidirectional Process Lattice:



Bidirectional Machines

Bidirectional Process Lattice:

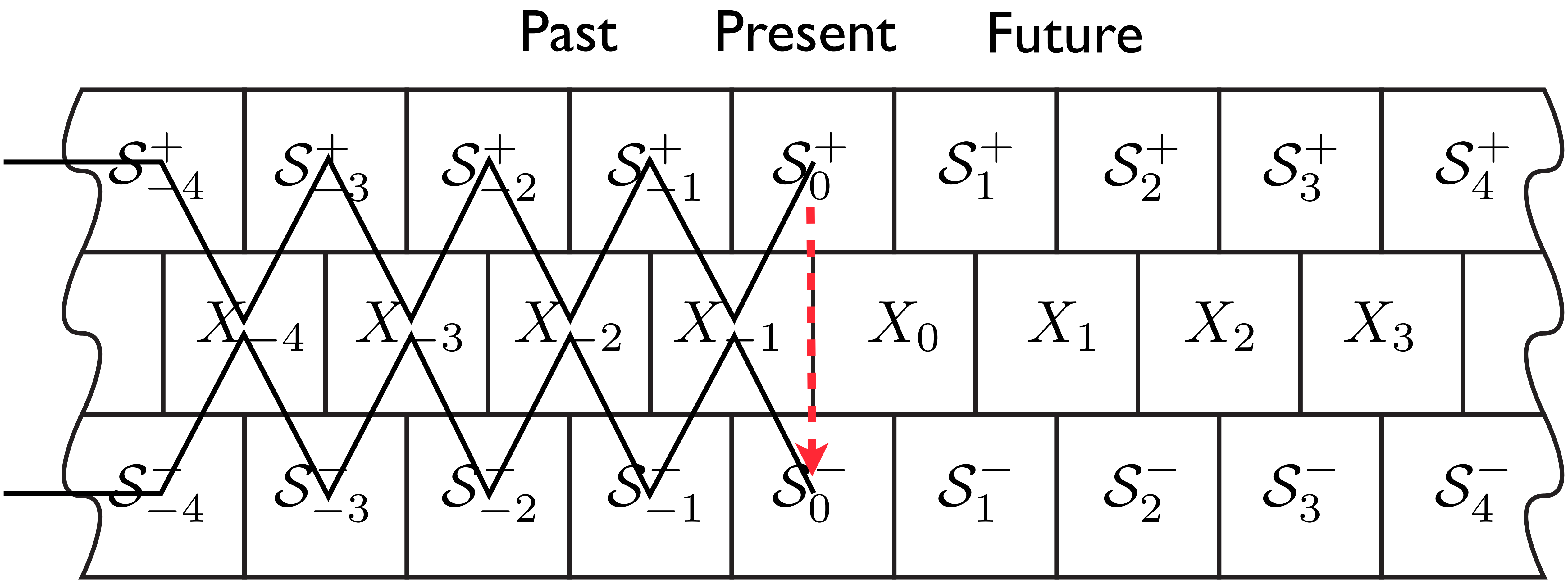


You can choose to reverse direction.

Path automaton over scan direction $\{+, -\}$.

Bidirectional Machines

Bidirectional Process Lattice:



You can choose to reverse direction.

Path automaton over scan direction $\{+, -\}$.

What describes this?

A bidirectional machine, with forward and reverse moves.

Previous forward & reverse machines are a subset of paths.

Bidirectional Machines

Bidirectional machine: $M^{\pm}$

Equivalence relation: $\sim^{\pm}$

$$\epsilon^{\pm}(\overleftrightarrow{x}) = \left\{ \overleftrightarrow{x}' = \overleftarrow{x}'\overrightarrow{x}' : \overleftarrow{x}' \in \epsilon^{+}(\overleftarrow{x}) \text{ and } \overrightarrow{x}' \in \epsilon^{-}(\overrightarrow{x}) \right\}$$

Bidirectional states:

$$\begin{aligned} \boldsymbol{\mathcal{S}}^{\pm} &= \Pr(\overleftarrow{X}, \overrightarrow{X})/\sim^{\pm} \\ &\subseteq \boldsymbol{\mathcal{S}}^{+} \times \boldsymbol{\mathcal{S}}^{-} \end{aligned}$$

A partition of $\overleftrightarrow{X}$

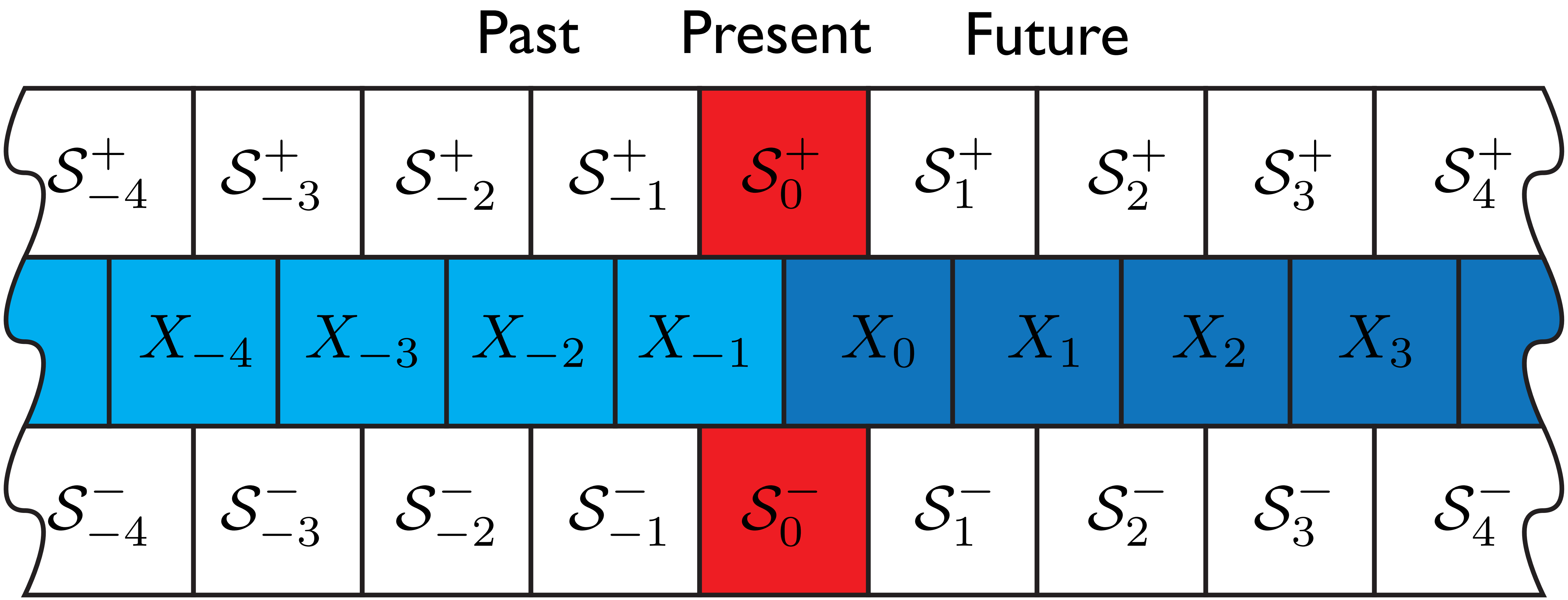
Bidirectional Machine:

$$M^{\pm} = \{\boldsymbol{\mathcal{S}}^{\pm}; \mathcal{T}^{(x)}, x \in \mathcal{A}\}$$

Bidirectional Machines

The bidirectional causal state the process is in at time t is

$$\mathcal{S}_t^{\pm} = (\epsilon^{+}(\overleftarrow{x}_t), \epsilon^{-}(\overrightarrow{x}_t)) \qquad \overleftrightarrow{x} = \overleftarrow{x}_t \overrightarrow{x}_t$$

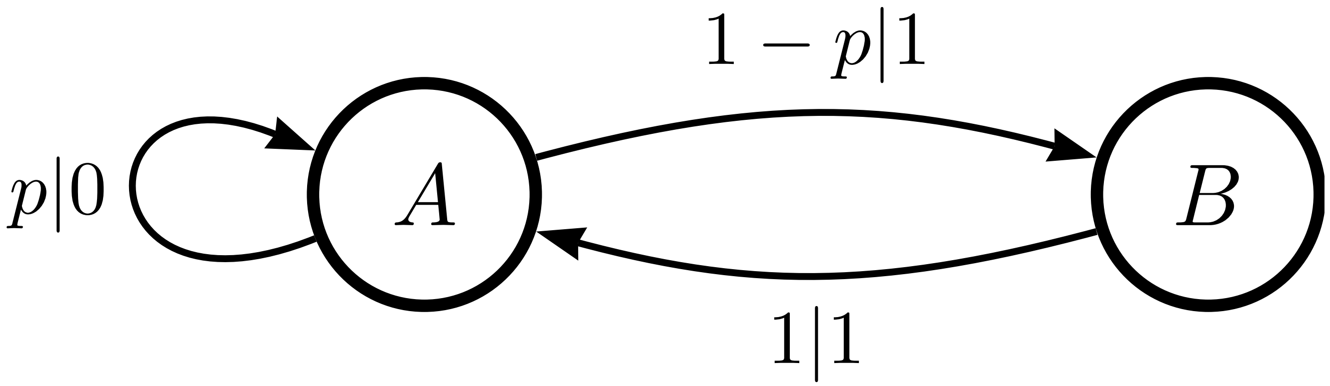


Bidirectional Machines

Example: Even Process

Forward ε-machine

M^+

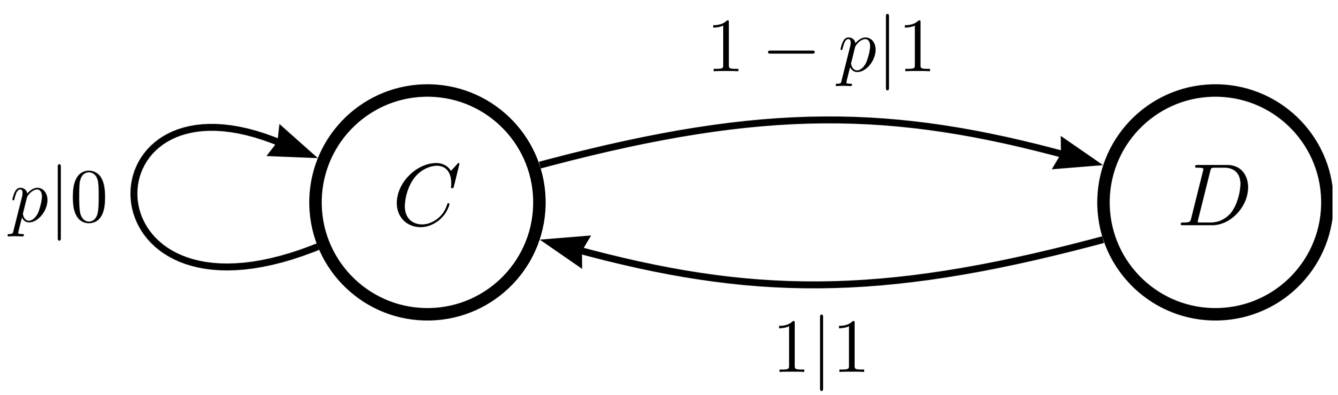


$$\Pr(\mathcal{S}^+) = \begin{array}{cc} A & B \\ \end{array}$$
$$\Pr(\mathcal{S}^+) = \begin{pmatrix} \frac{1}{2-p} & \frac{1-p}{2-p} \end{pmatrix}$$

$$\boldsymbol{\mathcal{S}}^+ = \{A, B\}$$

Reverse ε-machine

M^-



$$\Pr(\mathcal{S}^-) = \begin{pmatrix} \frac{1}{2-p} & \frac{1-p}{2-p} \end{pmatrix}$$

$$\boldsymbol{\mathcal{S}}^- = \{C, D\}$$

Bidirectional Machines

Example: Even Process ...

$$h_\mu = H(p)/(2 - p)$$

$$C_\mu^+ = H\left(1/(2 - p)\right)$$

$$C_\mu^- = H\left(1/(2 - p)\right)$$

$$\Xi = 0$$

Bidirectional Machines

Example: Even Process ...

Forward switching map:

$$\Pr(\mathcal{S}^+|\mathcal{S}^-) = \begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

Reverse switching map:

$$\Pr(\mathcal{S}^-|\mathcal{S}^+) = \begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

Identities!

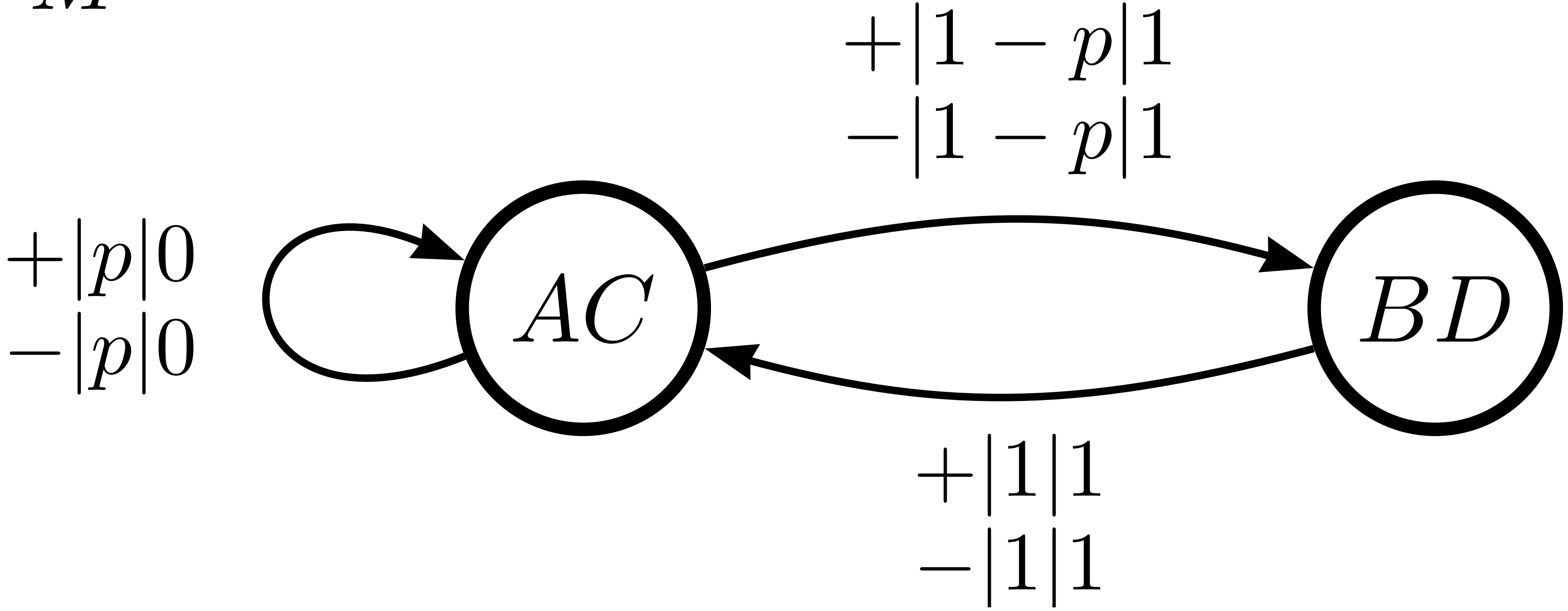
Bidirectional Machines

Example: Even Process ...

Bidirectional machine:

$$\boldsymbol{\mathcal{S}}^{\pm} = \{AC, BD\}$$

$M^{\pm}$



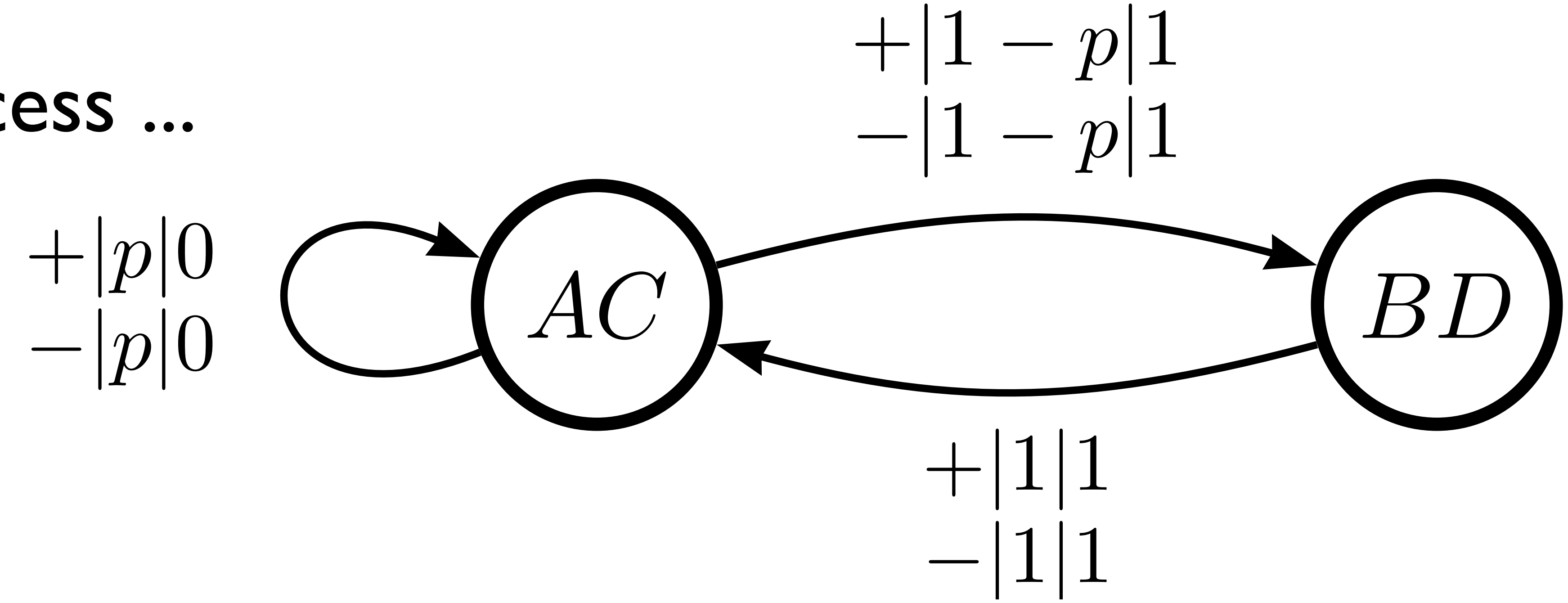
$(\overleftarrow{X}, \overrightarrow{X})$:

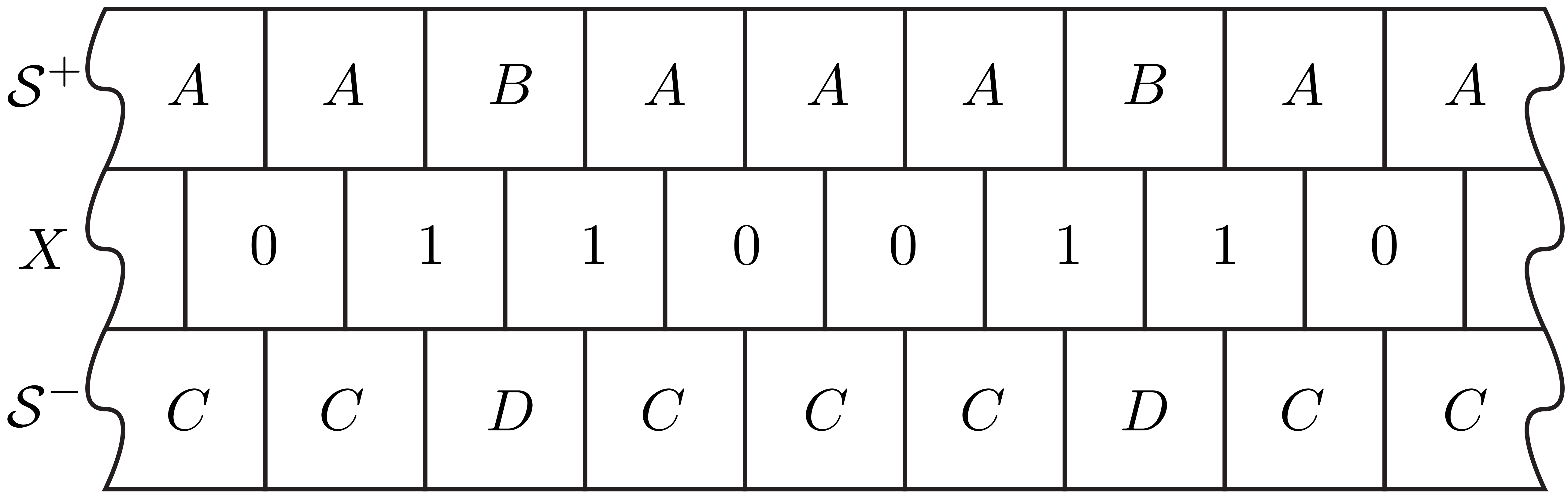
$(A, C) \sim (\{*(11)^k\}, \{(11)^k*\})$ Even # 1s

$(B, D) \sim (\{*(11)^k 1\}, \{1(11)^k*\})$ Odd # 1s

Bidirectional Machines

Example: Even Process ...





Example: Even Process ...

$$\chi^{+}(p) = \chi^{-}(p) = 0$$

$$\mathbf{E} = C_\mu^{+} = C_\mu^{-}$$

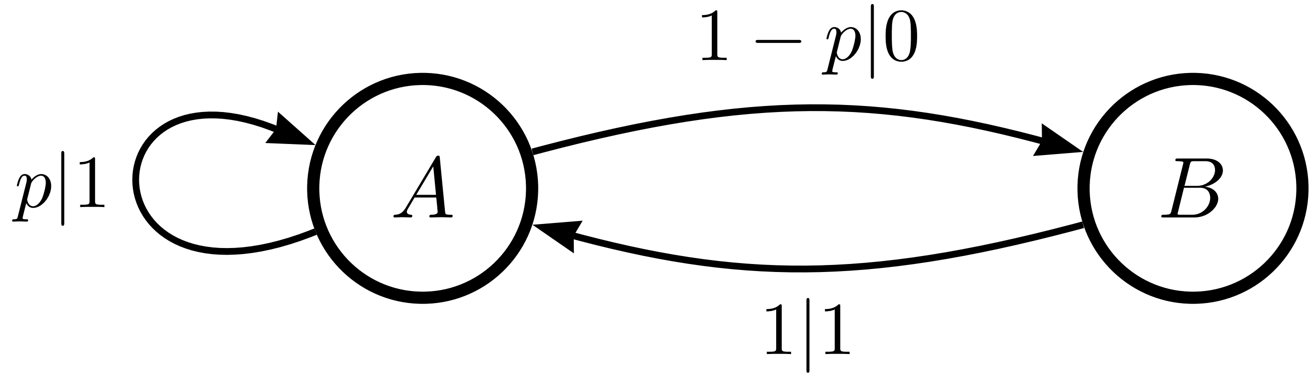
Sofic, non-Markov
But simple!
- Microscopically reversible
- Causally reversible
- Explicit process: non-cryptic

Bidirectional Machines

Example: Golden Mean Process

Forward ε-machine

M^+

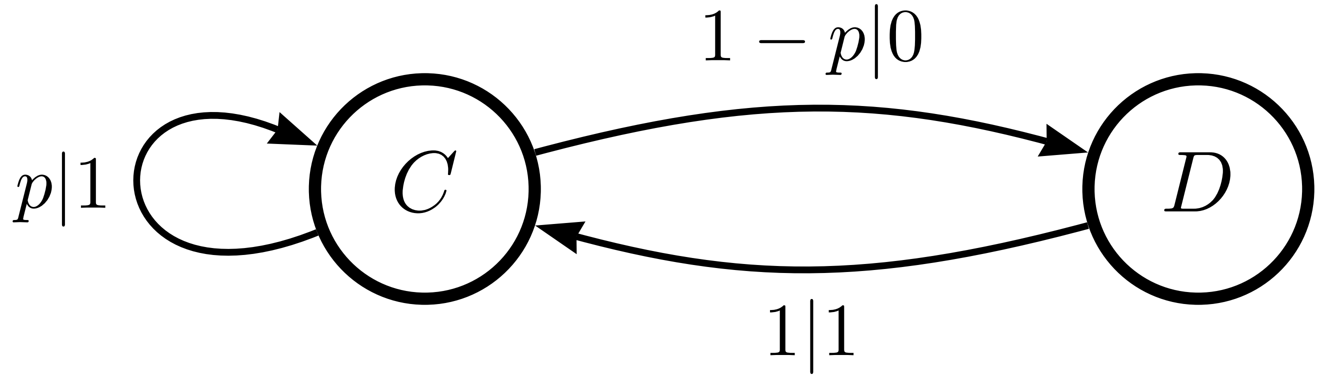


$$\begin{array}{ccc} & A & B \\ \Pr(\mathcal{S}^+) = & \Big(\frac{1}{2-p} & \frac{1-p}{2-p} \Big) \end{array}$$

$$\boldsymbol{\mathcal{S}}^+ = \{A, B\}$$

Reverse ε-machine

M^-



$$\begin{array}{ccc} & C & D \\ \Pr(\mathcal{S}^-) = & \Big(\frac{1}{2-p} & \frac{1-p}{2-p} \Big) \end{array}$$

$$\boldsymbol{\mathcal{S}}^- = \{C, D\}$$

Bidirectional Machines

Example: Golden Mean Process ...

$$h_\mu = H(p)/(2-p)$$

$$C_\mu^+ = H\left(1/(2-p)\right)$$

$$C_\mu^- = H\left(1/(2-p)\right)$$

$$\Xi = 0$$

Same as Even Process!

Example: Golden Mean Process ...

Forward switching map:

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \Pr(\mathcal{S}^+|\mathcal{S}^-) = \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} p & 1-p \\ 1 & 0 \end{pmatrix} \end{array}$$

Reverse switching map:

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \Pr(\mathcal{S}^-|\mathcal{S}^+) = \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} p & 1-p \\ 1 & 0 \end{pmatrix} \end{array}$$

Not identities!

Ambiguity and loss of information on switching.

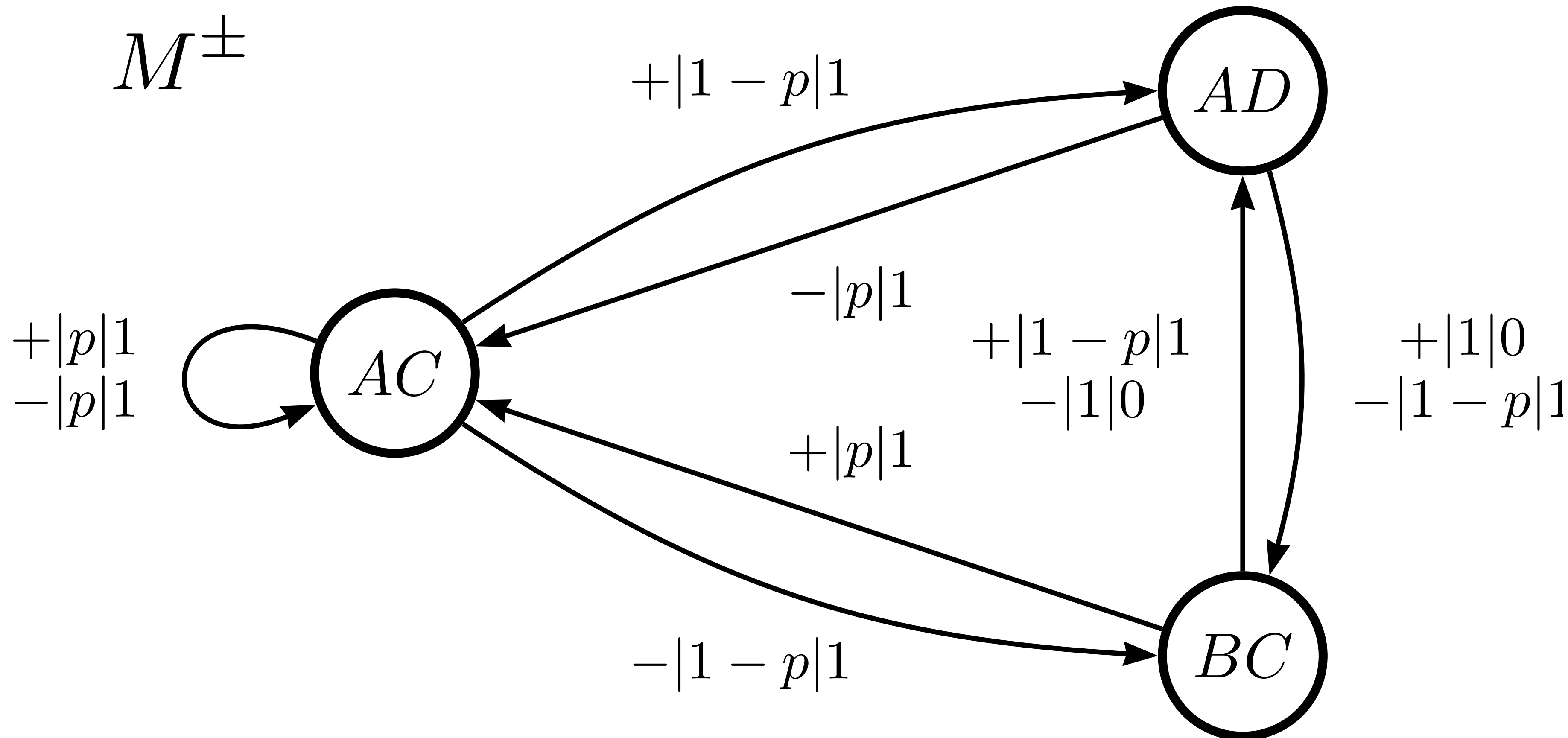
Bidirectional Machines

Example: Golden Mean Process ...

Bidirectional machine:

$$\boldsymbol{\mathcal{S}}^{\pm} = \{AC, AD, BC\}$$

$M^{\pm}$

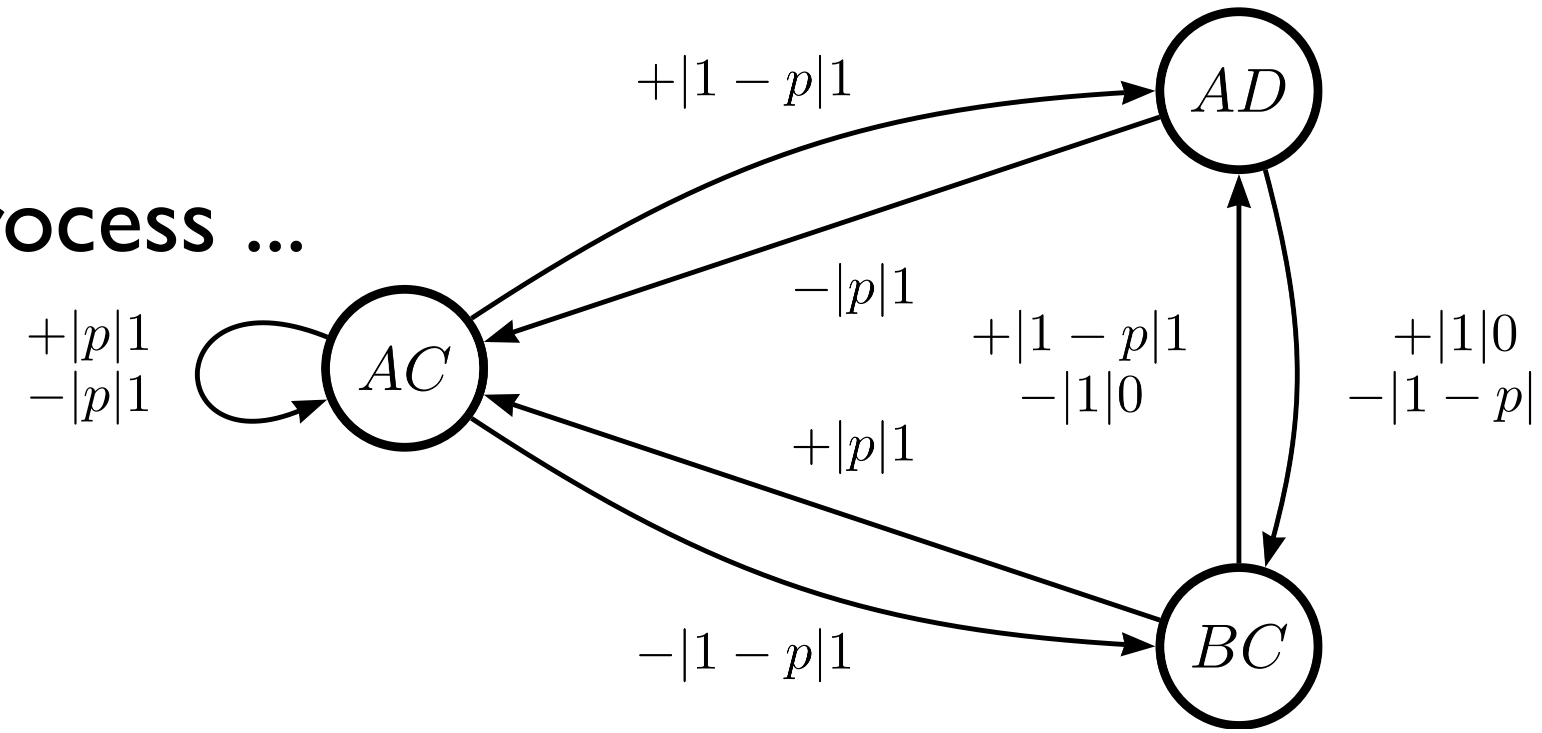


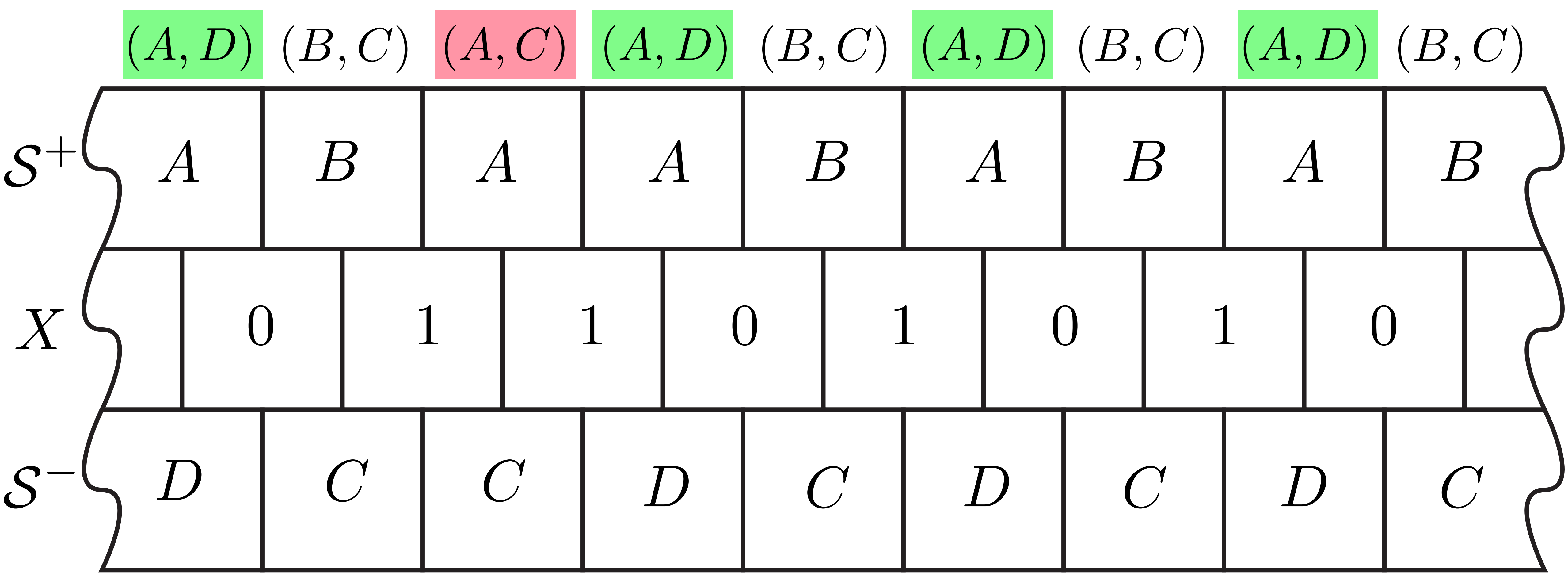
$$(\overleftarrow{X}, \overrightarrow{X}) :$$

$$\begin{aligned} (A, C) &\sim (\{*1\}, \{1*\}) \\ (A, D) &\sim (\{*1\}, \{0*\}) \\ (B, C) &\sim (\{*0\}, \{1*\}) \end{aligned}$$

Bidirectional Machines

Example: Golden Mean Process ...





Example: Golden Mean Process ...

Order-1 Markov:

$$\begin{aligned} \mathbf{E} &= C_\mu^+ - h_\mu \\ &= H\left(\frac{1}{2-p}\right) - \frac{H(p)}{2-p} \end{aligned}$$

Subshift of finite-type

But simple:

Causally reversible: $\Xi = 0$

and not simple:

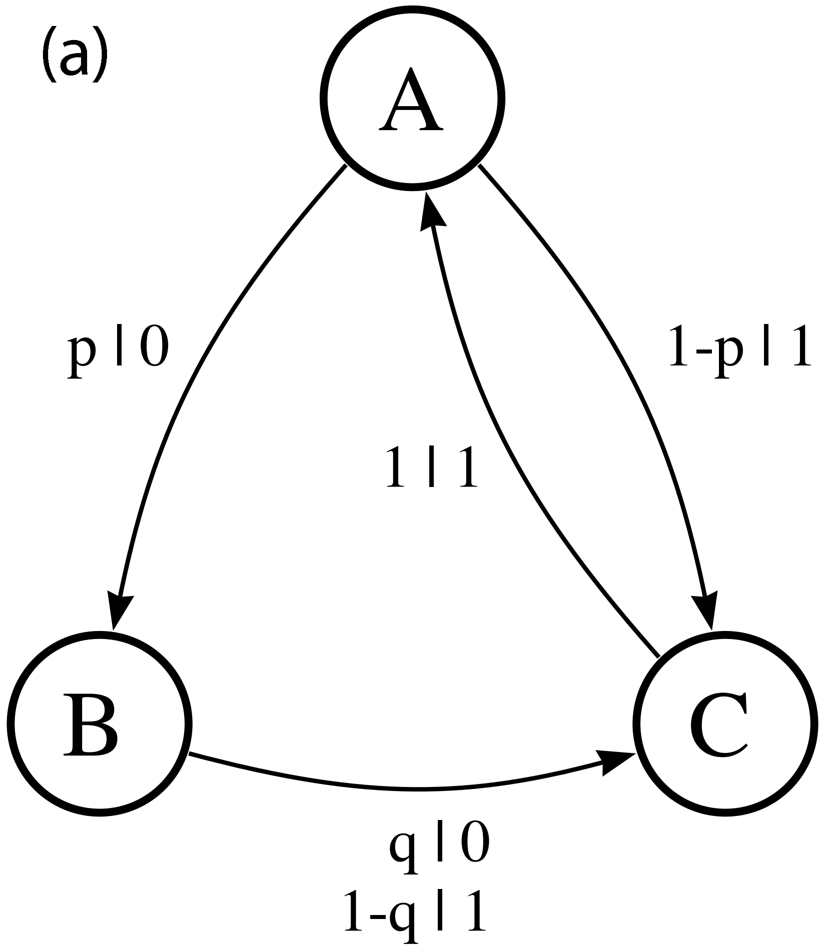
Cryptic process: $\chi^+ = \chi^- = \dfrac{H(p)}{2-p}$

Causal Irreversibility

Example: Random Insertion Process

Forward ε-machine:

(a)

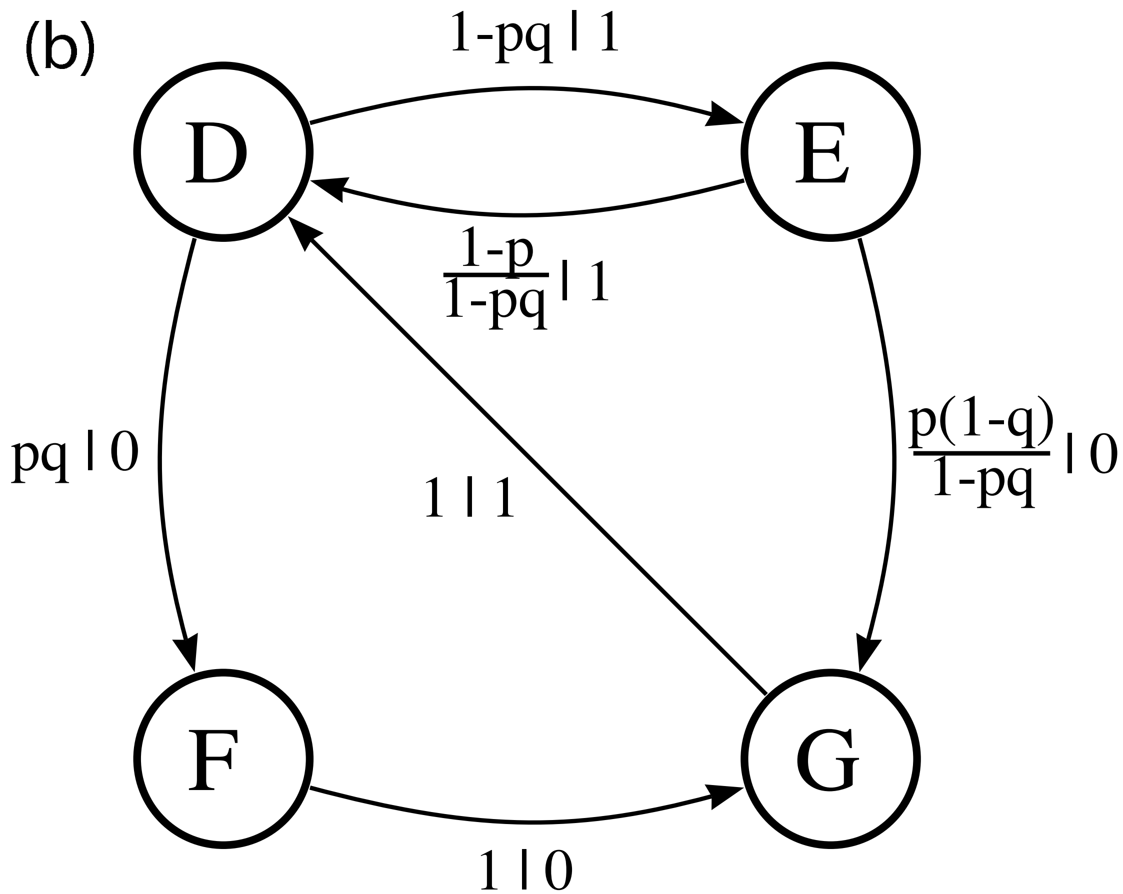


$$
\begin{array}{cccc}
 & A & B & C \\
\Pr(\mathcal{S}^+) \quad = & \Big(\frac{1}{p+2} & \frac{p}{p+2} & \frac{1}{p+2} \Big)
\end{array}
$$

Causal Irreversibility

Example: Random Insertion Process ...

Reverse ε-machine:



$$
\begin{array}{cccccc}
 & & D & E & F & G \\
\Pr(\mathcal{S}^-) & = & \Big(\frac{1}{p+2} & \frac{1-pq}{p+2} & \frac{pq}{p+2} & \frac{p}{p+2} \Big)
\end{array}
$$

Bidirectional Machines

Example: Random Insertion Process ...

Causally irreversible:

$$M^+ \neq M^-$$

$$C_\mu^+ = \log_2(p+2) - \frac{p \log_2 p}{p+2}$$

$$C_\mu^- = \log_2(p+2) + \frac{H(pq) - p \log_2 p}{p+2}$$

$$C_\mu^+ \neq C_\mu^-$$

$$\Xi = \frac{H(pq)}{p+2}$$

Bidirectional Machines

Example: Random Insertion Process ...

Forward switching map: $p = q = \frac{1}{2}$

$$
\Pr(\mathcal{S}^+ | \mathcal{S}^-) = \begin{array}{c c}
 & \begin{array}{ccc} A & B & C \end{array} \\
\begin{array}{c} D \\ E \\ F \\ G \end{array} & \begin{pmatrix} 0 & 0 & 1 \\ 2/3 & 1/3 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\end{array}
$$

Reverse switching map: $p = q = \frac{1}{2}$

$$
\Pr(\mathcal{S}^- | \mathcal{S}^+) = \begin{array}{c c}
 & \begin{array}{cccc} D & E & F & G \end{array} \\
\begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\end{array}
$$

Bidirectional Machines

Example: Random Insertion Process ...

Joint distribution (general p and q):

$$\Pr(\mathcal{S}^+, \mathcal{S}^-) = \frac{1}{(p+2)} \begin{array}{c} \\ A \\ B \\ C \end{array} \!\!\begin{array}{c} \begin{array}{cccc} D & E & F & G \end{array} \\ \begin{pmatrix} 0 & 1-p & 0 & p \\ 0 & p(1-q) & pq & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

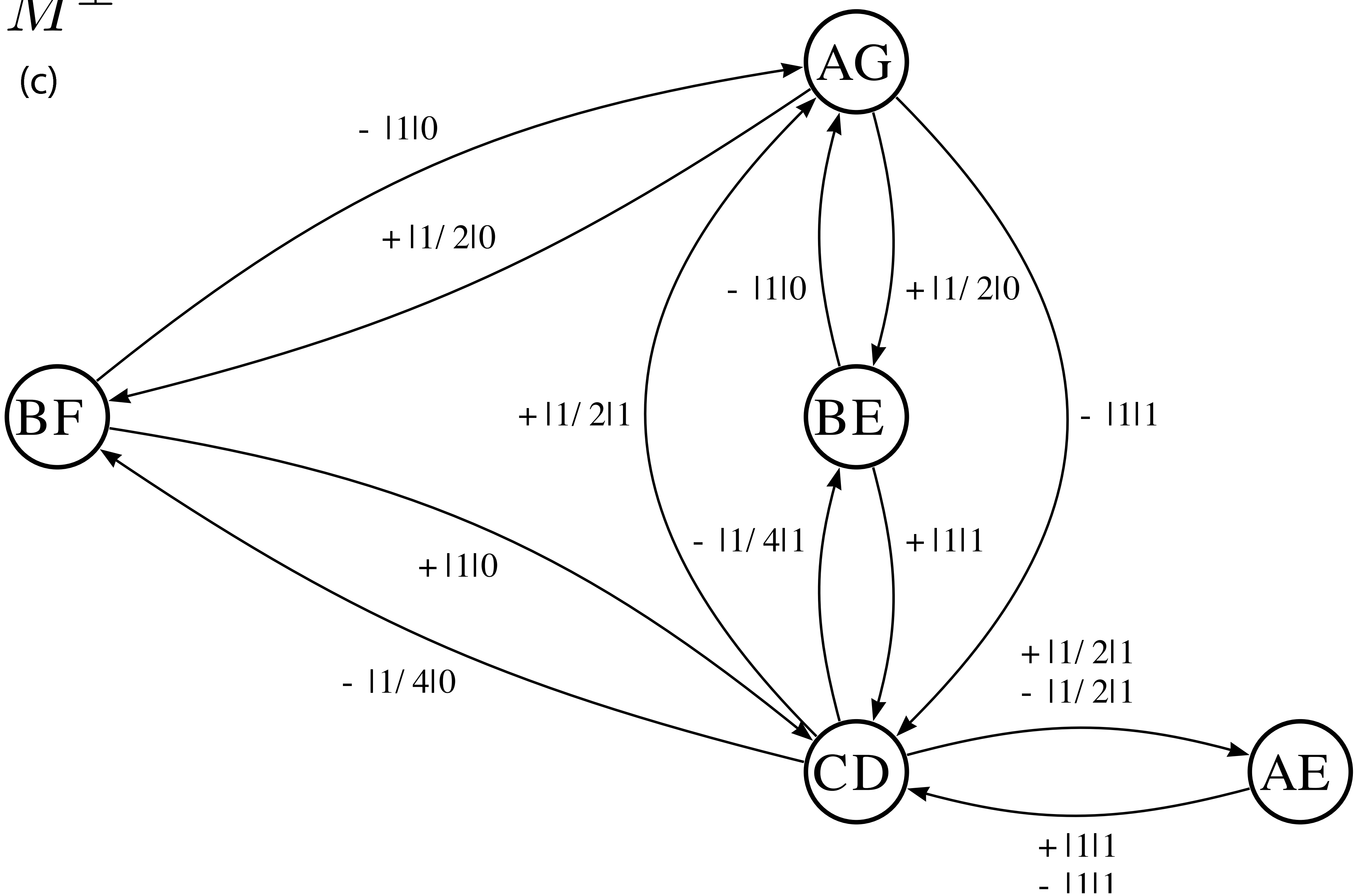
Bidirectional Machines

Example: Random Insertion Process ...

Bidirectional machine: $p = q = \frac{1}{2}$

$M^{\pm}$

(c)



Bidirectional Machines

Example: Random Insertion Process ...

$$\chi^+ = \frac{1 - pq}{p + 2} H\left(\frac{1 - p}{1 - pq}\right)$$

$$\chi^- = \frac{1 - pq}{p + 2} H\left(\frac{1 - p}{1 - pq}\right) + \frac{H(pq)}{p + 2}$$

$$\begin{aligned} \mathbf{E} &= C_\mu^+ - \chi^+ \\ &= \log_2(p + 2) - \frac{p \log_2 p}{p + 2} - \frac{1 - pq}{p + 2} H\left(\frac{1 - p}{1 - pq}\right) \end{aligned}$$

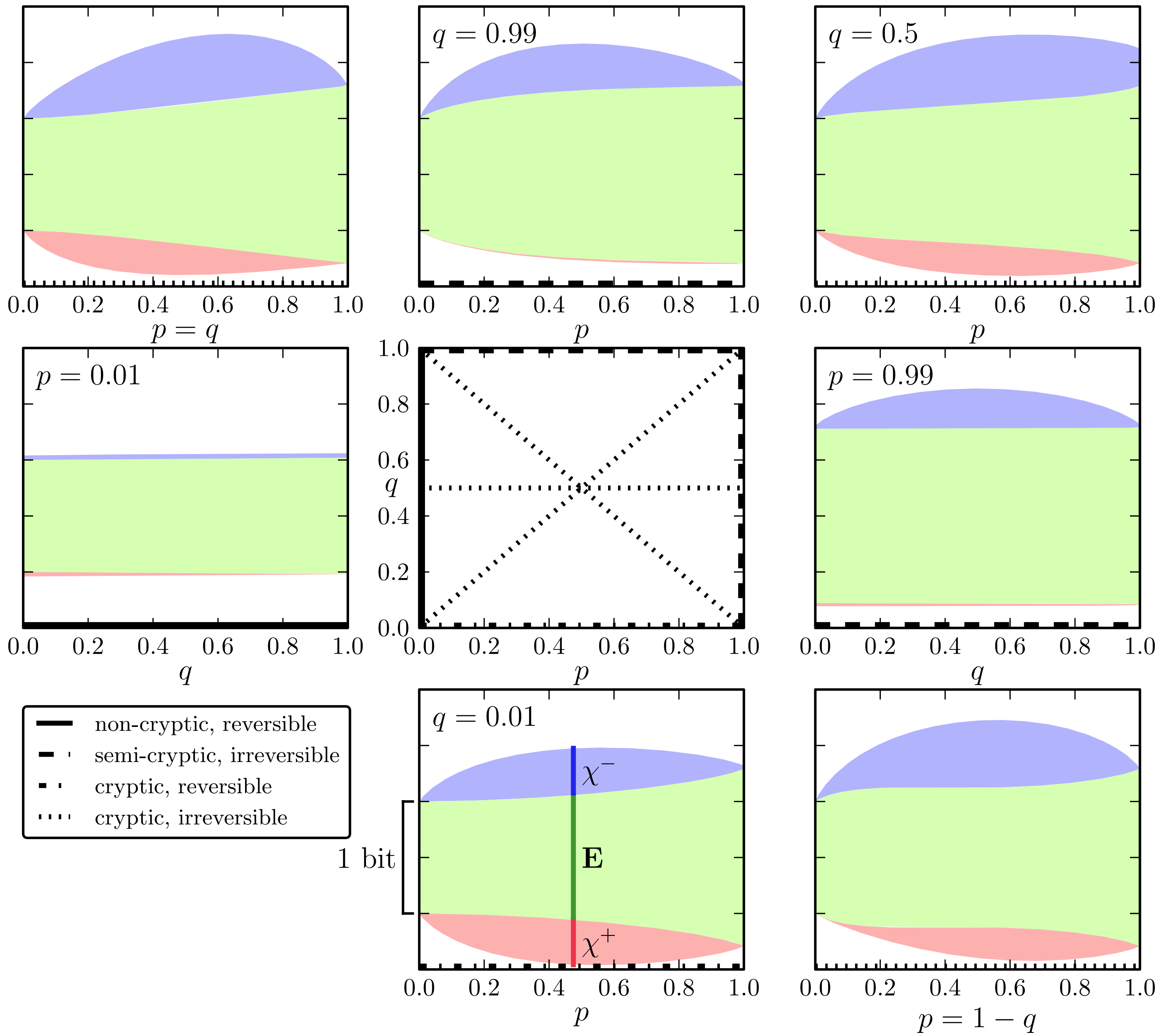
Bidirectional Machines

Example: Random Insertion Process ...

Generally, cryptic, irreversible process $(p, q) \in [0, 1]^2$

But ranges over:

- non-cryptic, reversible
- semi-cryptic, irreversible
- cryptic, reversible
- cryptic, irreversible



Comments:

Naive combination of forward and reverse machines
would be addition or product over states,
bidirectional machine is neither.

Bidirectional machine:
- Nonminimal
- Nonunifilar
- Not ε-machine.

Projections onto forward moves gives forward process,
but might be nonminimal machine.
Projection onto reverse moves gives reverse process,
but might be nonminimal machine.

Bidirectional Machines

Oddities of "prediction":

Predictive states are better retrodictors
than they are predictors (by χ^+).

Predictive states are better retrodictors
than retrodictive states (by χ^-).

Bidirectional Machines

Open questions:

What are bidirectional transient states?

Minimality?

Unifilar presentation?

Bidirectional Complexities

Bidirectional Complexities

Statistical Complexity of Bidirectional Machine:

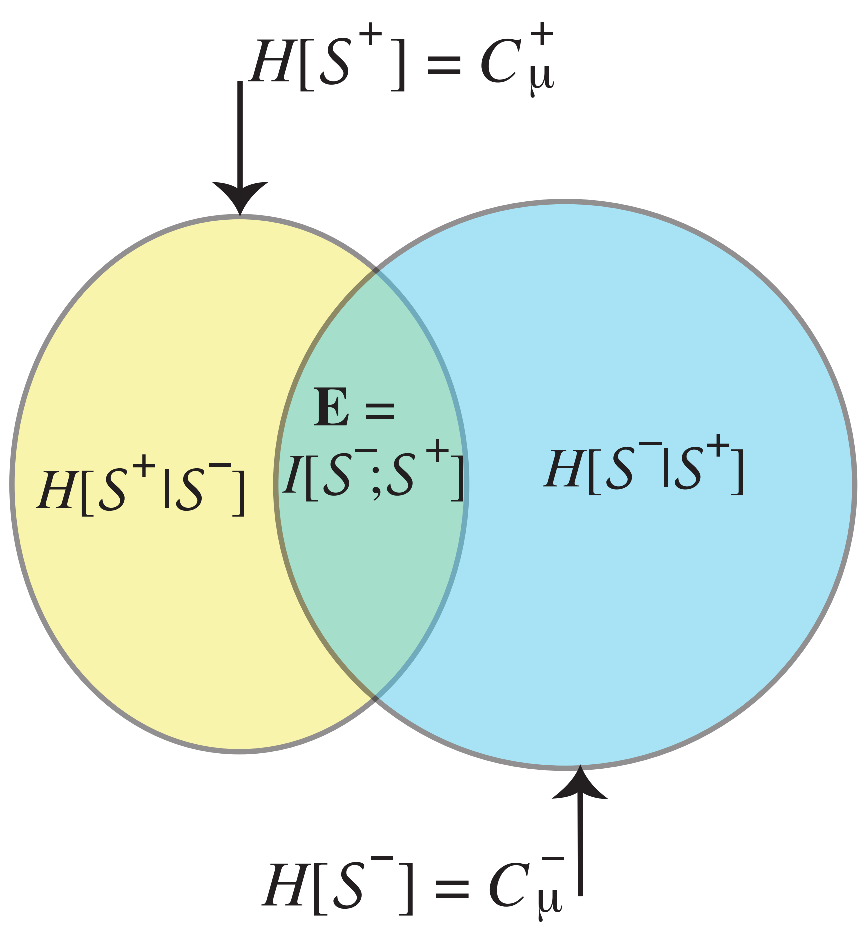
$$C_\mu^{\pm} \equiv H[\mathcal{S}^{\pm}] = H[\mathcal{S}^{+}, \mathcal{S}^{-}]$$

The stored information required to optimally predict *and* retrodict.

Bidirectional Complexities

Excess entropy:

$$\begin{aligned}
\mathbf{E} &= I[\mathcal{S}^+;\mathcal{S}^-] \\
&= H[\mathcal{S}^+] + H[\mathcal{S}^-] - H[\mathcal{S}^+,\mathcal{S}^-] \\
&= C_\mu^+ + C_\mu^- - C_\mu^\pm
\end{aligned}$$

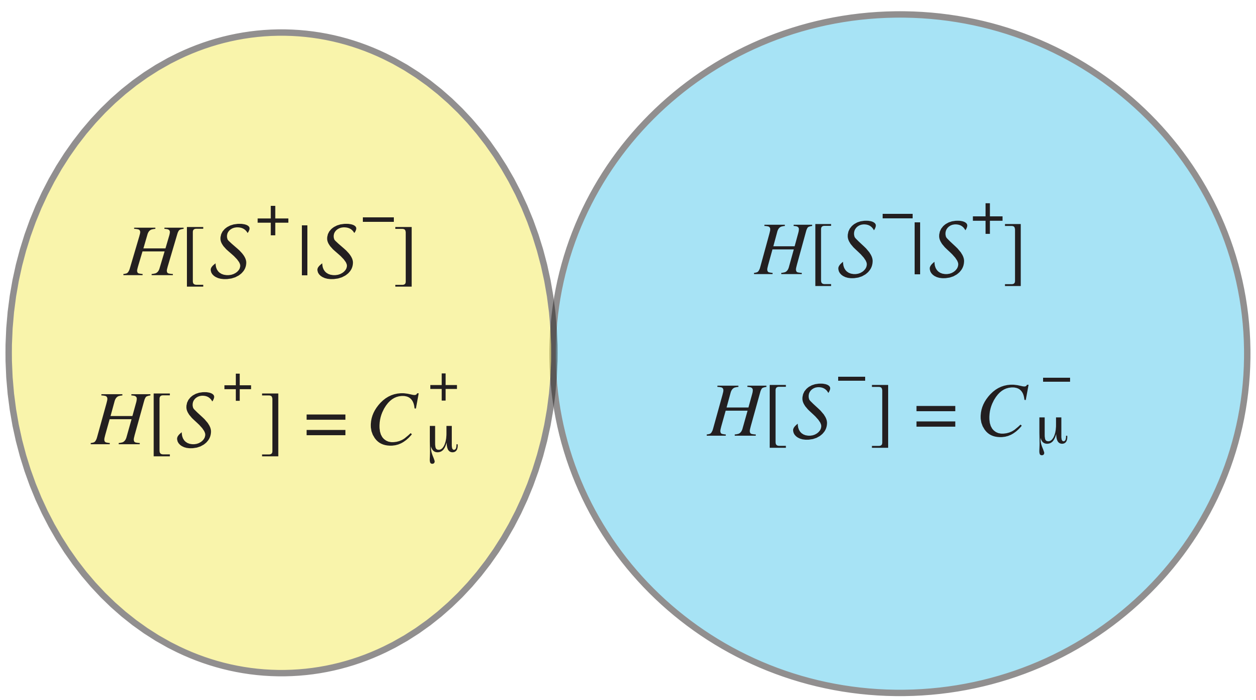


Bidirectional Complexities

Excess entropy ...

Only when $\mathbf{E} = 0$:

$$C_\mu^{\pm} = C_\mu^{+} + C_\mu^{-}$$



Bidirectional Complexities

Bounds:

Before:

$$\mathbf{E} \leq C_\mu \ (\equiv C_\mu^+)$$

Now, tighter bounds on excess entropy:

$$\mathbf{E} \leq C_\mu^+$$

and

$$\mathbf{E} \leq C_\mu^-$$

Bidirectional Complexities

Bounds ...

Bidirectional machine smaller than forward *and* reverse:

$$C_\mu^\pm \leq C_\mu^+ + C_\mu^-$$

$M^\pm$ is efficient representation.

Forward ε-machine smaller than bidirectional:

$$C_\mu^+ \leq C_\mu^\pm$$

Reverse ε-machine smaller than bidirectional:

$$C_\mu^- \leq C_\mu^\pm$$

Bidirectional Complexities

From I-diagram:

$$C_\mu^\pm = \mathbf{E} + \chi^\pm$$

where

Crypticity:
$$\begin{aligned} \chi^\pm &= \chi^+ + \chi^- \\ &= H[\mathcal{S}^+ | \mathcal{S}^-] + H[\mathcal{S}^- | \mathcal{S}^+] \end{aligned}$$

Distance between measurements & model: $C_\mu^\pm - \mathbf{E}$

Form of a true distance: $d(X,Y) = H[X|Y] + H[Y|X]$

Also, distance between forward and reverse processes.

Information inaccessibility:

Degree to which internal information is hidden.

Bidirectional Complexities

Bounds ...

$$\chi^{\pm} \leq C_\mu^{\pm}$$

Truly cryptic process:

$$\chi^{\pm} = C_\mu^{\pm}$$

(All state information is in crypticity.)

$$\mathbf{E} = 0$$

Nothing can be learned about a process's structure from measurements.

Summary

Bidirectionality
Excess entropy from ε-machine
Information diagrams for processes
Bidirectional machines
Bidirectional complexities