The Big Picture

Discuss

- Examples of unpredictability

- *Odds*, Stanisław Lem, The New Yorker (1974)

- *Chaos*, Scientific American (1986)

The Big Picture ...

Qualitative Dynamics (Reading: *NDAC*, Chapters 1 and 2)

What is it?

Analyze nonlinear systems *without* solving the equations.

Why is it needed?

In general, nonlinear systems cannot be solved in closed form.

Three tools:

Statistics

Computation: e.g., simulation

Mathematical: Dynamical Systems Theory

Why each is good.

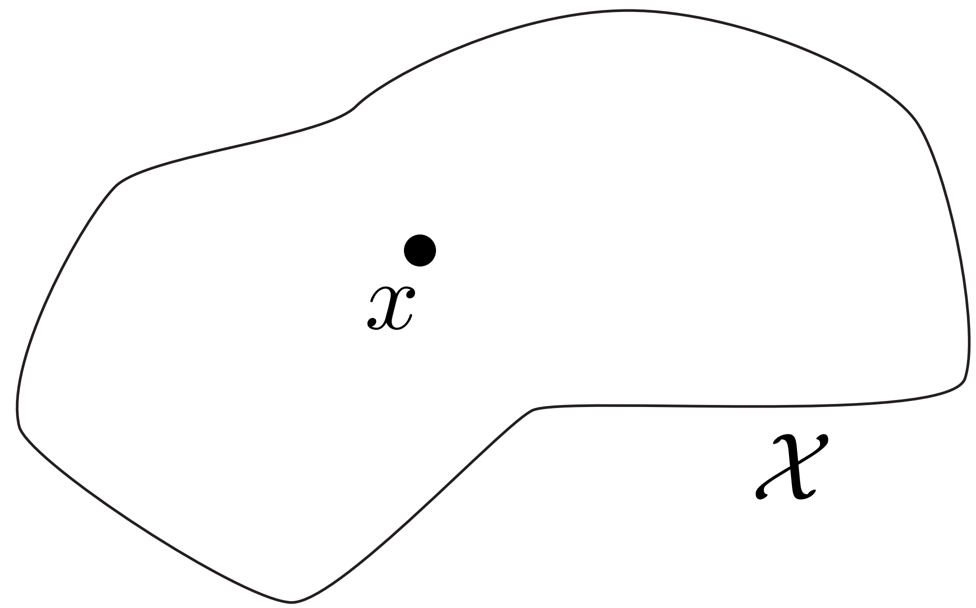
Why each fails in some way.

The Big Picture ...

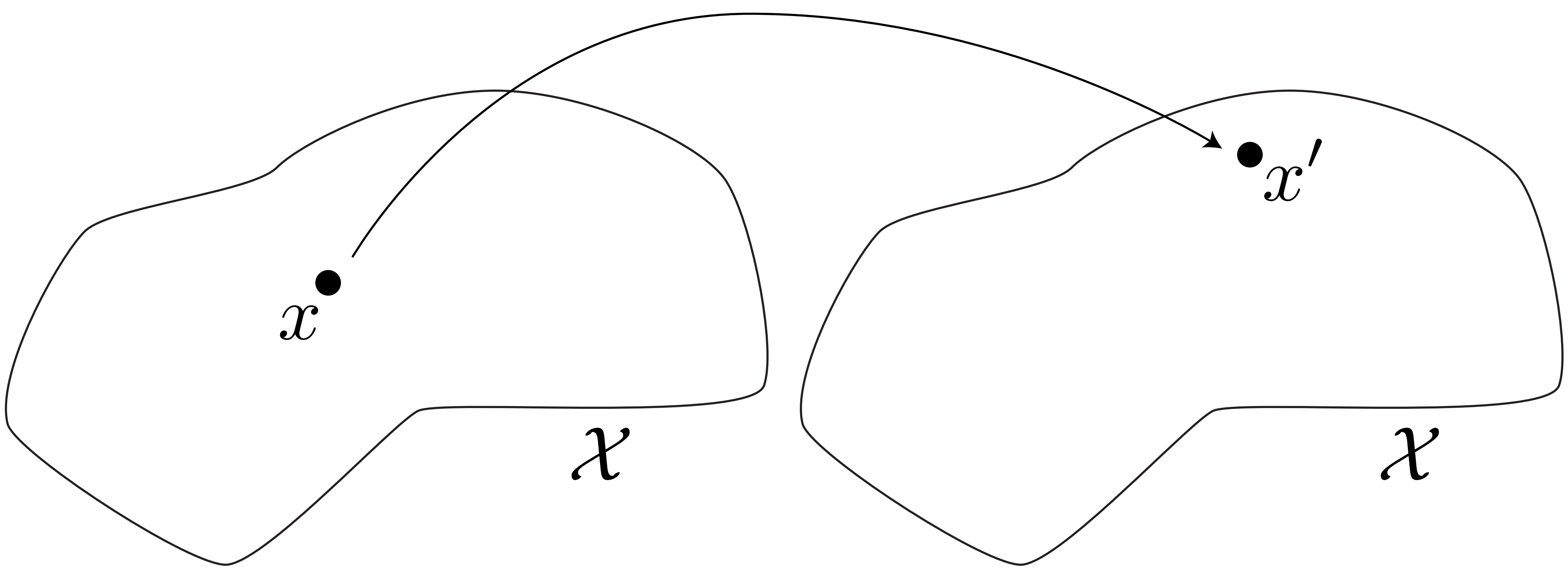
Dynamical System: $\{\mathcal{X}, \mathcal{T}\}$

State Space: $\mathcal{X}$

State: $x \in \mathcal{X}$



Dynamic: $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$



The Big Picture ...

Dynamical System ...

For example, continuous time ...

Ordinary differential equation: $\dot{\vec{x}} = \vec{F}(\vec{x})$ $\quad (\dot{} = \frac{d}{dt})$

State: $\vec{x}(t) \in \mathbf{R}^n$ $\quad \vec{x} = (x_1, x_2, \ldots, x_n)$

Initial Condition (IC): $\vec{x}(0)$

Dynamic: $\vec{F} : \mathbf{R}^n \to \mathbf{R}^n$ $\quad \vec{F} = (f_1, f_2, \ldots, f_n)$

Dimension: n

The Big Picture ...

Dynamical System ...

For example, discrete time ...

Map: $\vec{x}_{t+1} = \vec{F}(\vec{x}_t)$

State: $\vec{x}_t \in \mathbf{R}^n$ $\quad\quad$ $\vec{x} \; = \; (x_1, x_2, \ldots, x_n)$

Initial condition (IC): $\vec{x}_0$

Dynamic: $\vec{F} : \mathbf{R}^n \rightarrow \mathbf{R}^n$ $\quad\quad$ $\vec{F} \; = \; (f_1, f_2, \ldots, f_n)$

Dimension: n

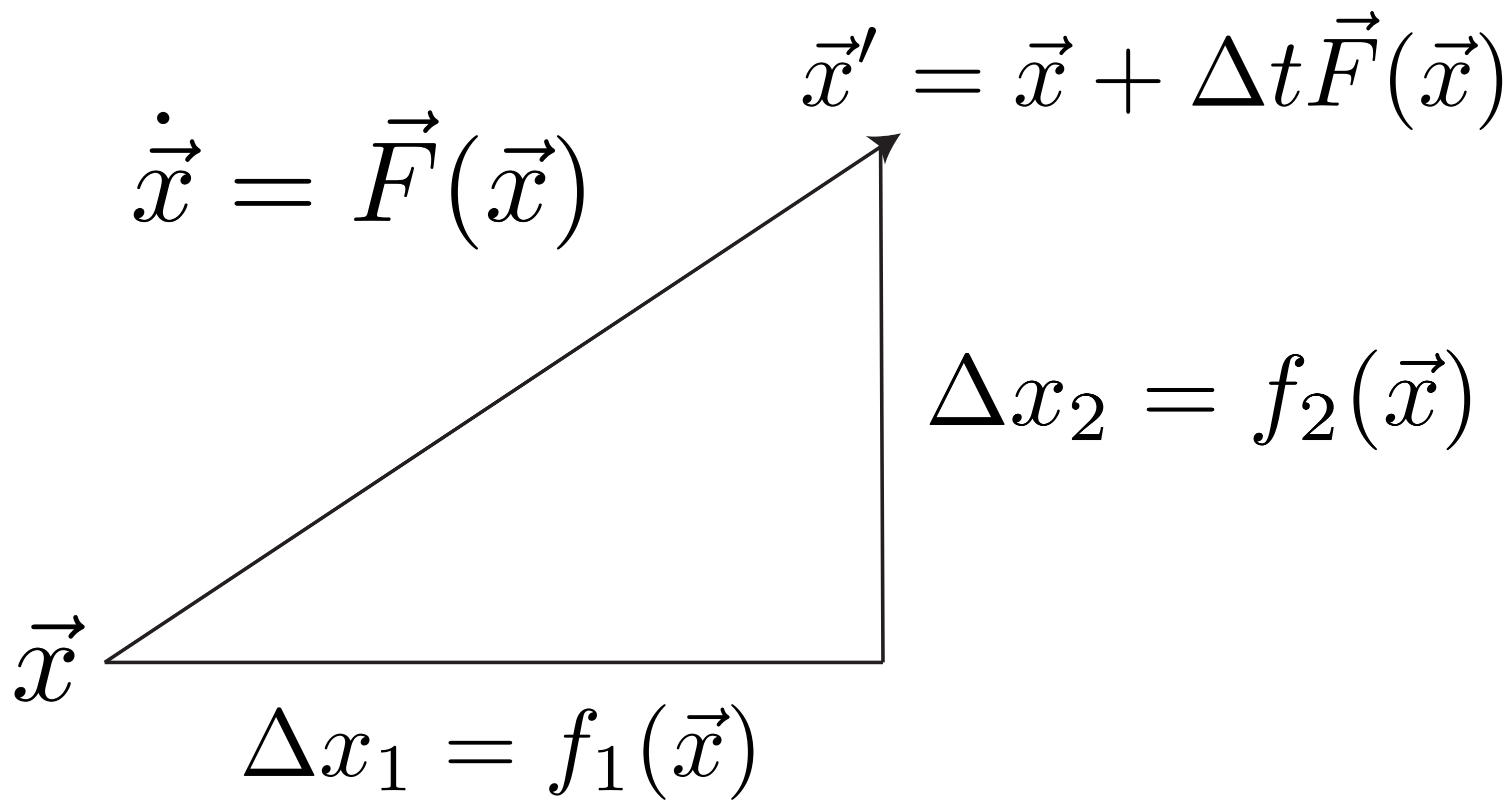
The Big Picture ...

Flow field for an ODE (aka Phase Portrait)

$$\dot{\vec{x}} = \vec{F}(\vec{x}) \qquad (\ \dot{} = \frac{d}{dt} \ )$$

Each state $\vec{x} = (x_1, x_2)$
has a vector attached

That says to what
next state $\vec{x}'$
to go



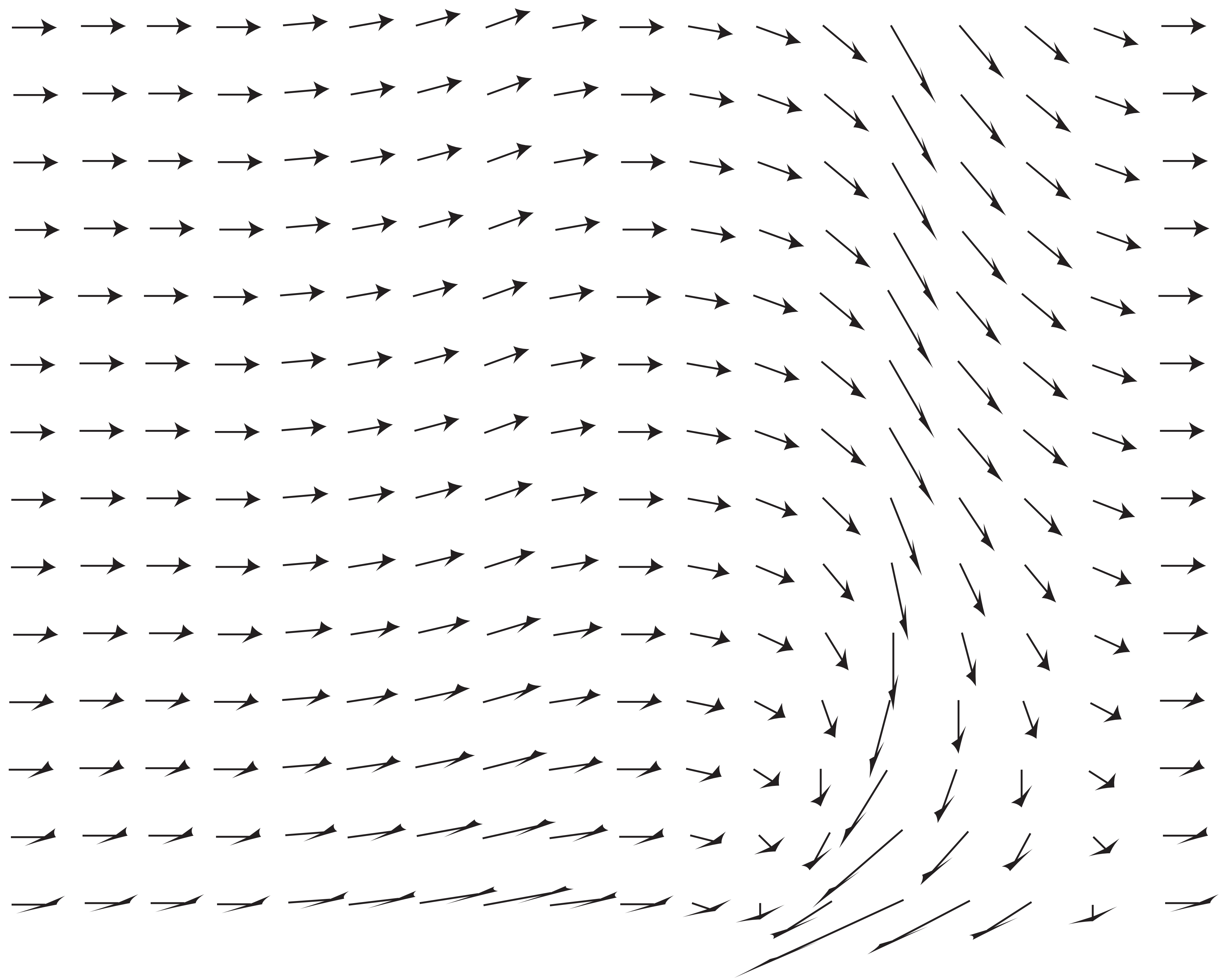
The Big Picture ...

Flow field for an ODE (aka Phase Portrait)

Vector field:

A set of rules

$$\mathcal{X} = \mathbf{R}^2$$



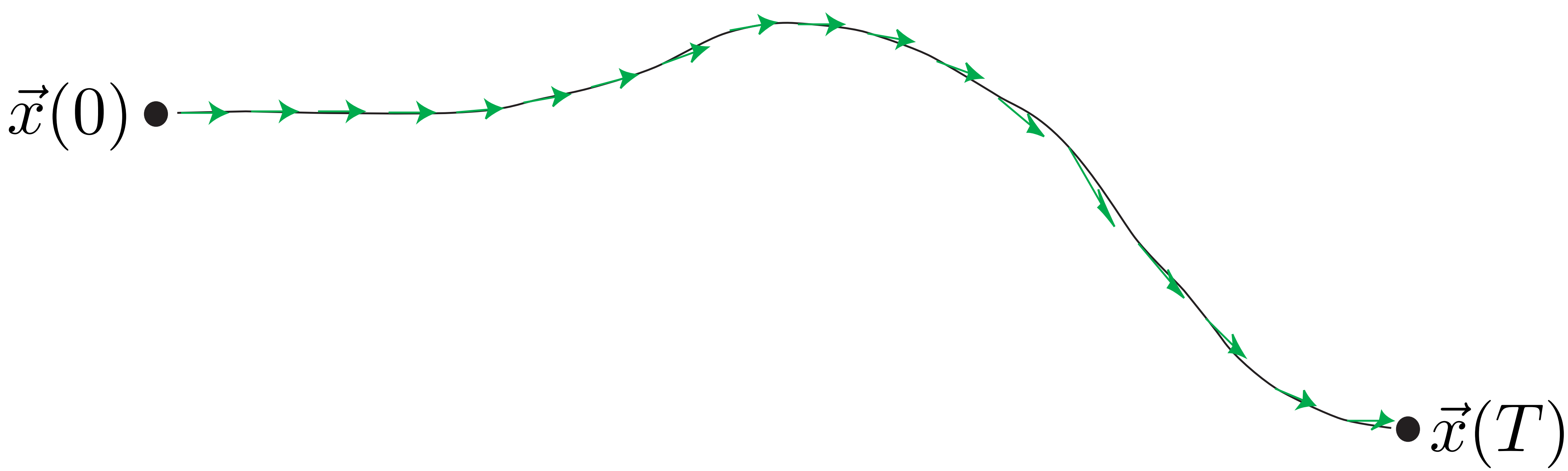
The Big Picture ...

Trajectory or Orbit:

the solution,

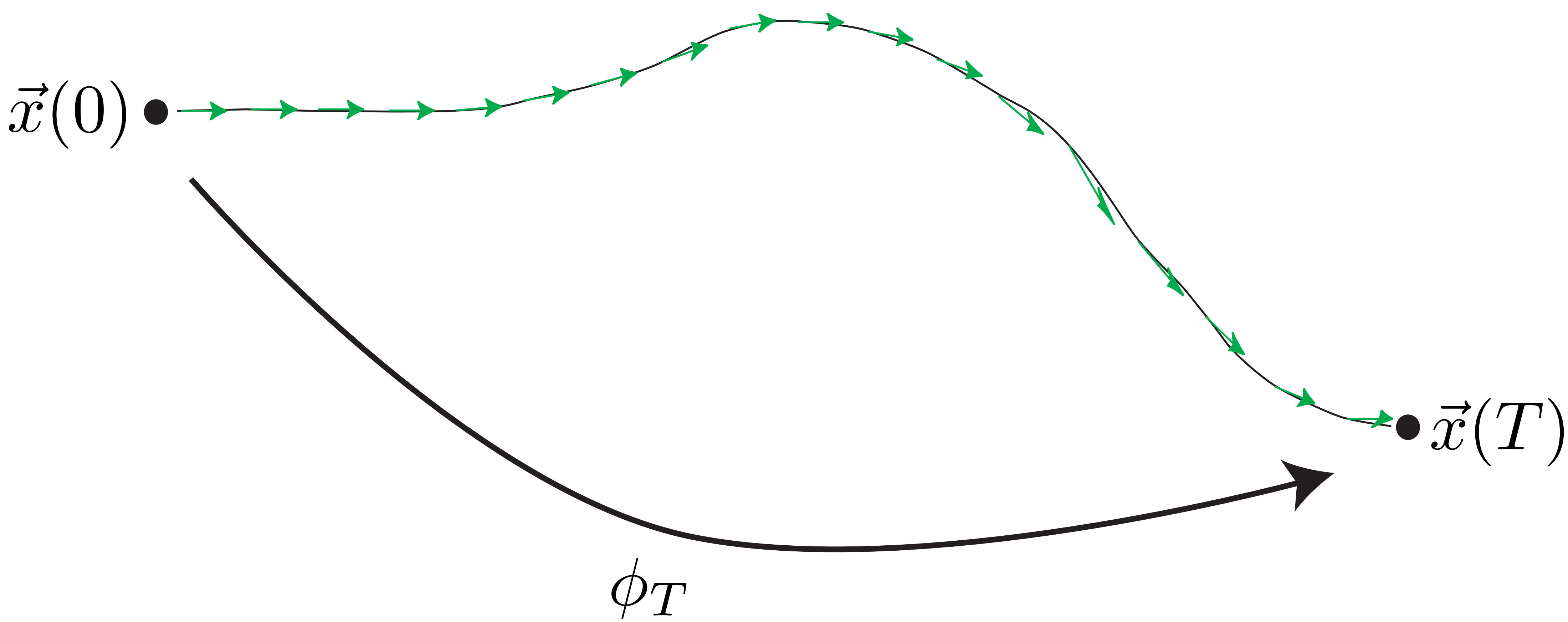
starting from some IC

simply follow the arrows



The Big Picture ...

Time-T Flow: $\vec{x}(T) = \phi_T(\vec{x}(0))$



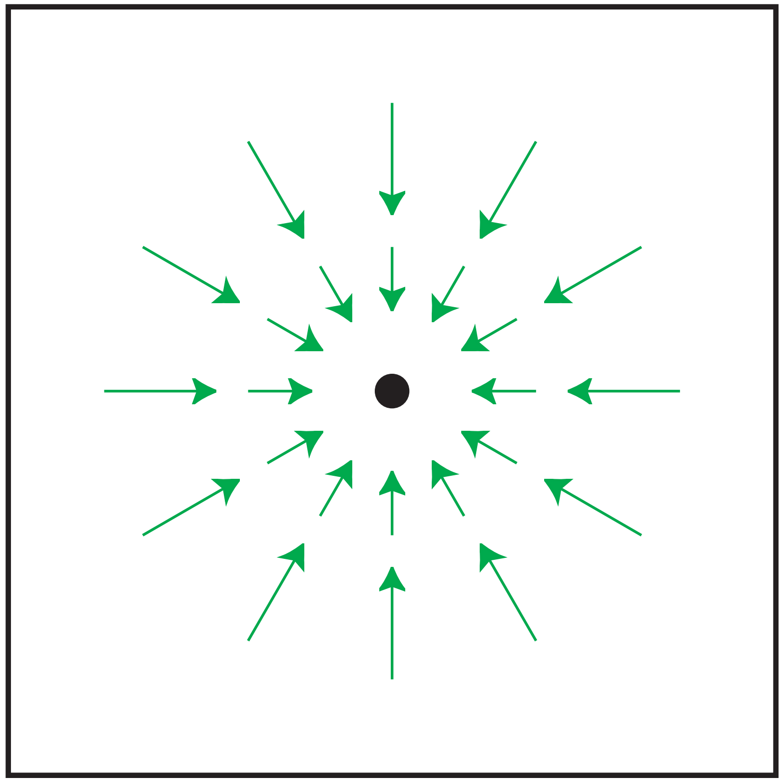
Point: ODE is only instantaneous,
flow gives state for *any* time t

The Big Picture ...

Invariant set: $\Lambda \subset \mathcal{X}$

Mapped into itself by the flow $\Lambda = \phi_T(\Lambda)$

Example: Invariant point



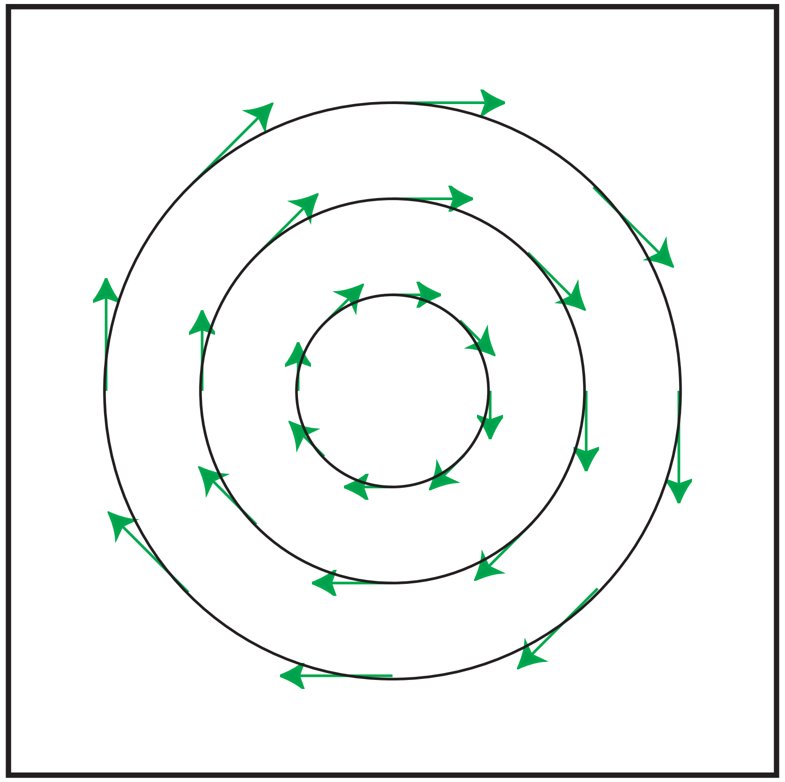
Fixed Point

The Big Picture ...

Invariant set: $\Lambda \subset \mathcal{X}$

Mapped into itself by the flow: $\Lambda = \phi_T(\Lambda)$

For example: Invariant circles



Pure Rotation
(Simple Harmonic Oscillator)

The Big Picture ...

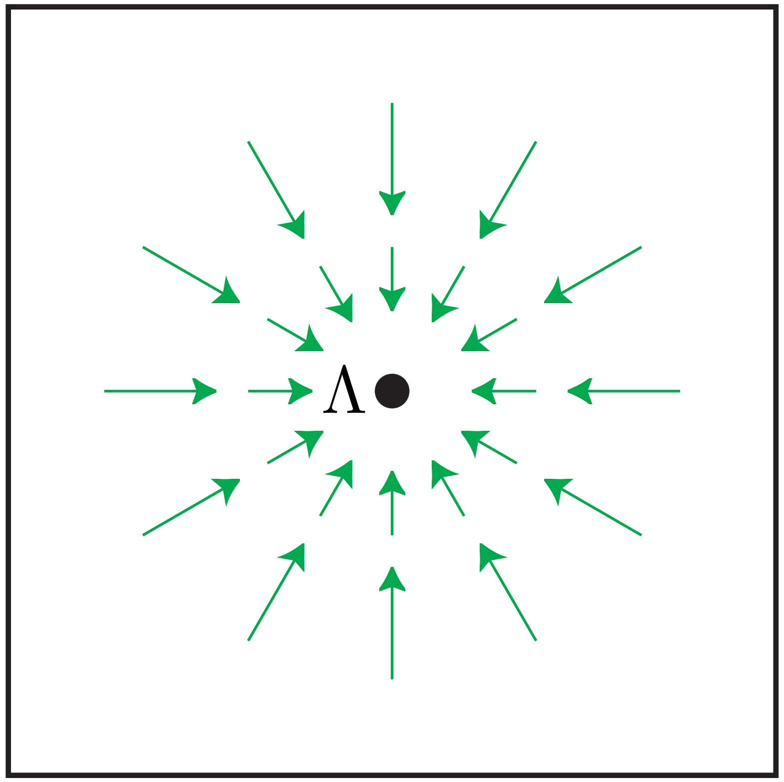
Attractor: $\Lambda \subset \mathcal{X}$

Where the flow goes at long times

(1) An invariant set

(2) A stable set: Perturbations off the set return to it

For example: Equilibrium

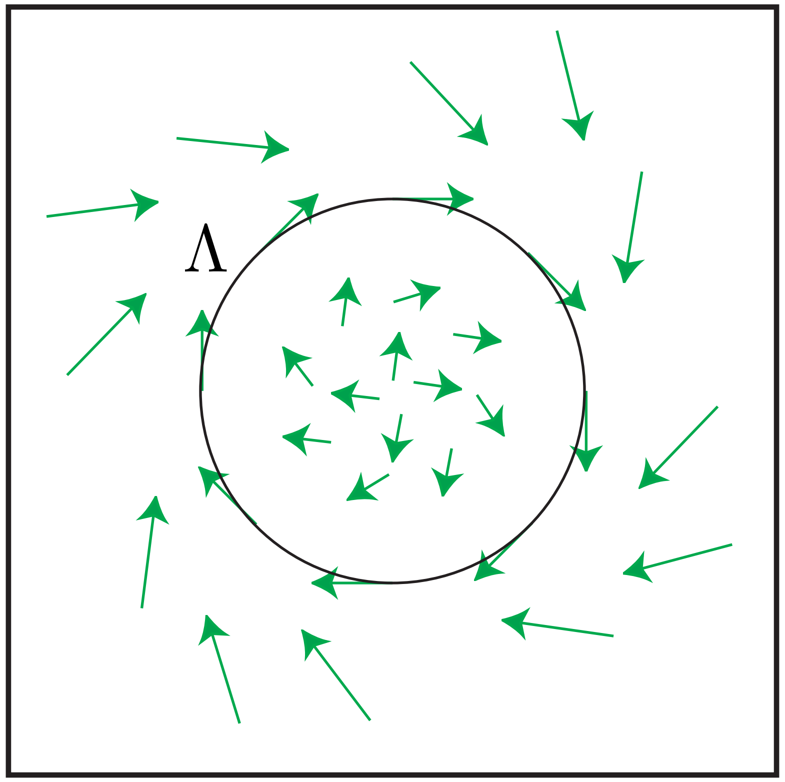


Stable Fixed Point

The Big Picture ...

Attractor: $\Lambda \subset \mathcal{X}$

For example: Stable oscillation



Limit Cycle

The Big Picture ...

Preceding:

A semi-local view ...

invariant sets and attractors in some region of the state space

Next:

A slightly Bigger Picture ...

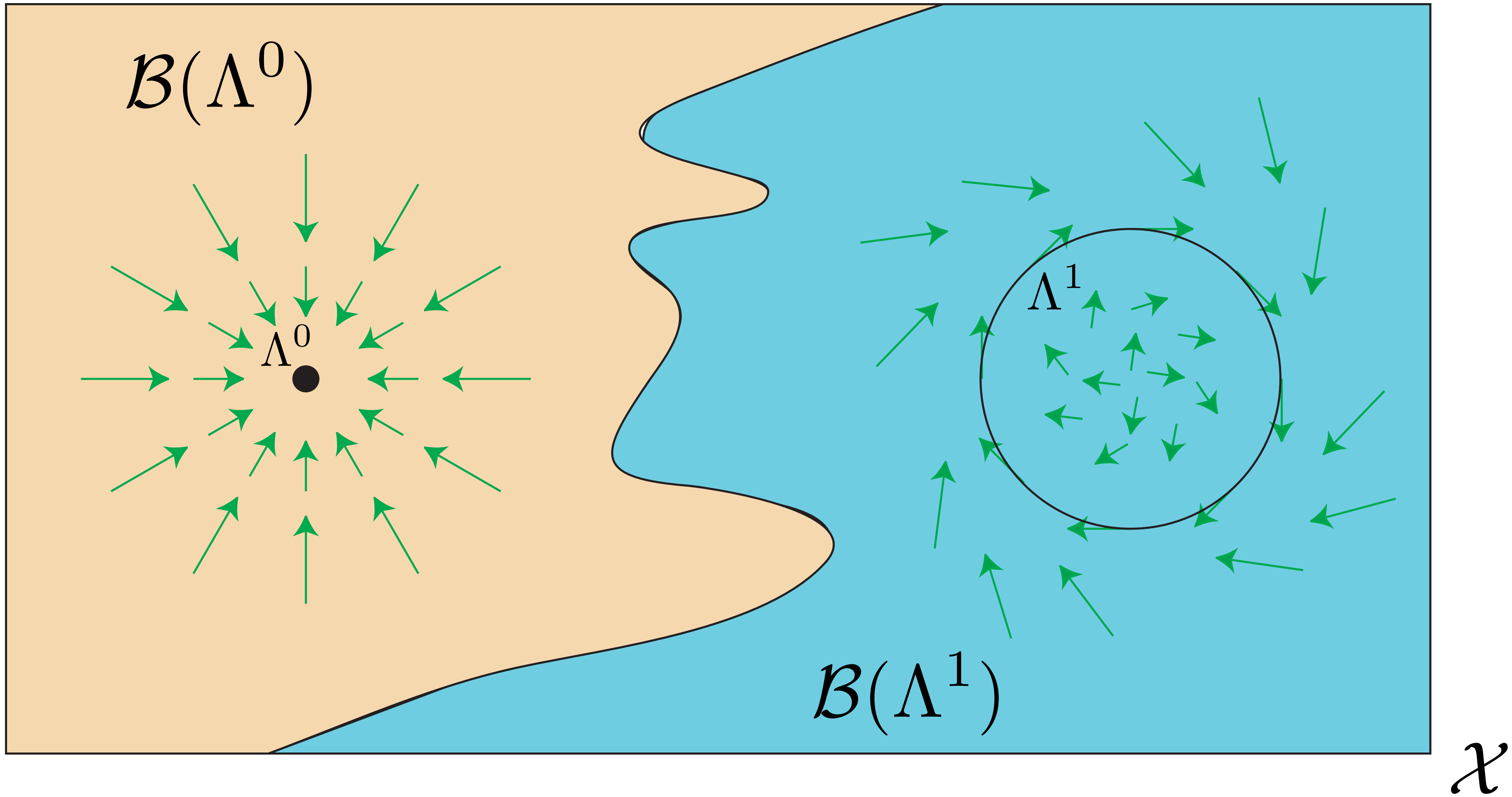
the full roadmap for the behavior of a dynamical system

The Big Picture ...

Basin of Attraction: $\mathcal{B}(\Lambda)$

The set of states that lead to an attractor

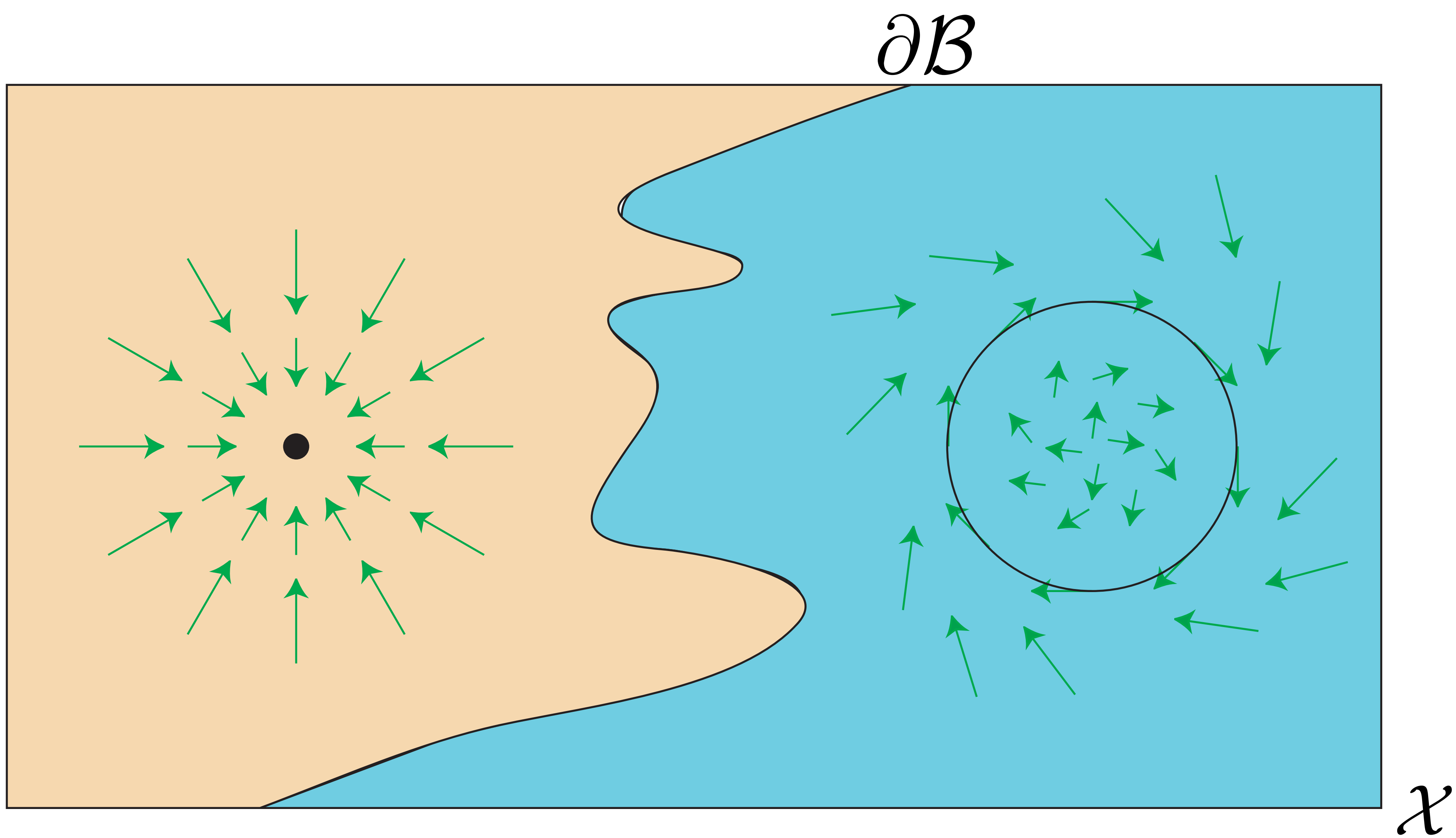
$$\mathcal{B}(\Lambda) = \{x \in \mathcal{X} : \lim_{t \to \infty} \phi_t(x) \in \Lambda\}$$



The Big Picture ...

Separatrix (aka Basin Boundary): $\partial\mathcal{B} = \mathcal{X} - \bigcup_i \mathcal{B}(\Lambda^i)$

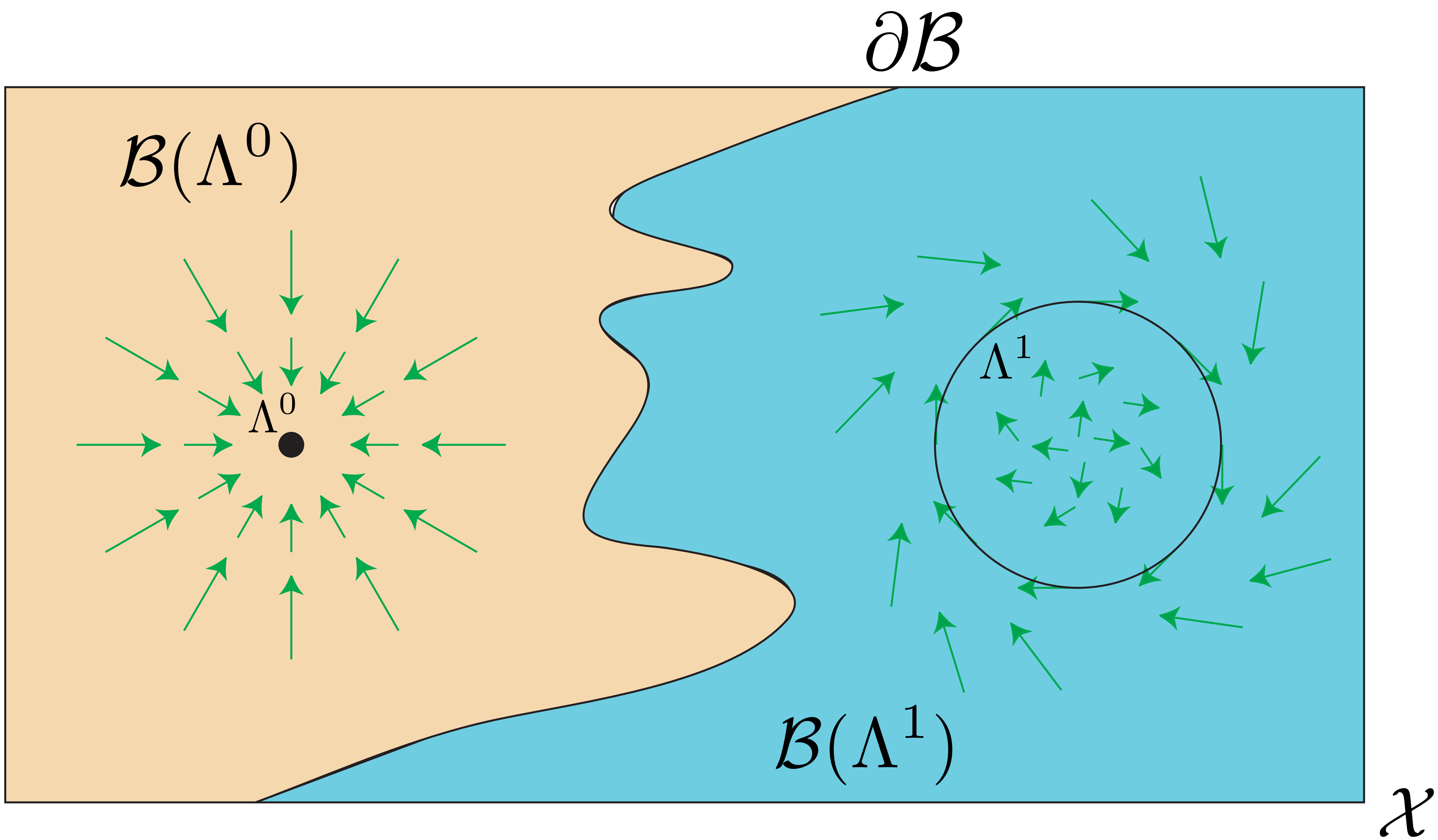
The set of states that do not go to an attractor
The set of states in no basin
The set of states dividing multiple basins



The Big Picture ...

The Attractor-Basin Portrait: $\Lambda^0, \Lambda^1, \mathcal{B}(\Lambda^0), \mathcal{B}(\Lambda^1), \partial\mathcal{B}$

The collection of attractors, basins, and separatrices



The Big Picture ...

Back to the local, again

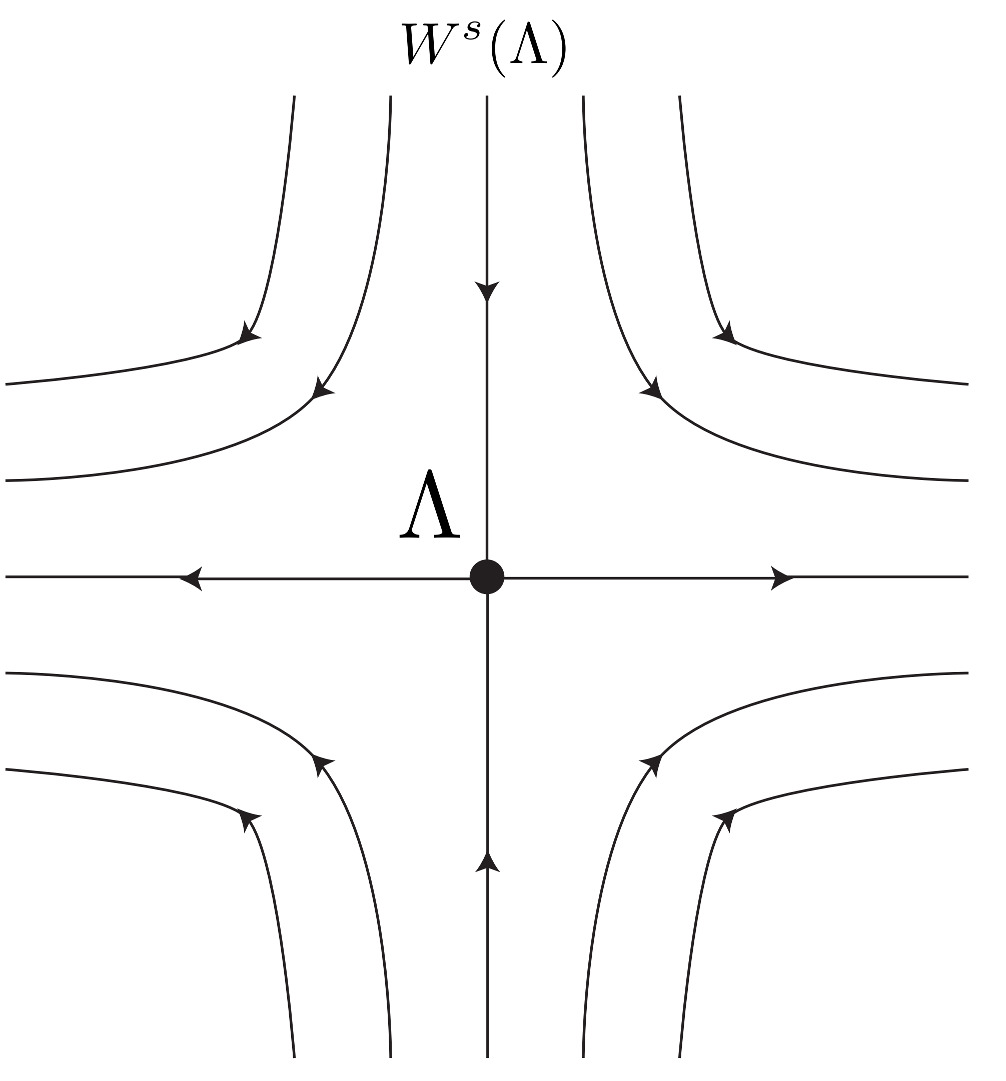
Submanifolds:

Split the state space into subspaces that track stability

Stable manifolds of an invariant set:

Points that go to the set

$$W^s(\Lambda) = \{x \in \mathcal{X} : \lim_{t \to \infty} \phi_t(x) \in \Lambda\}$$

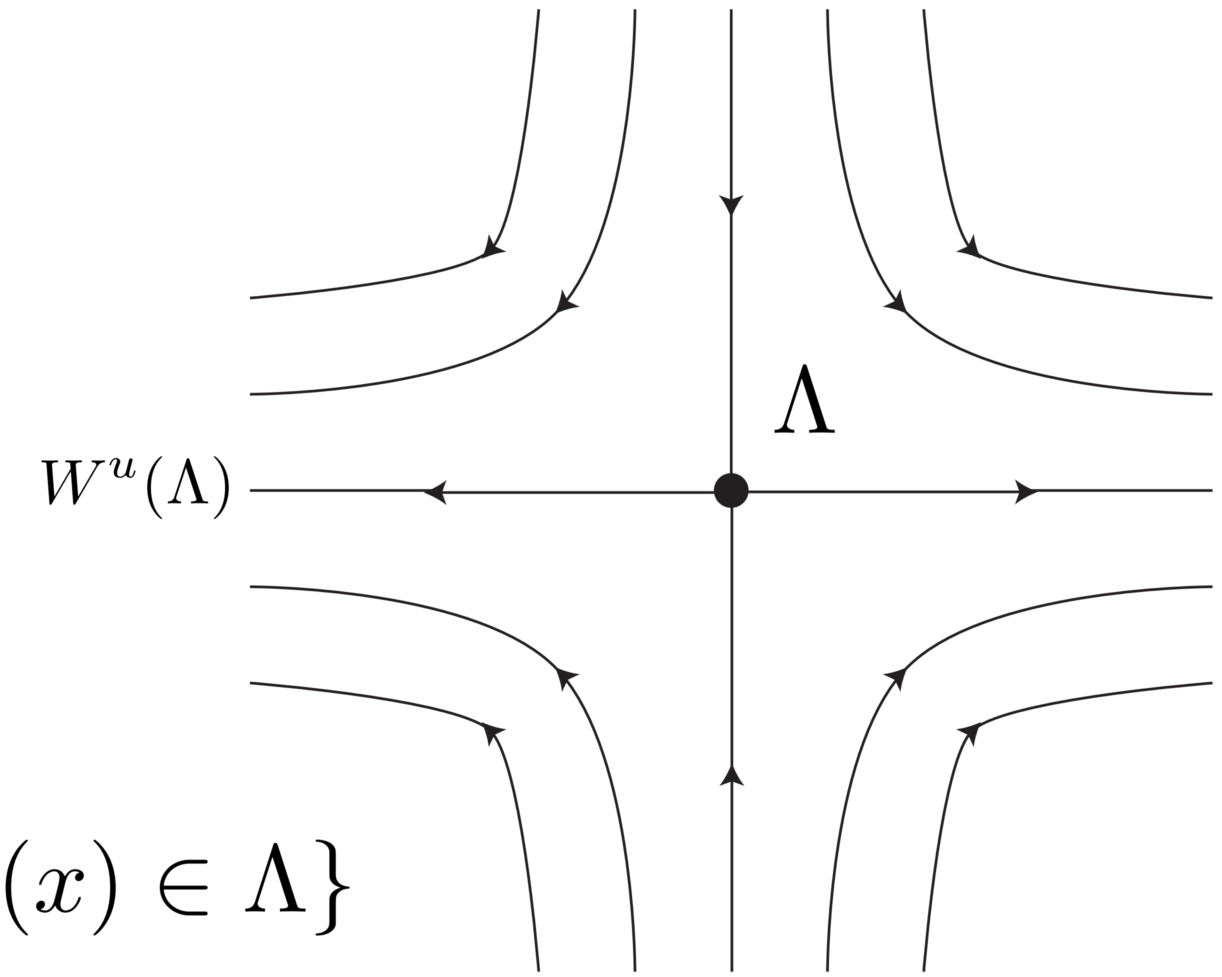


The Big Picture ...

Submanifolds ...

Unstable manifold:

Points that go to an invariant set in *reverse time*



$$W^u(\Lambda) = \{x \in \mathcal{X} : \lim_{t \to \infty} \phi_{-t}(x) \in \Lambda\}$$

The Big Picture ...

Example: 1D flows $\dot{x} = F(x)$

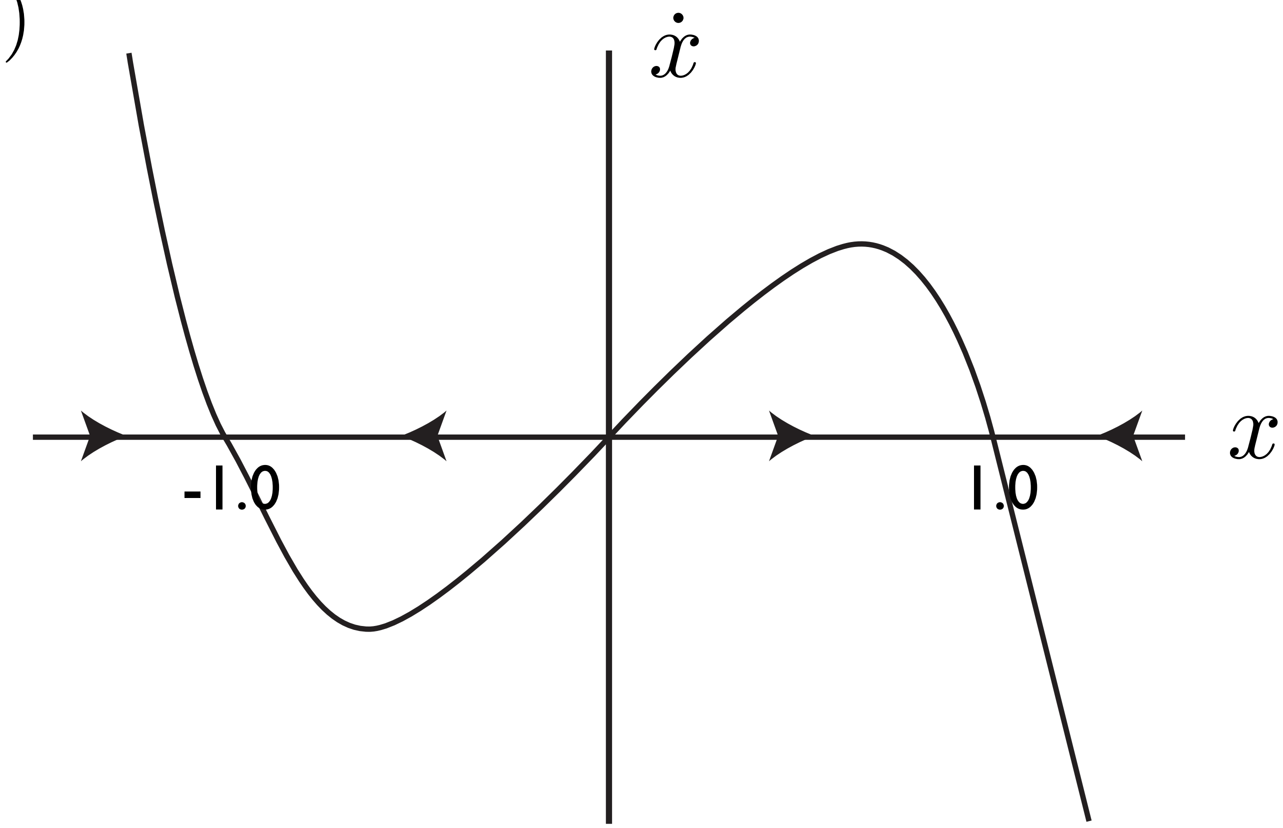
State Space: $\mathbf{R}$

State: $x \in \mathbf{R}$

Dynamic: $F : \mathbf{R} \to \mathbf{R}$

Flow: $x(T) = \phi_T(x(0)) = x(0) + \int_0^T dt F(x(t))$

Graph $F(x)$



$\Leftarrow$ Vector field

The Big Picture ...

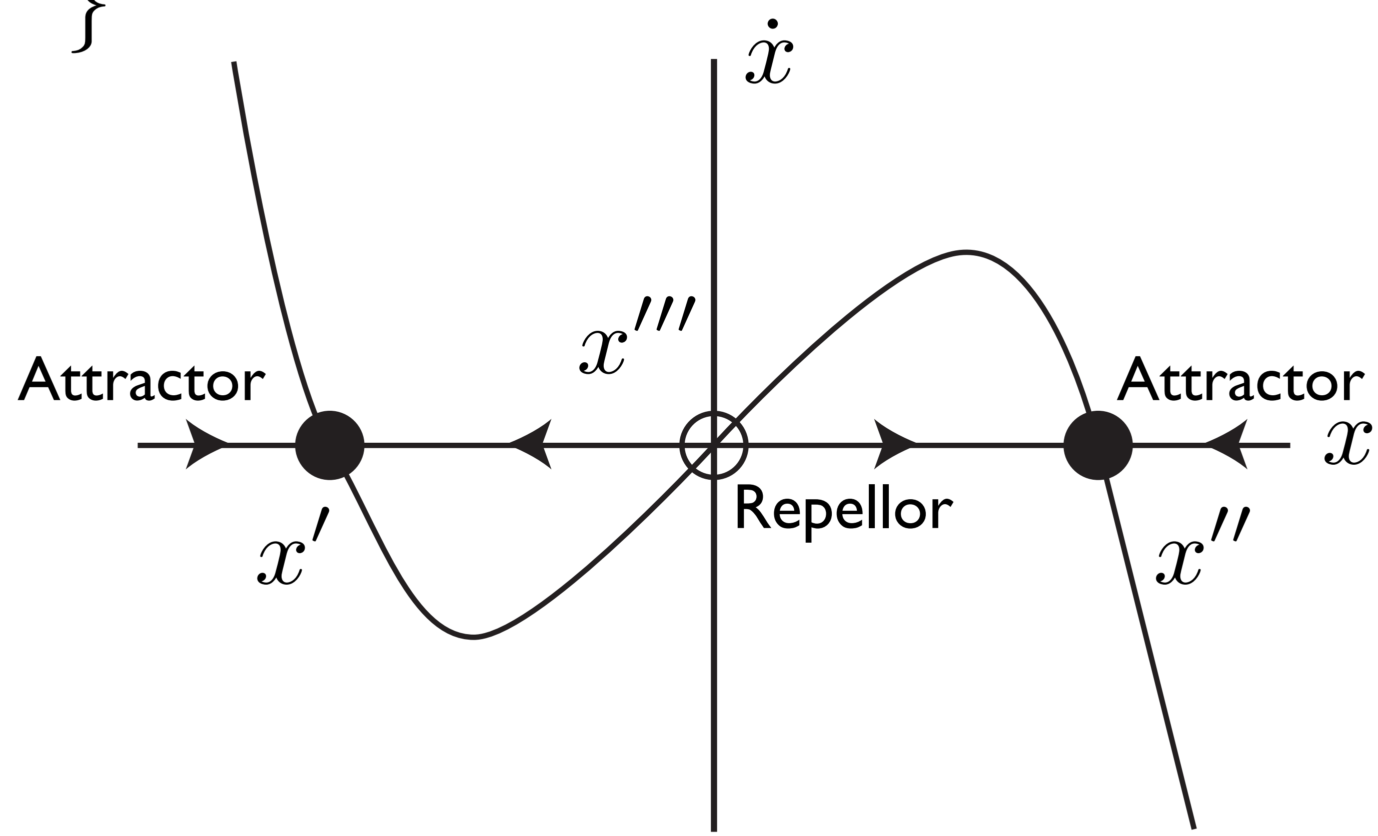
Example: 1D flows ...

Invariant sets: $\Lambda = \{x', x'', x'''\}$

Fixed points: $\dot{x} = 0$

Attractors: $x', x'' = \pm 1$

Repellor: $x''' = 0$

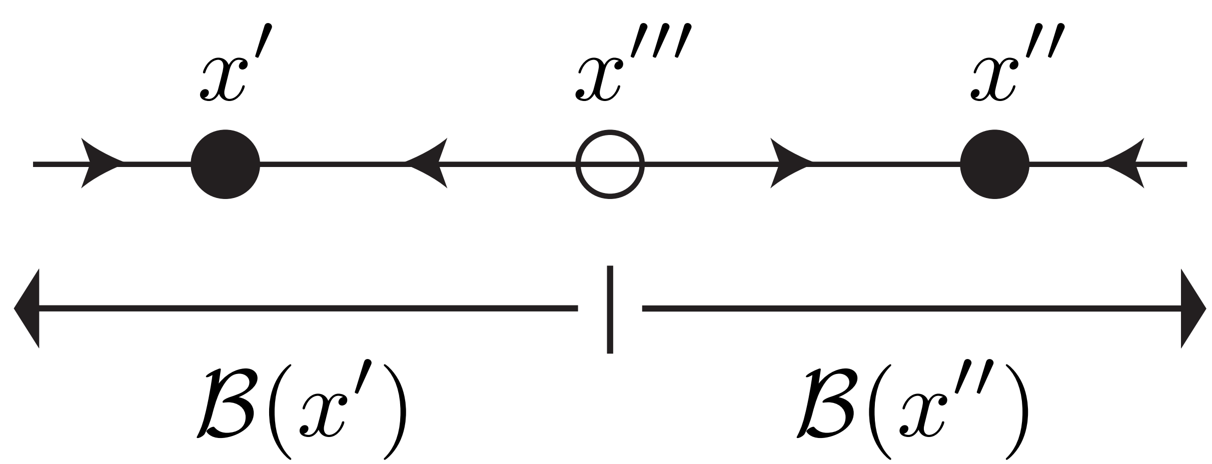


The Big Picture ...

Example: 1D flows ...

Basins: $\mathcal{B}(x') = [-\infty, 0)$
$\mathcal{B}(x'') = (0, \infty]$

Separatrix: $\partial\mathcal{B} = \{x'''\}$



Attractor-Basin Portrait:

$$x', x'', x''', \mathcal{B}(x'), \mathcal{B}(x''), \partial\mathcal{B}$$

The Big Picture ...

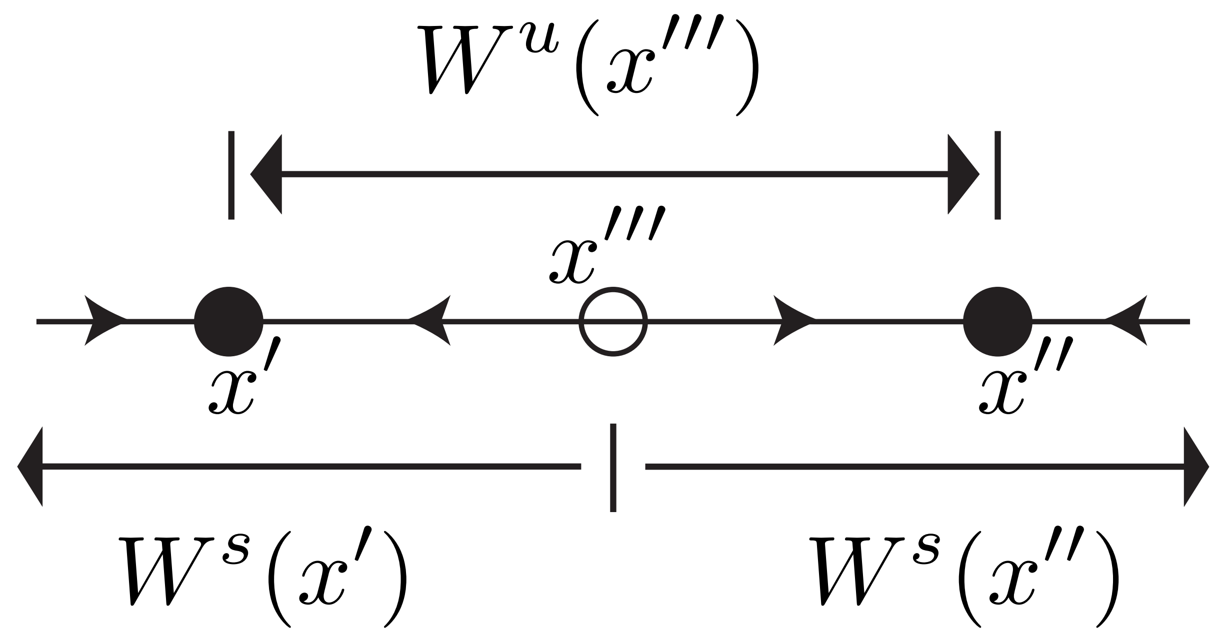
Example: 1D flows ...

Stable manifolds:

$$W^s(x') = [-\infty, x''')$$
$$W^s(x'') = (x''', -\infty]$$

Unstable manifold:

$$W^u(x''') = (x', x'')$$



Example: 1D flows ...

Hey, most of these dynamical systems are solvable!

For example, when the dynamic is polynomial
you can do the integral for the flow for all times

What's the point of all this abstraction?

The Big Picture ...

Reading for next lecture:

NDAC, Sec. 6.0-6.4, 7.0-7.3, & 9.0-9.4