Directional
Computational Mechanics I

Reading for this lecture: CMR articles

TBA
PRATISP
MSP
IACP, IACPLCOCS

Directional Computational Mechanics

Agenda:

- Forward and reverse processes
- Forward and reverse ε-machines
- Causal irreversibility
- Joint process lattice
- Calculating reverse ε-machine from forward

Forward and Reverse Processes

Forward and Reverse Processes

Chain: $\overleftrightarrow{X} = \overleftarrow{X}_t \overrightarrow{X}_t$

Past: $\overleftarrow{X}_t = \ldots X_{t-3} X_{t-2} X_{t-1}$

Future: $\overrightarrow{X}_t = X_t X_{t+1} X_{t+2} \ldots$

Future L-Block: $\overrightarrow{X}_t^L = X_t X_{t+1} \ldots X_{t+L-1}$

Past L-Block: $\overleftarrow{X}_t^L = X_{t-L} \ldots X_{t-2} X_{t-1}$

Process: $\mathcal{P} \sim \Pr(\overleftrightarrow{X})$

$$\Pr(\overleftrightarrow{X}) = \Pr(\ldots X_{-2} X_{-1} X_0 X_1 X_2 \ldots)$$

Forward and Reverse Processes

Previously,

$$\overleftrightarrow{X} = \ldots X_{-2} X_{-1} X_0 X_1 X_2 \ldots$$

Scan direction →

Forward process: Same as before.

$$\mathcal{P}^+ \sim \Pr(\overleftrightarrow{X})$$

Forward entropy rate ~ Prediction:

$$h_\mu^+ = h_\mu(\mathcal{P}^+) = H[X_0 | \overleftarrow{X}_0]$$

Forward and Reverse Processes

Now, reverse process:

$$\overleftrightarrow{X} = \ldots X_{-2} X_{-1} X_0 X_1 X_2 \ldots$$

← Scan direction

Reverse process: Scan measurements in reverse order

$$\mathcal{P}^- \sim \Pr(\overleftrightarrow{Y})$$

where $Y_t \equiv X_{-t}$

Typically, we will still write in terms of X_t.

Reverse entropy rate ~ Retrodiction:

$$h_\mu{}^- = h_\mu(\mathcal{P}^-) = H[Y_0 | \overleftarrow{Y}_0]$$

$$= H[X_{-1} | \overrightarrow{X}_0]$$

(Note index shift)

Forward and Reverse Processes

In which time direction is process most predictable?

Neither!

Theorem: Entropy rate is time symmetric

$$h_\mu^- = h_\mu^+$$

Forward and Reverse Processes

In which time direction is process most predictable?

Proof:
$$h_\mu^+ = H[X_0|\overleftarrow{X}_0]$$
$$= \lim_{L\to\infty} H[X_0|X_{-L},\ldots,X_{-1}]$$
$$= \lim_{L\to\infty} [H[X_{-L},\ldots,X_0] - H[X_{-L},\ldots,X_{-1}]]$$

Stationarity: $H[X_{-L},\ldots,X_{-1}] = H[X_{-L+1},\ldots,X_0]$

$$h_\mu^+ = \lim_{L\to\infty} [H[X_{-L},\ldots,X_0] - H[X_{-L+1},\ldots,X_0]]$$
$$= \lim_{L\to\infty} H[X_{-L}|X_{-L+1},\ldots,X_0]$$
$$= \lim_{L\to\infty} H[X_{-1}|X_0,\ldots,X_{L-1}]$$
$$= H[X_{-1}|\overrightarrow{X}_0]$$
$$= h_\mu^- \qquad (= H[Y_0|\overleftarrow{Y}_0])$$

(Stationarity, again)

Forward and Reverse Processes

Does the communicated information differ?

Theorem: Excess entropy is time symmetric

$$\mathbf{E}(\mathcal{P}^+) = \mathbf{E}(\mathcal{P}^-)$$

Forward and Reverse Processes

Does the communicated information differ?

Proof sketch:

$$
\begin{aligned}
\mathbf{E}\left(\mathcal{P}^+\right) &= I\left[\overleftarrow{X}_0 ; \overrightarrow{X}_0\right] \\
&= \lim_{L \to \infty} I\left[\overleftarrow{X}{}_0^L ; \overrightarrow{X}{}_0^L\right] \\
&= \lim_{L \to \infty} I\left[\overrightarrow{X}{}_0^L ; \overleftarrow{X}{}_0^L\right] \\
&= \lim_{L \to \infty} I\left[\overleftarrow{Y}{}_0^L ; \overrightarrow{Y}{}_0^L\right] \\
&= I\left[\overleftarrow{Y}_0 ; \overrightarrow{Y}_0\right] \\
&= \mathbf{E}\left(\mathcal{P}^-\right)
\end{aligned}
$$

$I[W;Z] = I[Z;W]$

Definition of Y_t

Conclusion:

Neither entropy rate nor excess entropy detect temporal asymmetry

Does the stored information differ?

First, we need the ε-machine of the reverse process: $M(\overleftarrow{\mathcal{P}})$

Forward and Reverse ε-Machines

Forward and Reverse ε-Machines

Previously, $\overleftrightarrow{X} = \ldots X_{-2} X_{-1} X_0 X_1 X_2 \ldots$

Scan direction →

Forward ε-Machine: M^+

Equivalence Relation $\overleftarrow{x} \sim^+ \overleftarrow{x}'$: $\epsilon^+(\overleftarrow{x})$

Forward Causal States: $\boldsymbol{\mathcal{S}}^+ = \mathcal{P}^+ / \sim^+$

Complexity measures:

Forward entropy rate: $h_\mu^+ = H[X_0 | \mathcal{S}_0^+] = h_\mu$

Statistical complexity: $C_\mu^+ = H[\mathcal{S}^+]$

Forward crypticity: $\chi^+ = H[\mathcal{S}_0^+ | \overrightarrow{X}_0]$

Mystery wedge!
$C_\mu - \mathbf{E}$

Forward and Reverse ε-Machines

Now, "reverse" ε-Machines: $\overleftrightarrow{X} = \ldots X_{-2} X_{-1} X_0 X_1 X_2 \ldots$

← Scan direction

Retrodictive equivalence relation: $\overrightarrow{x} \sim^- \overrightarrow{x}'$

$$\epsilon^-(\overrightarrow{x}) = \{\overrightarrow{x}' : \Pr(\overleftarrow{X}|\overrightarrow{x}) = \Pr(\overleftarrow{X}|\overrightarrow{x}')\}$$

Retrodictive causal states: $\boldsymbol{\mathcal{S}}^- = \mathcal{P}^- / \sim^-$

Reverse ε-Machine: $M^- = \{\boldsymbol{\mathcal{S}}^-, T^{(s)} \in \mathcal{A}\}$

Forward and Reverse ε-Machines

Reverse ε-Machines ...

$$\overleftrightarrow{X} = \ldots X_{-2} X_{-1} X_0 X_1 X_2 \ldots$$

← Scan direction

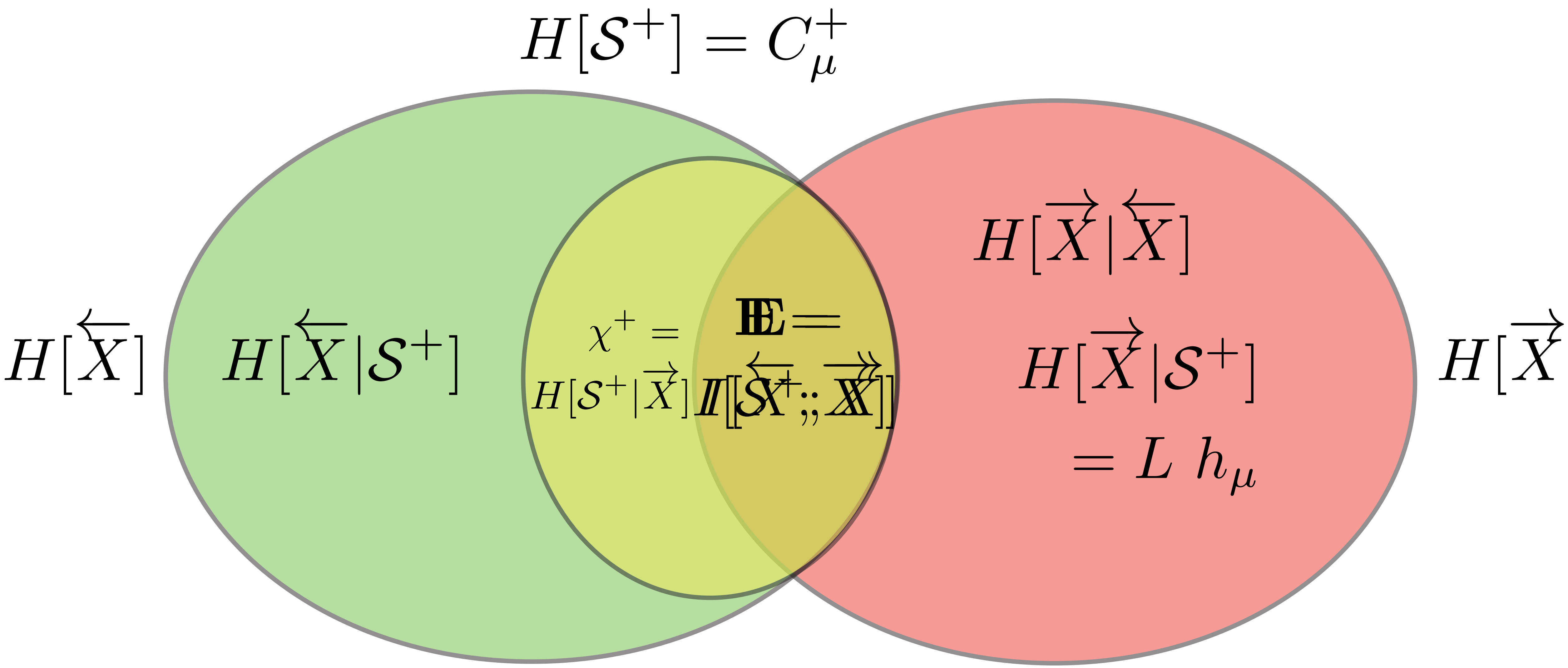
Retrodictive entropy rate: $h_\mu^- = H[X_{-1} | \mathcal{S}_0^-] = h_\mu$

Reverse statistical complexity: $C_\mu^- = H[\mathcal{S}^-]$

Reverse crypticity: $\chi^- = H[\mathcal{S}_0^- | \overleftarrow{X}_0]$

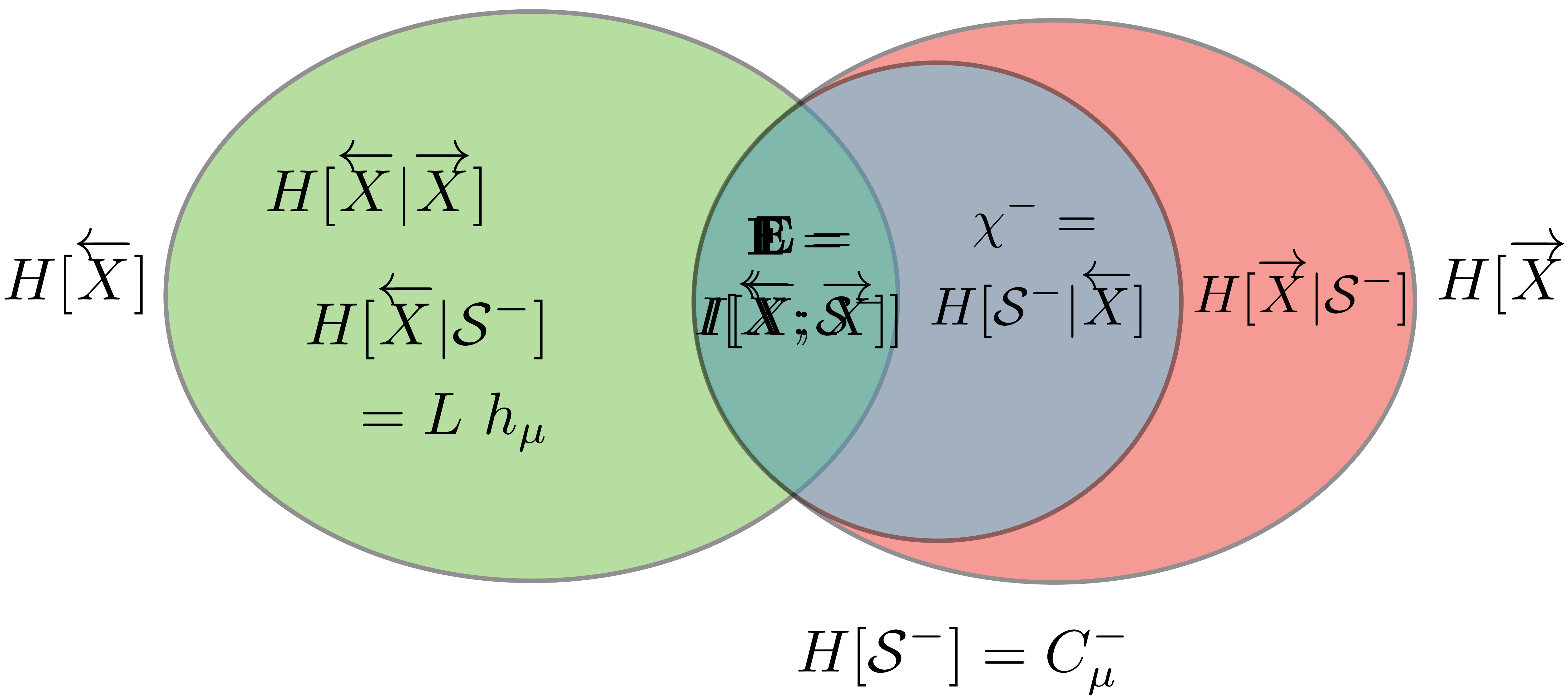
Forward and Reverse ε-Machines

Forward process ε-machine I-diagram:



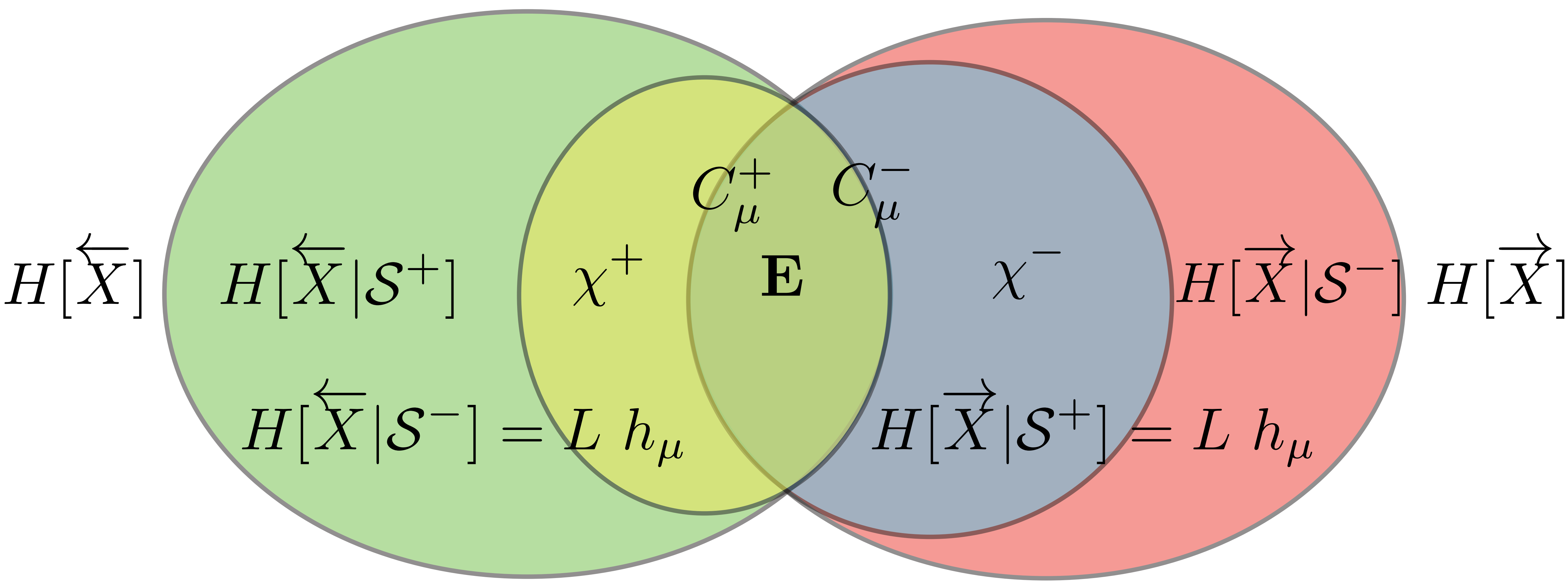
Forward and Reverse ε-Machines

Reverse process ε-machine I-diagram:



Forward and Reverse ε-Machines

Forward and reverse process ε-machine I-diagram:



Forward and Reverse ε-Machines

Two remaining mystery wedges!

"Error" associated with retrodiction using forward ε-machines:

$$H[\overleftarrow{X} | \mathcal{S}^{+}]$$

"Error" associated with prediction using reverse ε-machines:

$$H[\overrightarrow{X} | \mathcal{S}^{-}]$$

Forward and Reverse ε-Machines

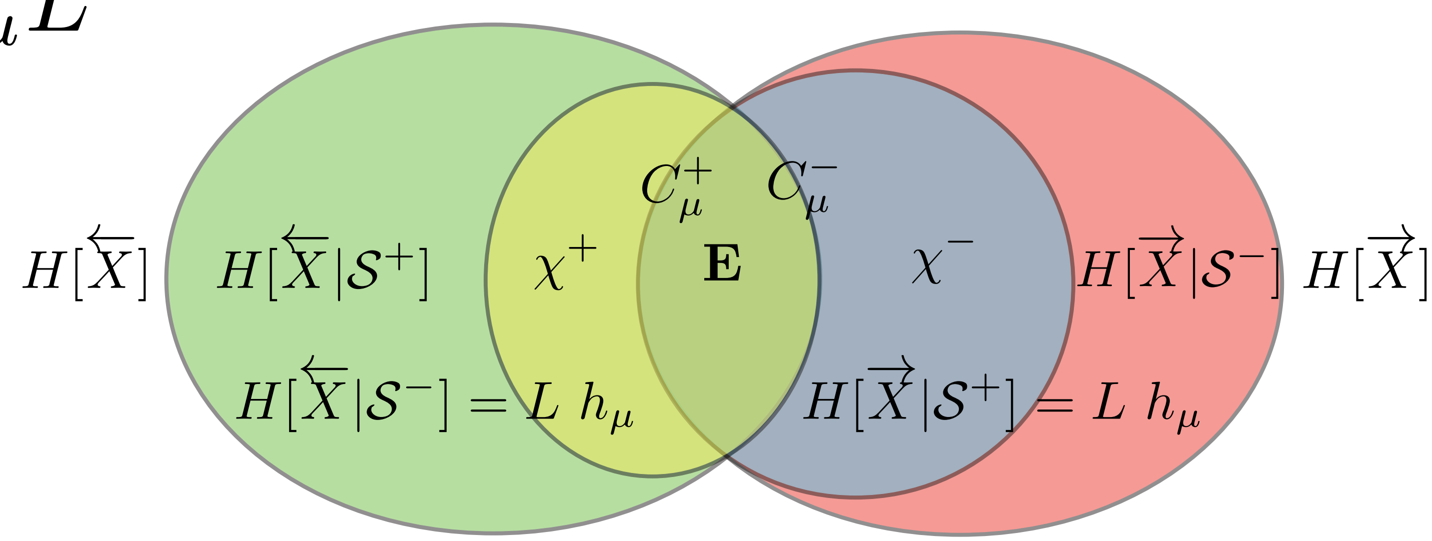
Mysterious, perhaps, but well characterized.

Corollary:

$$H[\overleftarrow{X}^L|\mathcal{S}^+] = h_\mu L - \chi^+$$

Proof sketch:

$$\begin{aligned}\chi^+ + H[\overleftarrow{X}^L|\mathcal{S}^+] &= H[\overleftarrow{X}^L|\overrightarrow{X}] \\ &= H[\overleftarrow{X}^L|\mathcal{S}^-] \\ &= h_\mu L\end{aligned}$$



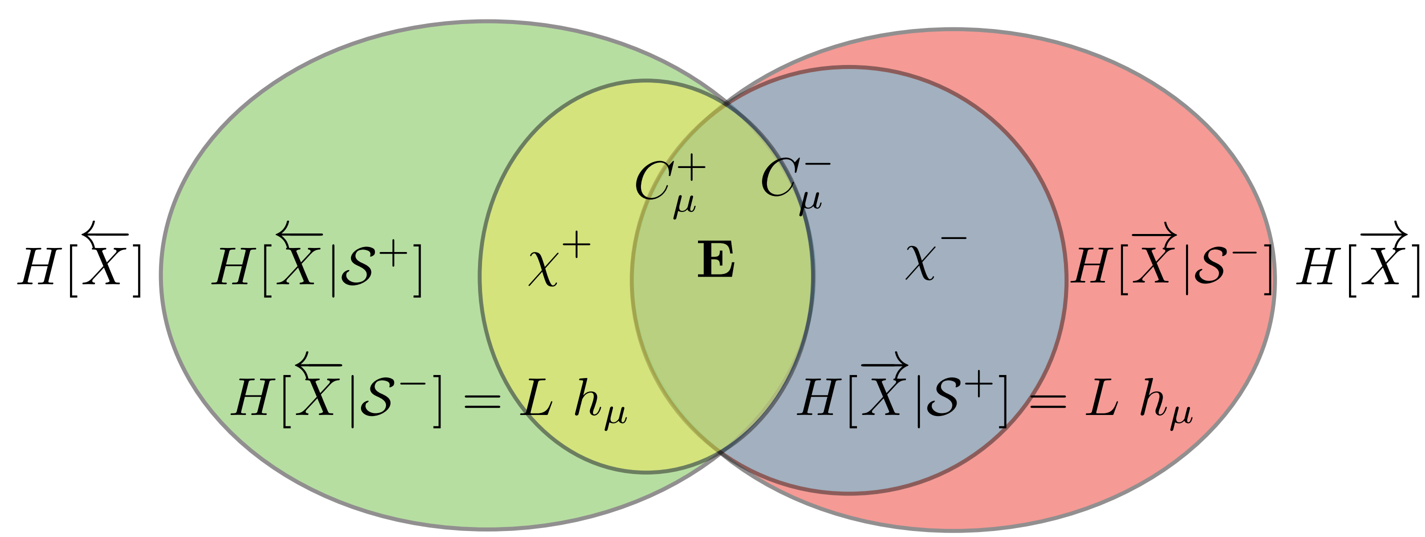
Forward and Reverse ε-Machines

Corollary:

$$H[\overrightarrow{X}^L|\mathcal{S}^-] = h_\mu L - \chi^-$$

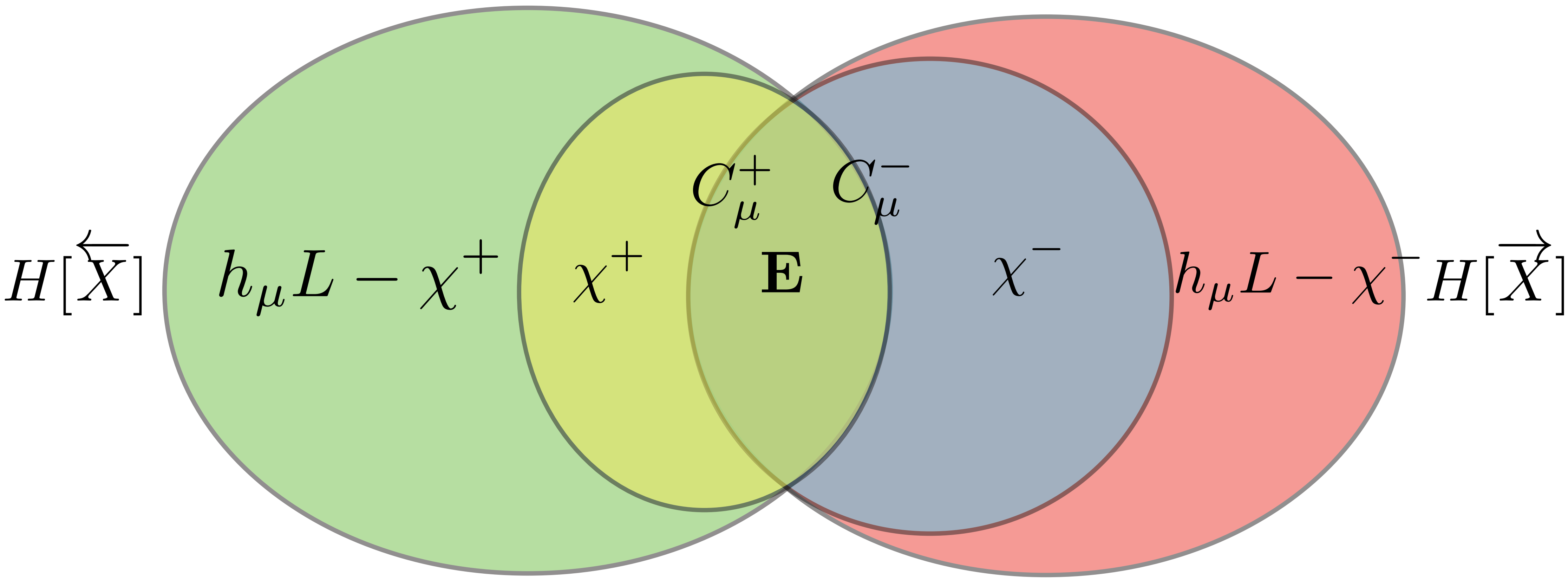
Proof sketch:

$$\begin{aligned}\chi^- + H[\overrightarrow{X}^L|\mathcal{S}^-] &= H[\overrightarrow{X}^L|\overleftarrow{X}] \\ &= H[\overrightarrow{X}^L|\mathcal{S}^+] \\ &= h_\mu L\end{aligned}$$



Forward and Reverse ε-Machines

Result: Characterized the entire I-diagram



Forward and Reverse Joint Processes

Joint Process Lattice:

A graphi-notational interlude

Forward and Reverse Joint Processes

Joint Processes:

Various:

- State-symbol:

$$\Pr(\overleftrightarrow{\mathcal{S}}, \overleftrightarrow{X}) = (\ldots, (\sigma_t, x_t), \ldots)$$

- Forward state-symbol:

$$\Pr(\overleftrightarrow{\mathcal{S}^+}, \overleftrightarrow{X}) = \left(\ldots, (\sigma_t^+, x_t), \ldots\right)$$

- Reverse state-symbol:

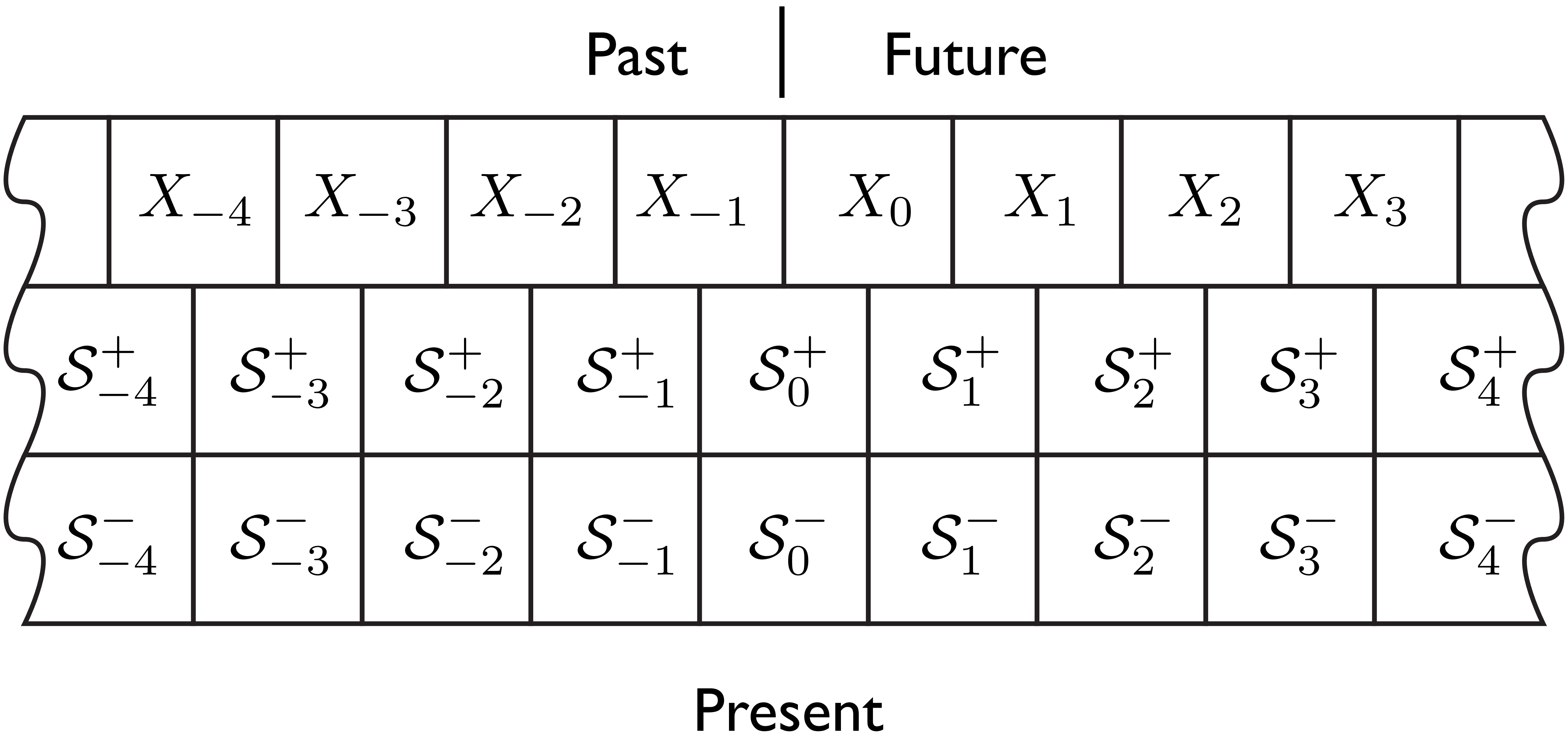
$$\Pr(\overleftrightarrow{\mathcal{S}^-}, \overleftrightarrow{X}) = \left(\ldots, (\sigma_t^-, x_t), \ldots\right)$$

- Forward state, reverse state, symbol:

$$\Pr(\overleftrightarrow{\mathcal{S}^-}, \overleftrightarrow{\mathcal{S}^+}, \overleftrightarrow{X}) = \left(\ldots, (\sigma_t^-, \sigma_t^+, x_t), \ldots\right)$$

Forward and Reverse Joint Processes

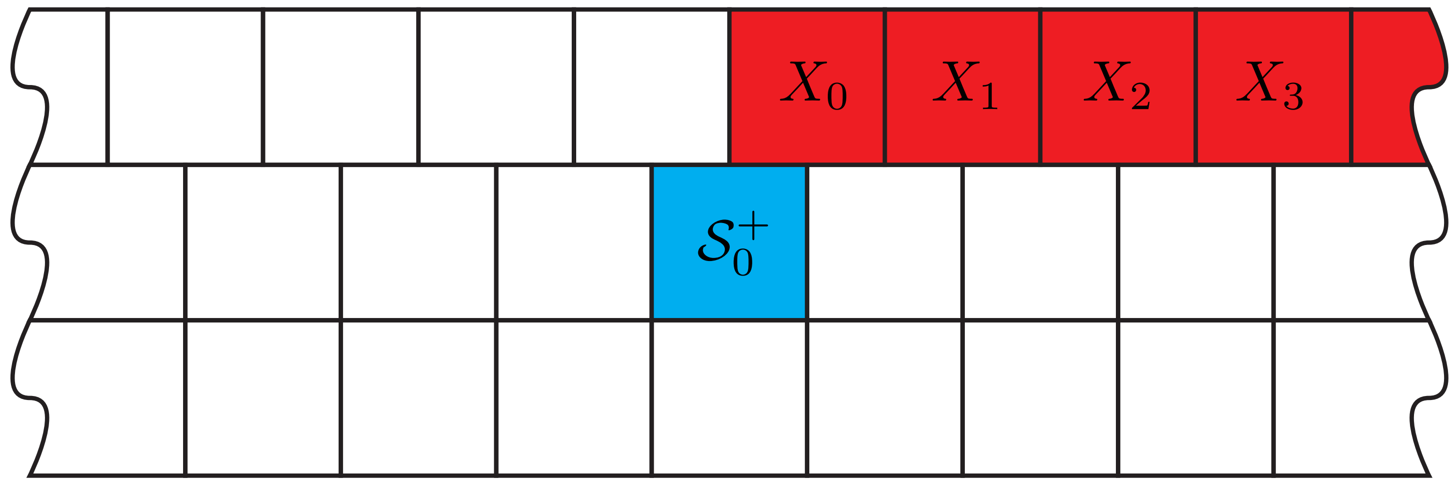
Joint Process Lattice: Guide book



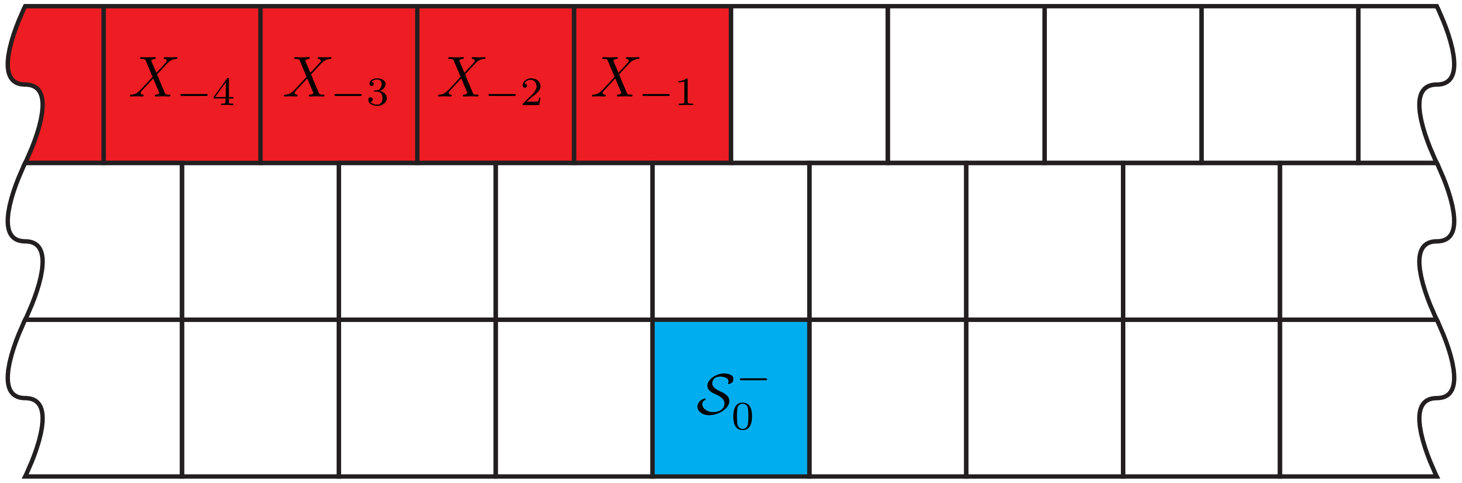
Forward and Reverse Joint Processes

Joint Process Lattice ...

Prediction using forward ε-machines: $H[\overrightarrow{X} | \mathcal{S}^+]$



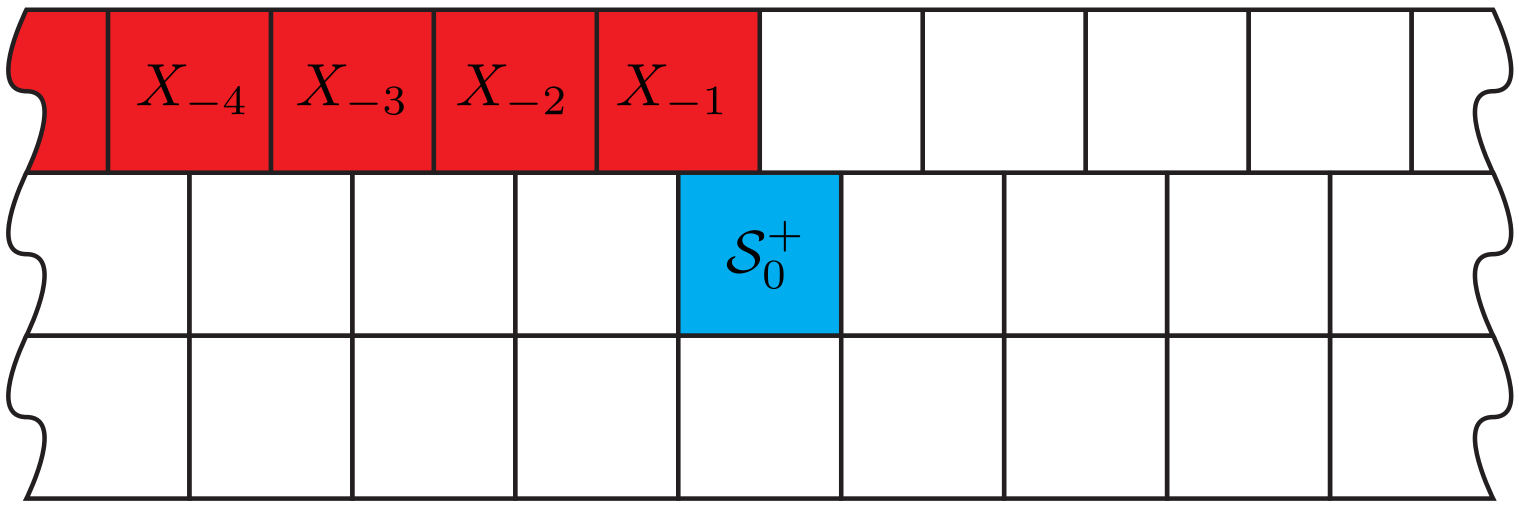
Retrodiction using reverse ε-machines: $H[\overleftarrow{X} | \mathcal{S}^-]$



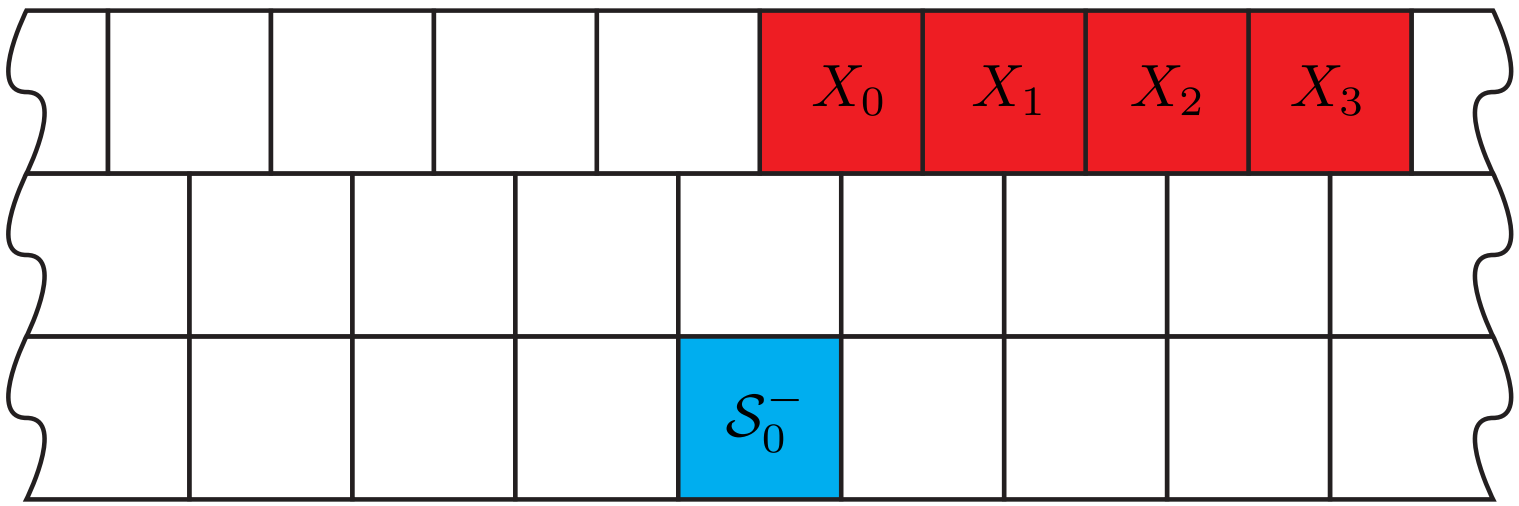
Forward and Reverse Joint Processes

Joint Process Lattice ...

Retrodiction using forward ε-machines: $H[\overleftarrow{X}^L | \mathcal{S}^+]$



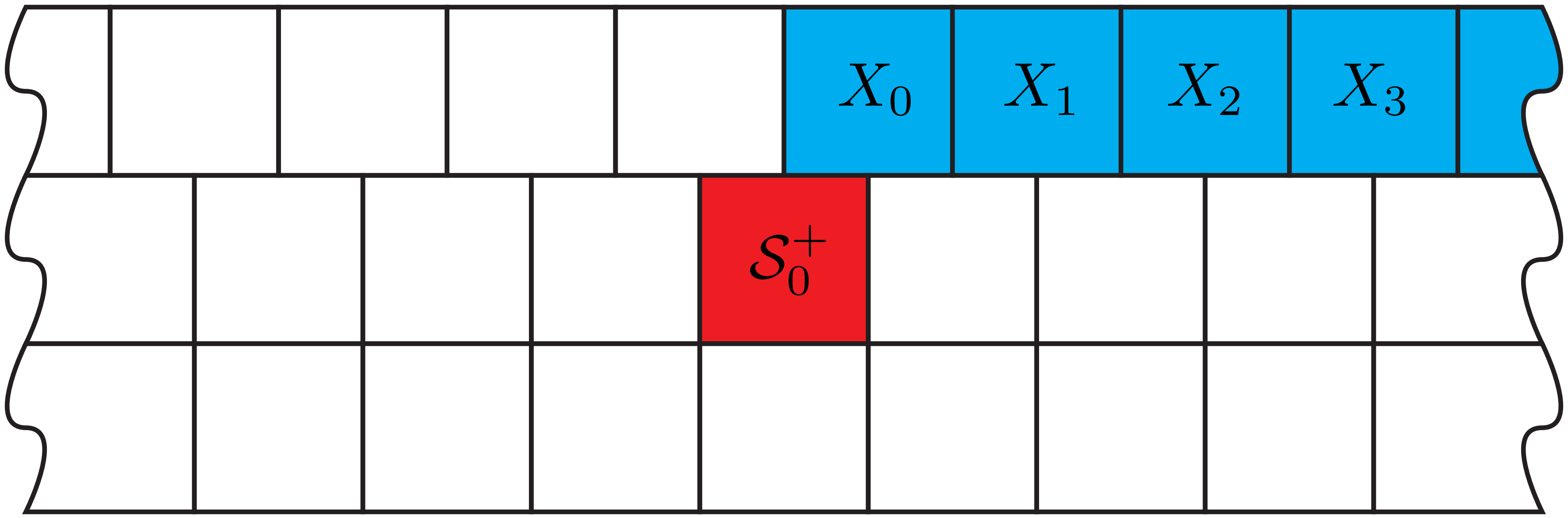
Prediction using reverse ε-machines: $H[\overrightarrow{X}^L | \mathcal{S}^-]$



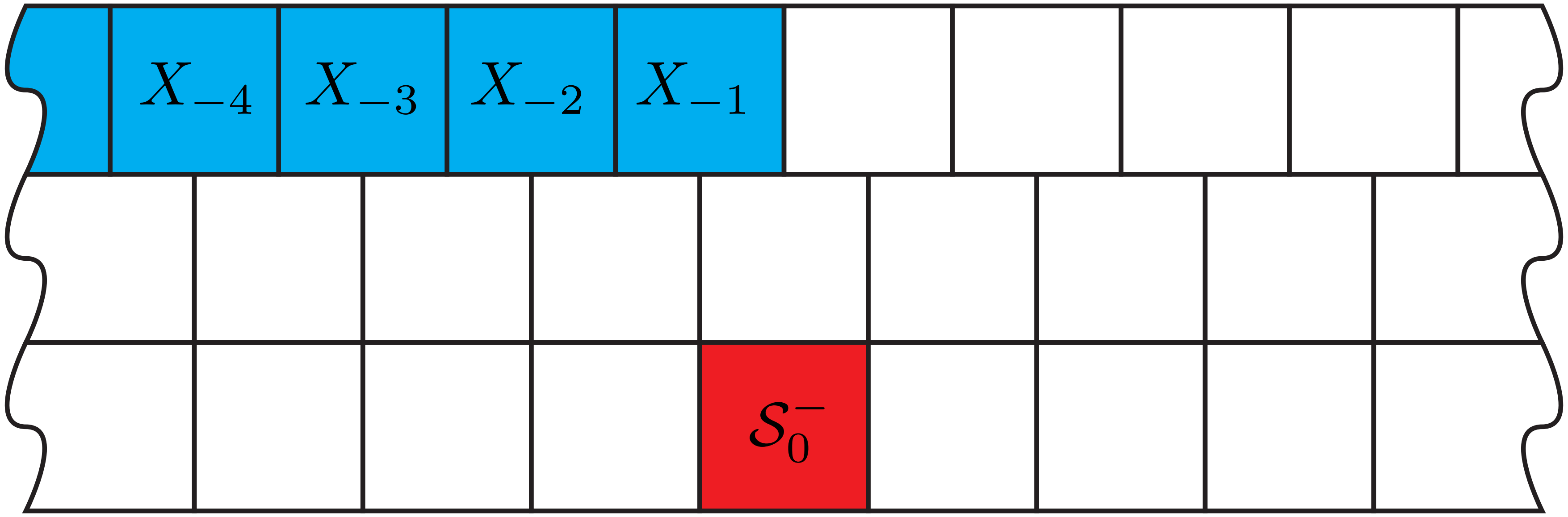
Forward and Reverse Joint Processes

Joint Process Lattice ...

Forward crypticity: $\chi^+ = H[\mathcal{S}^+ | \overrightarrow{X}]$



Reverse crypticity: $\chi^- = H[\mathcal{S}^- | \overleftarrow{X}]$



Causal Irreversibility

Causal Irreversibility

Time symmetric properties:

Forward and reverse entropy rates:

$$h_\mu = h_\mu^+ = h_\mu^-$$

Excess entropy of forward and reverse processes.

$$\mathbf{E} = \mathbf{E}(\mathcal{P}^+) = \mathbf{E}(\mathcal{P}^-)$$

Causal Irreversibility

Can the stored information differ?

Yes!

Theorem: ε-Machines need not be time symmetric

$$\overleftarrow{M} \neq \overrightarrow{M}$$

Corollary: Statistical complexity need not be time symmetric

$$C_\mu^- \neq C_\mu^+$$

Causal Irreversibility

Can the stored information differ?

Proof: By example ...

Misiurewicz parameter in the Logistic map:

First root, $r < 4$, where critical point is periodic and

$$f^4(\tfrac{1}{2}) = f^5(\tfrac{1}{2})$$

Find

$$r \approx 3.9277370017867516$$

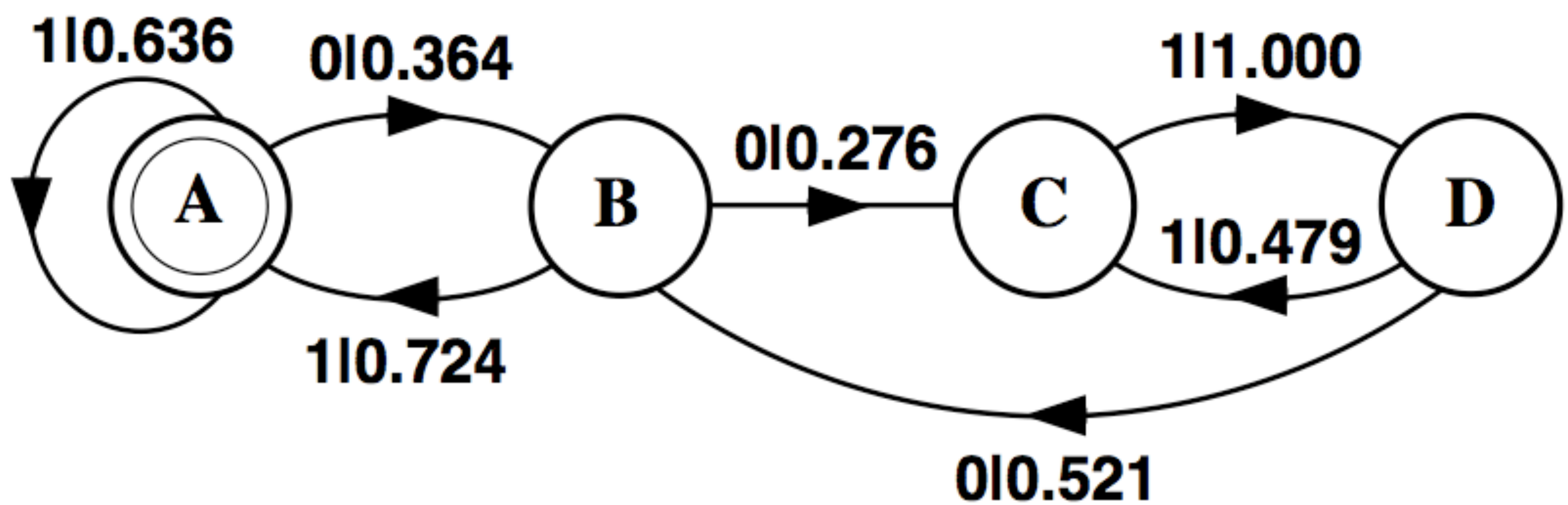
Use binary generating partition.

Causal Irreversibility

Can the stored information differ?

Proof: By example ... Misiurewicz parameter for Logistic map ...

Forward ε-machine:



$$h_\mu(\overrightarrow{\mathcal{P}}) \approx 0.81 \text{ bits/symbol}$$

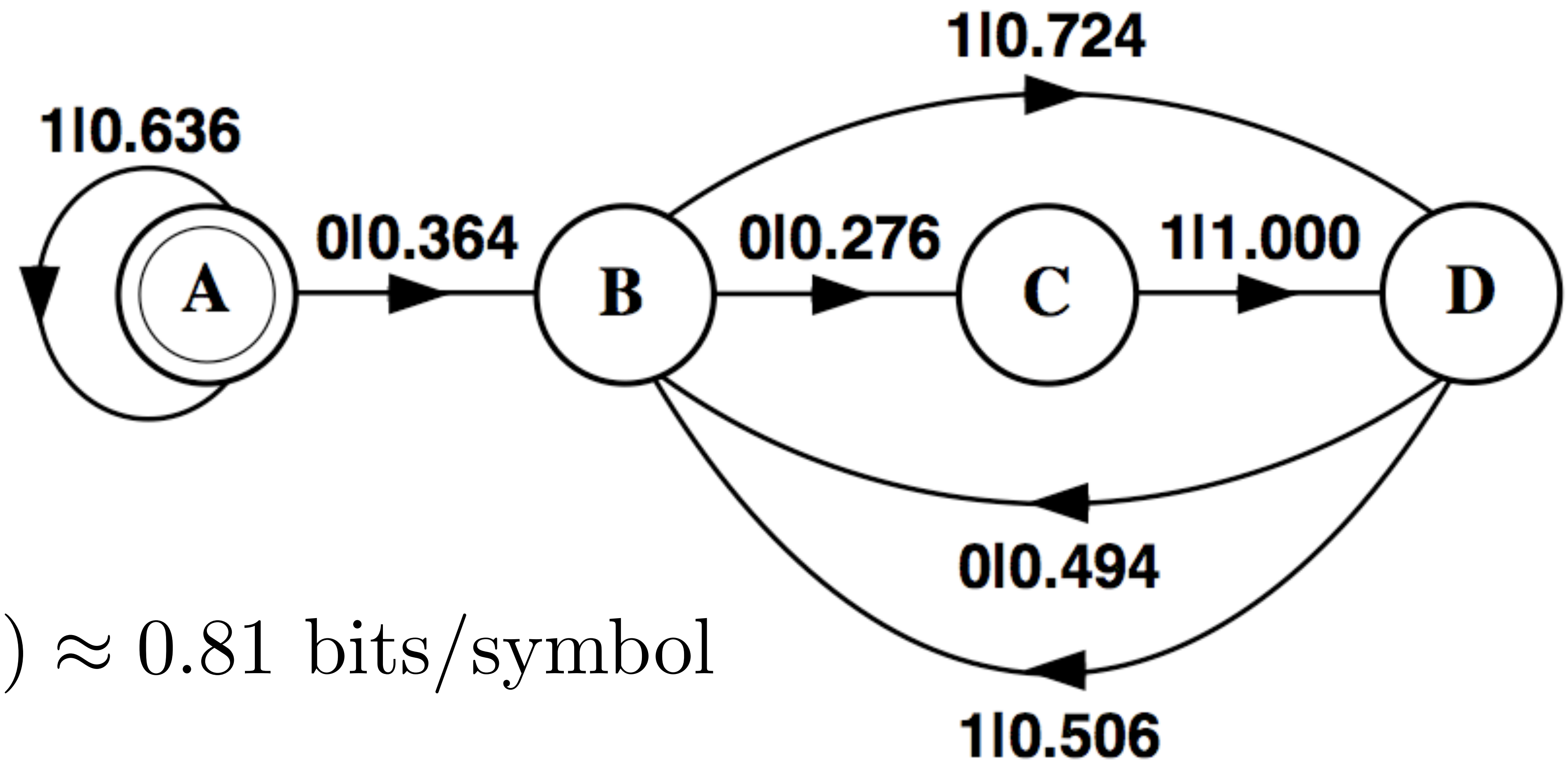
$$C_\mu(\overrightarrow{\mathcal{P}}) \approx 1.77 \text{ bits}$$

Causal Irreversibility

Can the stored information differ?

Proof: By example ... Misiurewicz parameter for Logistic map ...

Reverse ε-machine:



$$h_\mu(\overleftarrow{\mathcal{P}}) \approx 0.81 \text{ bits/symbol}$$

$$C_\mu(\overleftarrow{\mathcal{P}}) \approx 1.41 \text{ bits}$$

Causal Irreversibility

Temporal asymmetry in process structure.

Even if in (statistical) equilibrium!

Interpretation:

Stored information required to do optimal prediction can differ, depending on scan (time) direction.

Causal Irreversibility

Alternative notions of reversibility:

Most familiar: Microscopically reversible processes

Physics invariant under $t \Leftrightarrow -t$.

Here, what would this mean?

Given measurement sequence:

$$w = x_0 x_1 \ldots x_n$$

Reverse time order of measurements:

$$\begin{aligned} \tilde{w} &= y_0 y_1 \ldots y_n \ , \ y_i = x_{n-i} \\ &= x_n x_{n-1} \ldots x_0 \end{aligned}$$

Causal Irreversibility

Alternative notions of reversibility ...

Microscopically reversible process:

$$\mathcal{P}^+ = \mathcal{P}^-$$

Given $w = \mathrm{supp}\ \Pr(w)$

$$\Pr(w) = \Pr(\tilde{w})$$

$\mathcal{P}^+$'s word distribution

Causal Irreversibility

Alternative notions of reversibility ...

Causal reversibility:

$$C_\mu^+ = C_\mu^-$$

Microscopic reversibility ⇒ Causal reversibility

$$\mathcal{P}^+ = \mathcal{P}^- \quad \Leftrightarrow \quad M^+ = M^-$$

$$\Rightarrow \quad C_\mu^+ = C_\mu^-$$

Causal Irreversibility

Causal Irreversibility:

$$\begin{aligned} \Xi &= C_\mu(\mathcal{P}^+) - C_\mu(\mathcal{P}^-) \\ &= C_\mu^+ - C_\mu^- \end{aligned}$$

A measure of a process's time asymmetry

Misiurewicz process causal irreversibility:

$$\Xi(\mathcal{P}) \approx 0.36 \text{ bits}$$

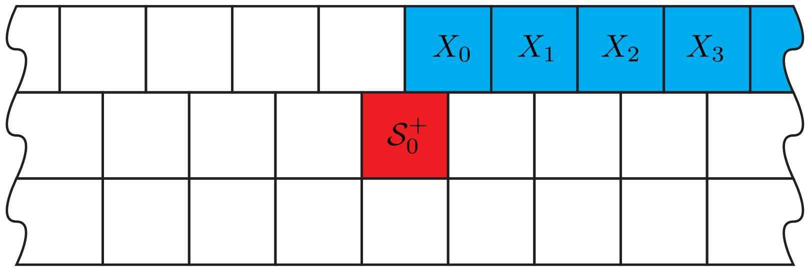
Causal irreversibility ⇒ Microscopic irreversibility

$$\begin{aligned} \Xi \neq 0 \quad &\Rightarrow \quad C_\mu^+ \neq C_\mu^- \\ &\Rightarrow \quad M^+ \neq M^- \\ &\Rightarrow \quad \mathcal{P}^+ \neq \mathcal{P}^- \end{aligned}$$

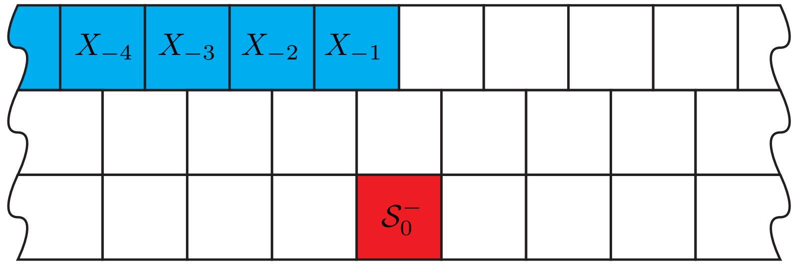
Causal irreversibility less restrictive: Structural similarity

Causal Irreversibility

Crypticity and reversibility:



Forward: $\chi^+ = H[\mathcal{S}^+ | \overrightarrow{X}]$



Reverse: $\chi^- = H[\mathcal{S}^- | \overleftarrow{X}]$

Proposition: $\Xi \neq 0 \Rightarrow \chi^+ \neq \chi^-$

Proof sketch:
$$\chi^+ = C_\mu^+ - \mathbf{E}$$
$$\chi^- = C_\mu^- - \mathbf{E}$$

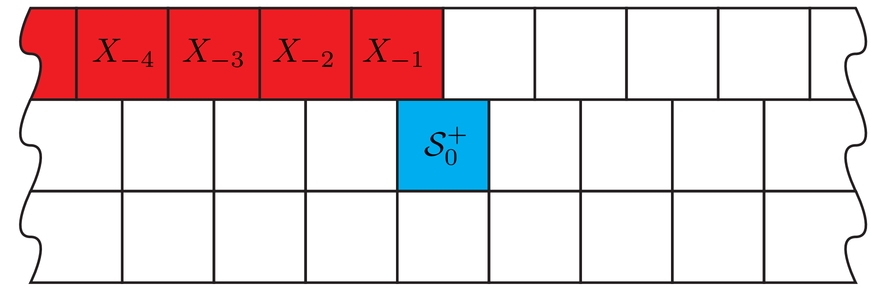
$$C_\mu^- \neq C_\mu^+$$

$$\Rightarrow \chi^+ \neq \chi^-$$

Causal Irreversibility

More corollaries:

Retrodiction with forward states



$$\begin{aligned} H[\overleftarrow{X}^L|\mathcal{S}^+] &= h_\mu L - \chi^+ \\ &= h_\mu L - C_\mu^+ + \mathbf{E} \\ &= \mathbf{E} + h_\mu L - C_\mu^+ \end{aligned}$$

$H[\overleftarrow{X}^L|\mathcal{S}^+] > 0$ RHS positive, only when: $L \geq \left\lceil \frac{\chi^+}{h_\mu} \right\rceil$

~ cryptic order

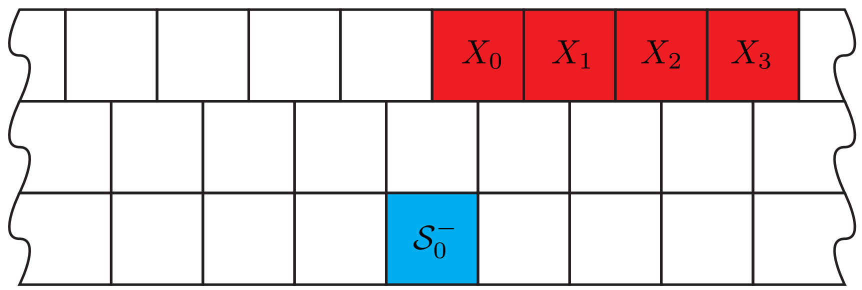
In addition,

$$\begin{aligned} H[\overleftarrow{X}^L|\mathcal{S}^+] + C_\mu^+ &= \mathbf{E} + h_\mu L \\ H[\overleftarrow{X}^L, \mathcal{S}^+] &= \mathbf{E} + h_\mu L \end{aligned}$$

Block-State Entropy

Causal Irreversibility

Similarly, prediction with reverse states



$$
\begin{aligned}
H[\overrightarrow{X}^L|\mathcal{S}^-] &= h_\mu L - \chi^- \\
&= h_\mu L - C_\mu^- + \mathbf{E} \\
&= \mathbf{E} + h_\mu L - C_\mu^-
\end{aligned}
$$

$H[\overrightarrow{X}^L|\mathcal{S}^-] > 0$ RHS positive, only when: $L \geq \left\lceil \frac{\chi^-}{h_\mu} \right\rceil$

~ cryptic order

And also,

$$
\begin{aligned}
H[\overrightarrow{X}^L|\mathcal{S}^-] + C_\mu^- &= \mathbf{E} + h_\mu L \\
H[\overrightarrow{X}^L, \mathcal{S}^-] &= \mathbf{E} + h_\mu L
\end{aligned}
$$

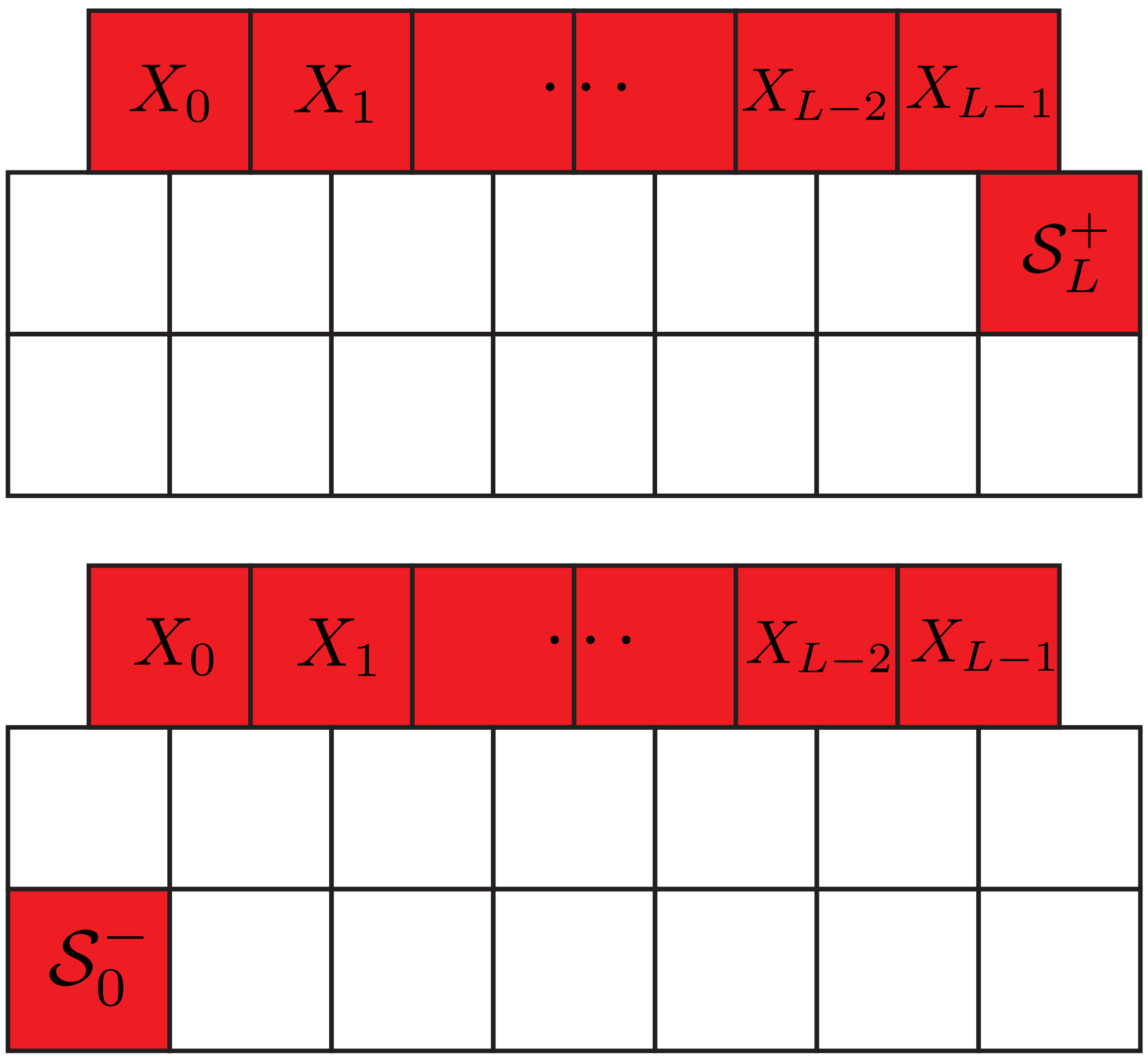
Block-State Entropy

Causal Irreversibility

So, in addition:

$$H[\overrightarrow{X}^L, \mathcal{S}^-] = H[\overleftarrow{X}^L, \mathcal{S}^+]$$

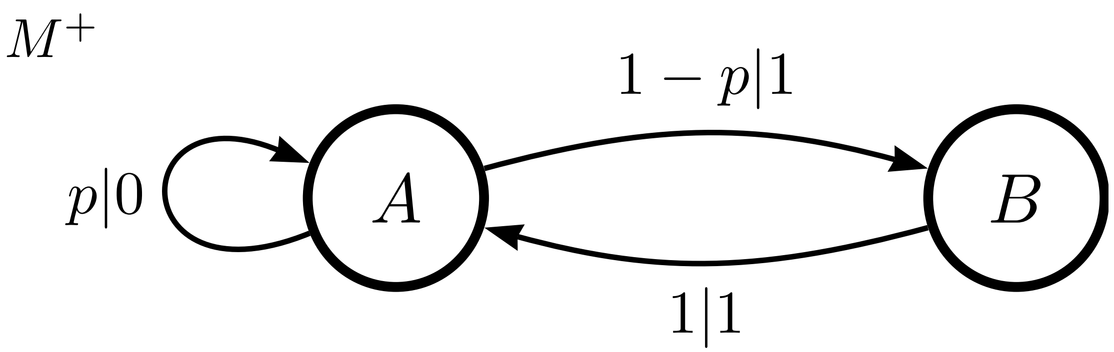
$$\overrightarrow{X}_0^L = \overleftarrow{X}_{L-1}^L$$



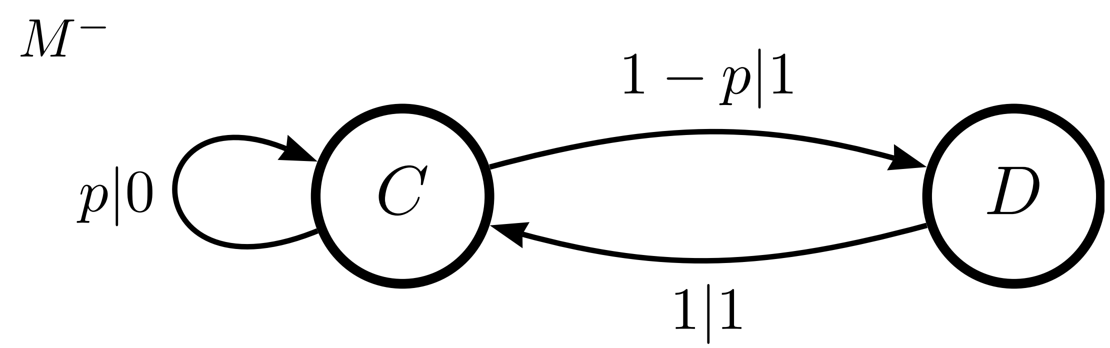
Causal Irreversibility

Example: Even Process

Forward ε-machine



Reverse ε-machine



Noncryptic: $\chi^+ = 0$

Noncryptic: $\chi^- = 0$

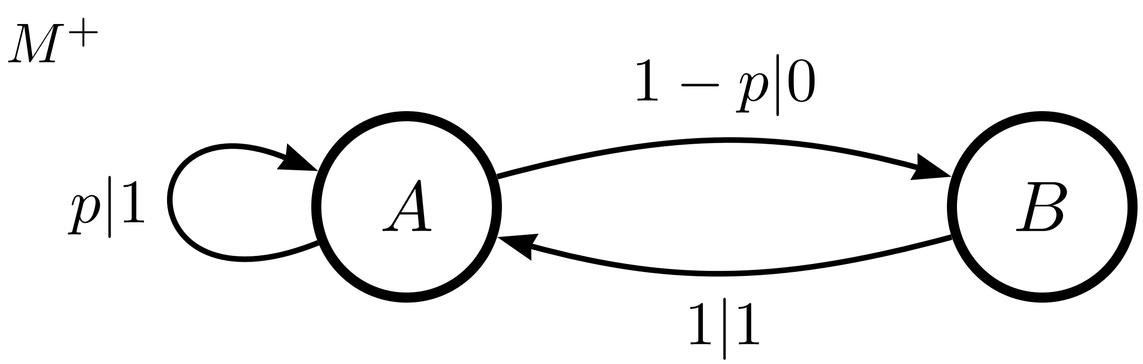
Microscopic reversible: $M^+ = M^- \Rightarrow \mathcal{P}^+ = \mathcal{P}^-$

Causally reversible: $\Xi = 0$

Causal Irreversibility

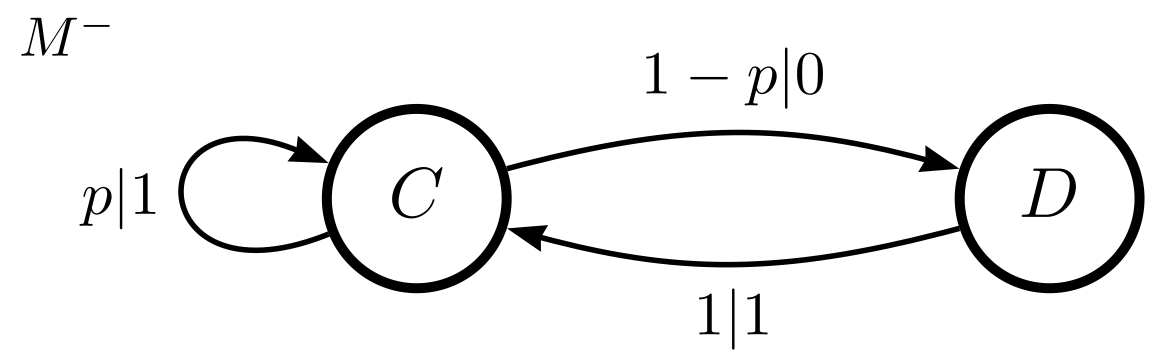
Example: Golden Mean Process

Forward ε-machine



Cryptic: $\chi^+ > 0$

Reverse ε-machine



Cryptic: $\chi^- > 0$

Microscopic reversible: $M^+ = M^- \Rightarrow \mathcal{P}^+ = \mathcal{P}^-$

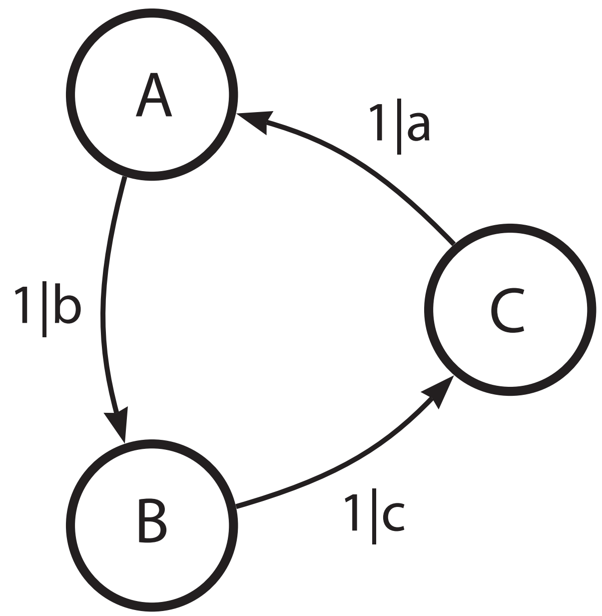
Causally reversible: $\Xi = 0$

So, crypticity not same as causal irreversibility.

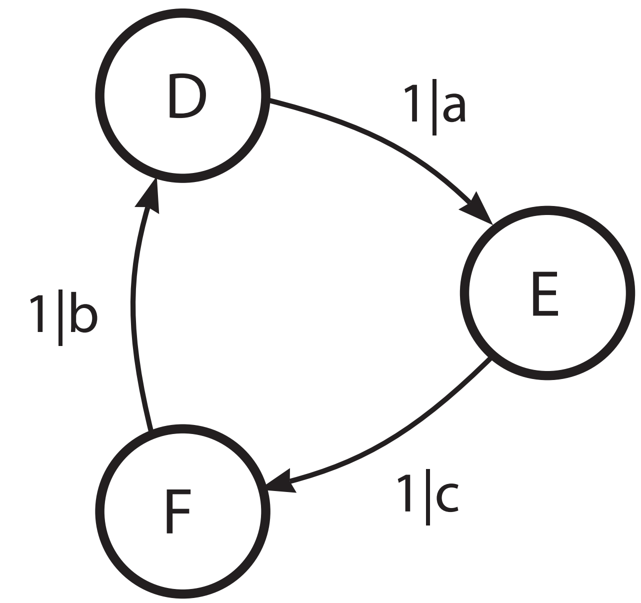
Causal Irreversibility

Example: a-b-c Process

Forward ε-machine



Reverse ε-machine



Noncryptic: $\chi^+ = 0$

Noncryptic: $\chi^- = 0$

Microscopically irreversible:

$$\begin{aligned} \Pr(ab|\mathcal{P}^+) &= 1/3 \\ \Pr(ab|\mathcal{P}^-) &= 0 \end{aligned}$$

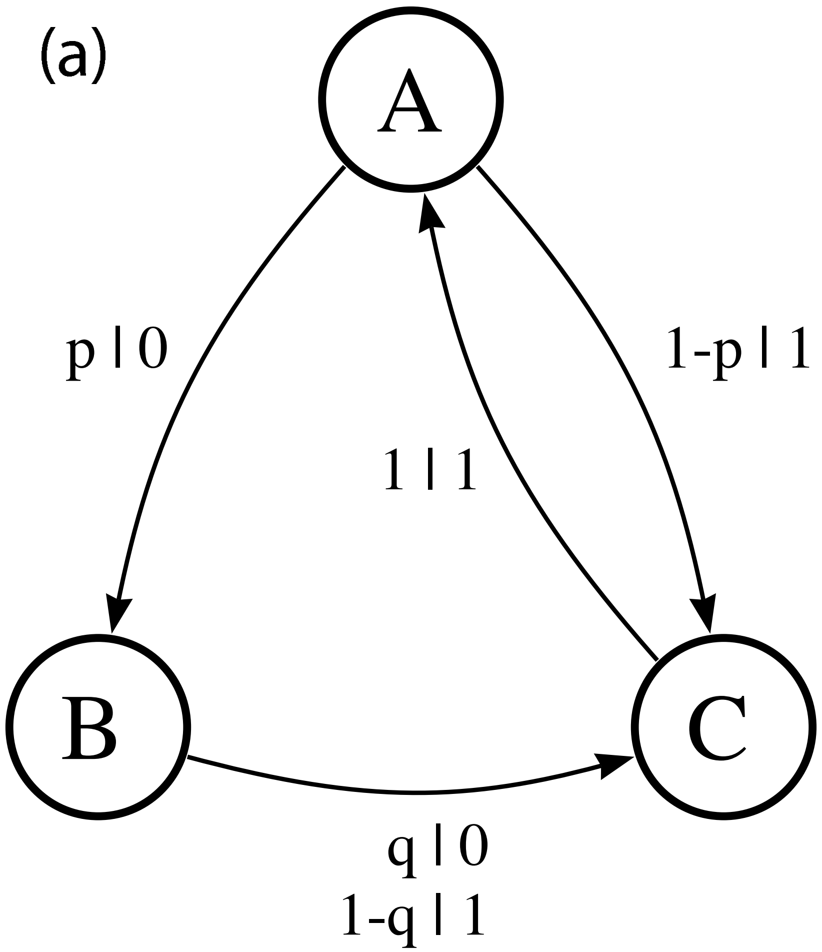
Causally reversible:

$$\Xi = 0$$

Causal Irreversibility

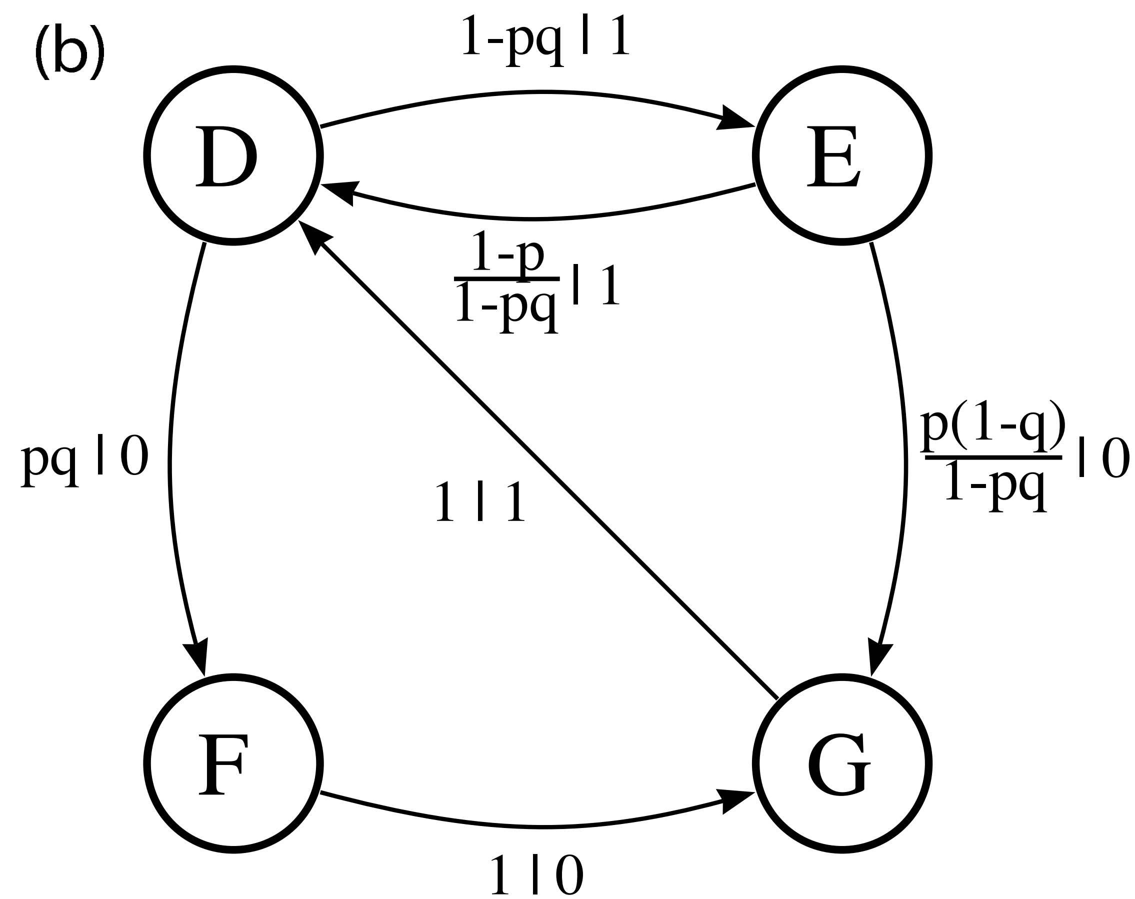
Example: Random Insertion Process

Forward ε-machine



Cryptic: $\chi^{+} > 0$

Reverse ε-machine



Cryptic: $\chi^{-} > 0$

Example: Random Insertion Process ...

Forward ε-machine

$$\Pr(\mathcal{S}^+) = \begin{pmatrix} \overset{A}{\frac{1}{p+2}} & \overset{B}{\frac{p}{p+2}} & \overset{C}{\frac{1}{p+2}} \end{pmatrix}$$

Reverse ε-machine

$$\Pr(\mathcal{S}^-) = \begin{pmatrix} \overset{D}{\frac{1}{p+2}} & \overset{E}{\frac{1-pq}{p+2}} & \overset{F}{\frac{pq}{p+2}} & \overset{G}{\frac{p}{p+2}} \end{pmatrix}$$

Example: Random Insertion Process ...

Set $p = q = 1/2$

$$h_\mu = \frac{3}{5} \text{ bits/symbol}$$

$$C_\mu^+ \approx 1.5219 \text{ bits}$$

$$C_\mu^- \approx 1.8464 \text{ bits}$$

Causally irreversible:

$$\Xi \approx -0.3245 \text{ bits}$$

Microscopically irreversible:

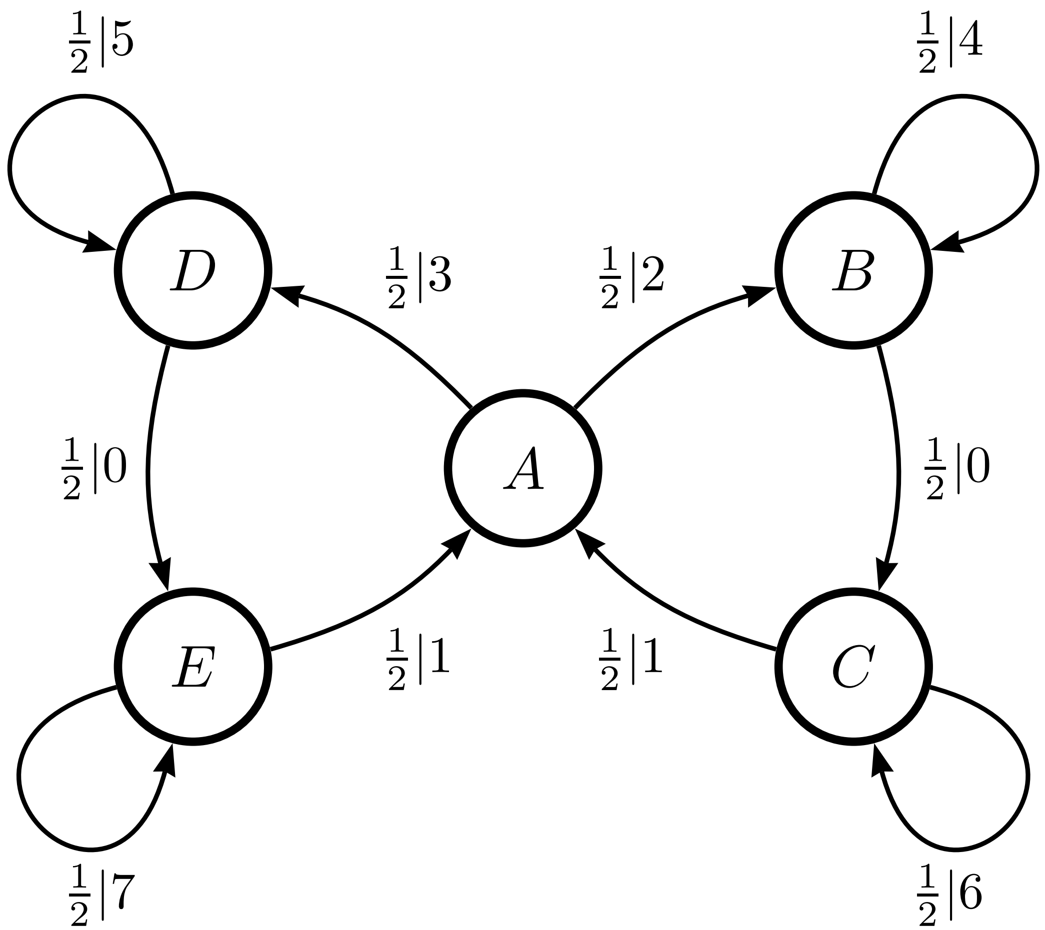
$$M^+ \neq M^- \Rightarrow \mathcal{P}^+ \neq \mathcal{P}^-$$

Causal Irreversibility

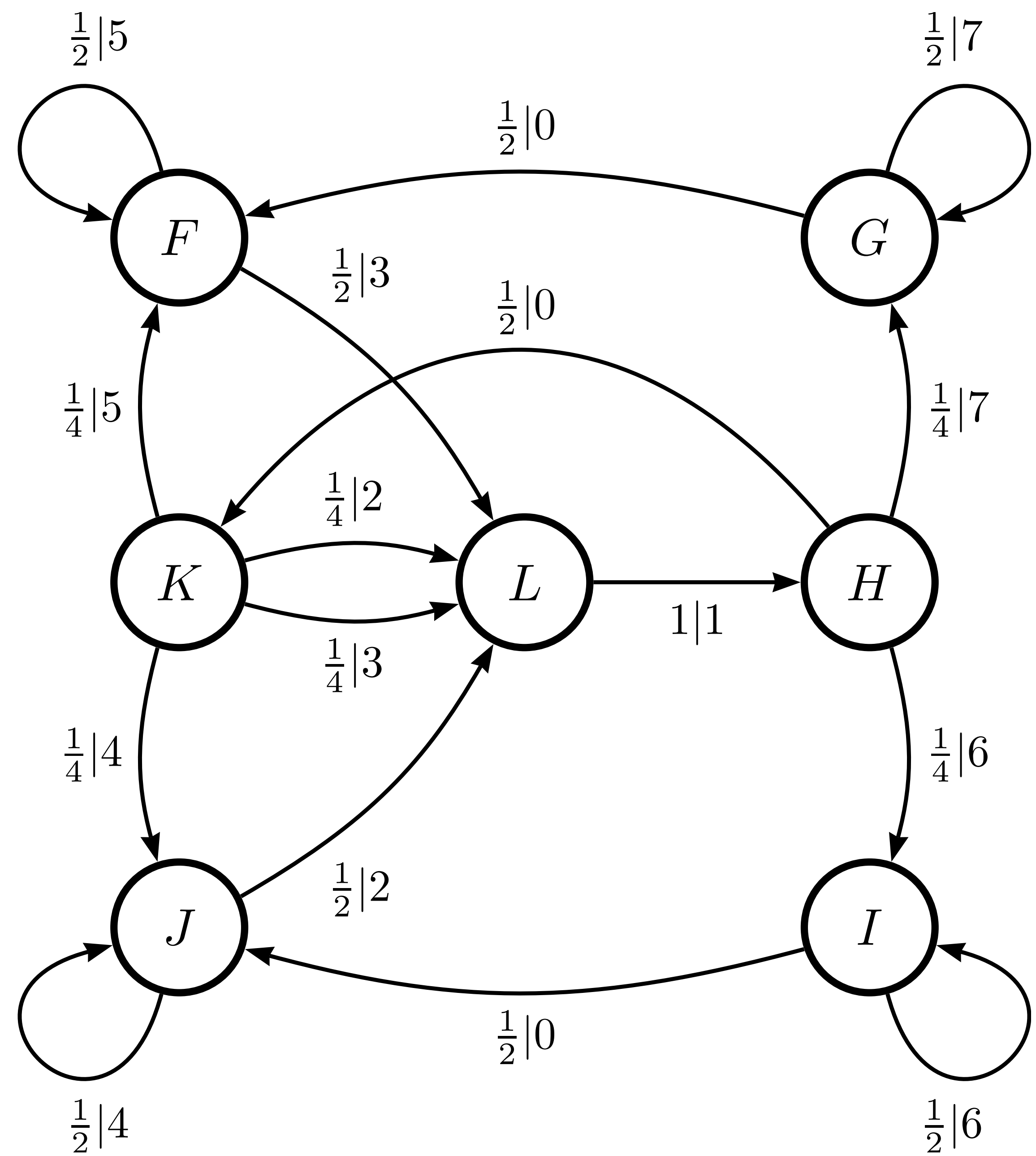
Example: Butterfly Process

$$\mathcal{A} = \{0, 1, \ldots, 6\}$$

Forward ε-machine



Reverse ε-machine



Causal Irreversibility

Example: Butterfly Process

Microscopically irreversible: $M^+ \neq M^-$

$$\Pr(\mathcal{S}^+) = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right) \qquad C_\mu^+ = \log_2 5 \text{ bits}$$

$$\Pr(\mathcal{S}^-) = (0.1, 0.2, 0.2, 0.15, 0.15, 0.1, 0.1)$$

$$C_\mu^- = 2.7464 \text{ bits}$$

Causally irreversible: $\Xi = -0.4245 \text{ bits}$

2-Cryptic: $\chi^+ = \frac{3}{10} \text{ bits}$

$$\mathbf{E} = C_\mu^+ - \chi^+ = 2.0219 \text{ bits}$$

$$\chi^- = C_\mu^- - \mathbf{E} = 0.7245 \text{ bits}$$

Causal Irreversibility

Examples Summary

Process	Microscopically	Causally	Cryptic
Even	Rev	Rev	No
Golden Mean	Rev	Rev	Yes
a-b-c	Irrev	Rev	No
RIP	Irrev	Irrev	Yes
Butterfly	Irrev	Irrev	Yes

Directional ε-Machines

Directional ε-Machines

Two questions:

How to calculate reverse ε-machine given forward?

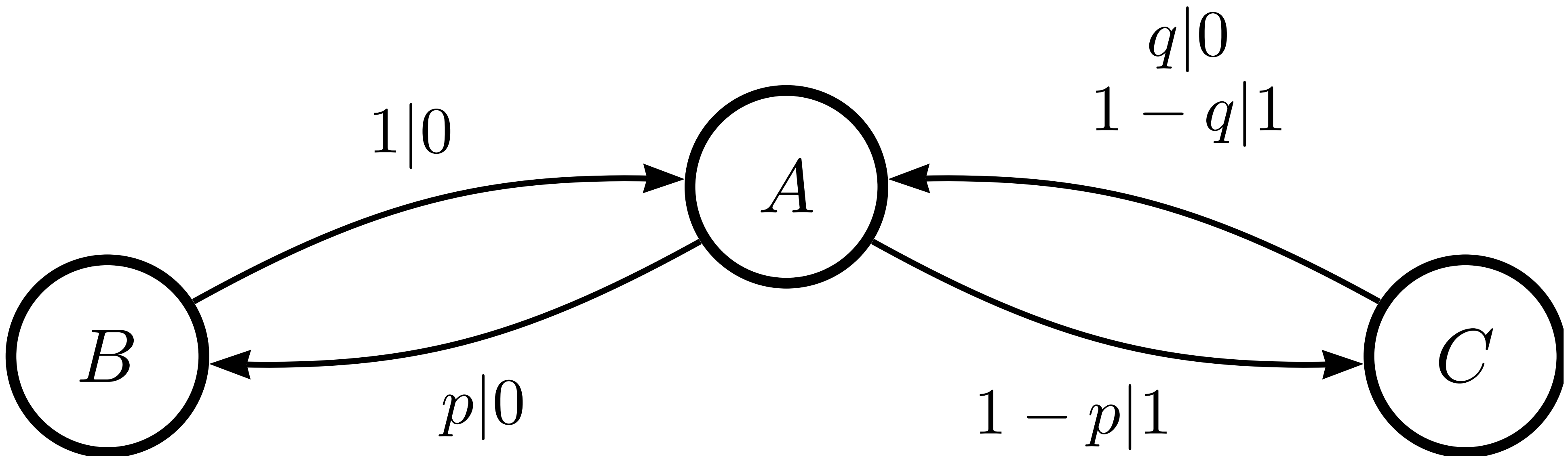
How to calculate forward and reverse state dependencies?

Answer:

1. Reverse time: transition arrow directions reversed.
2. Calculate mixed-state presentation of reversed machine.
3. Minimize to get ε-machine of reverse process.

Directional ε-Machines

Procedure by example:
Random Noisy Copy Process (RnC)



Directional ε-Machines

Procedure by example:

Random Noisy Copy Process (RnC)

$$\boldsymbol{\mathcal{S}}^+ = \{A, B, C\}$$

$$T^{(0)} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & p & 0 \\ 1 & 0 & 0 \\ q & 0 & 0 \end{pmatrix} \end{array} \quad T^{(1)} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 0 & 1-p \\ 0 & 0 & 0 \\ 1-q & 0 & 0 \end{pmatrix} \end{array}$$

$$\pi = \Pr(\mathcal{S}^+) = \frac{1}{2} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & p & 1-p \end{pmatrix} \end{array}$$

Directional ε-Machines

Time-reversed presentation: $\widetilde{M}^+ = \mathcal{T}(M^+)$

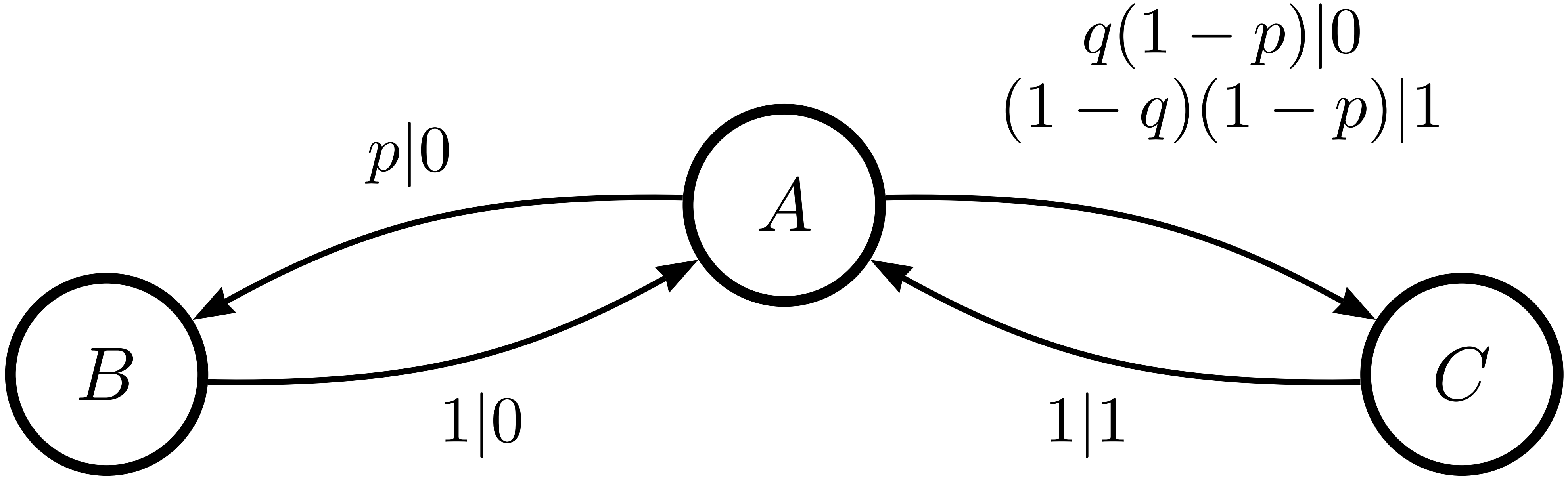
1. Reverse transitions.
2. Normalize to produce stochastic matrices:

$$\widetilde{T}^{(x)}_{\mathcal{R}'\mathcal{R}} \equiv \Pr(X = x, \mathcal{R} | \mathcal{R}')$$
$$= T^{(x)}_{\mathcal{R}\mathcal{R}'} \frac{\Pr(\mathcal{R})}{\Pr(\mathcal{R}')}$$

$$\widetilde{T} = \sum_{\{x\}} \widetilde{T}^{(x)}$$

Directional ε-Machines

Time-reversed presentation: $\widetilde{M}^+ = \mathcal{T}(M^+)$



$$\widetilde{T}^{(0)} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & p & q(1-p) \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \widetilde{T}^{(1)} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 0 & (1-q)(1-p) \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

Stationary distribution of $\widetilde{M}^+$: $\widetilde{\pi} = \pi$

Nonunifilar!

Directional ε-Machines

Mixed-state presentation: $M^- = \mathcal{U}(\widetilde{M^+})$

$$\{\mathcal{S}^-\} \sim \{\nu(w) = \Pr(\mathcal{S}^+ | \overrightarrow{w}) : \quad \overrightarrow{w} \in \mathcal{A}^*\}$$

Will generate transient and recurrent states.
Ignore transient states.

Start with $w = \lambda$

$$\nu(\lambda) = \Pr(\mathcal{S}^+) = \tfrac{1}{2} \left(1 \quad p \quad 1 - p\right)$$

Append symbols 0 and 1, calculate induced states:

$$\nu(0) \quad \nu(1)$$

Directional ε-Machines

Mixed-state presentation: $M^- = \mathcal{U}(\widetilde{M^+})$

$$
\begin{aligned}
\nu(0) &= \Pr(\mathcal{S}_0^+ | X_0 = 0) \\
&= \frac{\widetilde{\pi}\widetilde{T}^{(0)}}{\widetilde{\pi}\widetilde{T}^{(0)}\mathbf{1}} \\
&= \frac{\big(p, p, q(1-p)\big)}{2p + q(1-p)}
\end{aligned}
$$

$$
\begin{aligned}
\nu(1) &= \Pr(\mathcal{S}_0^+ | X_0 = 1) \\
&= \frac{\widetilde{\pi}\widetilde{T}^{(1)}}{\widetilde{\pi}\widetilde{T}^{(1)}\mathbf{1}} \\
&= \frac{\big(1, 0, 1-q\big)}{2-q}
\end{aligned}
$$

Directional ε-Machines

Mixed-state presentation: $M^- = \mathcal{U}(\widetilde{M^+})$

Extend words with 0 and 1;

calculate induced states that follow $\nu(0)$ and $\nu(1)$:

$$
\begin{aligned}
\nu(00) &= \Pr(\mathcal{S}_0^+ | X_0^2 = 00) = \frac{\widetilde{\pi}\widetilde{T}^{(0)}\widetilde{T}^{(0)}}{\widetilde{\pi}\widetilde{T}^{(0)}\widetilde{T}^{(0)}\mathbf{1}} \\
\nu(01) &= \Pr(\mathcal{S}_0^+ | X_0^2 = 01) = \frac{\widetilde{\pi}\widetilde{T}^{(1)}\widetilde{T}^{(0)}}{\widetilde{\pi}\widetilde{T}^{(1)}\widetilde{T}^{(0)}\mathbf{1}} \\
\nu(10) &= \Pr(\mathcal{S}_0^+ | X_0^2 = 10) = \frac{\widetilde{\pi}\widetilde{T}^{(0)}\widetilde{T}^{(1)}}{\widetilde{\pi}\widetilde{T}^{(0)}\widetilde{T}^{(1)}\mathbf{1}} \\
\nu(11) &= \Pr(\mathcal{S}_0^+ | X_0^2 = 11) = \frac{\widetilde{\pi}\widetilde{T}^{(1)}\widetilde{T}^{(1)}}{\widetilde{\pi}\widetilde{T}^{(1)}\widetilde{T}^{(1)}\mathbf{1}}
\end{aligned}
$$

Mixed-state presentation: $M^- = \mathcal{U}(\widetilde{M}^+)$

Continue, until no new mixed states are produced.

For example,

$$\nu(1001) = \nu(111001)$$

The result is the set of mixed states for the reverse process.

$$\boldsymbol{\mathcal{S}}^- = \{D, E, F\}$$

$$D \equiv \nu(1001) = \overset{\begin{matrix} A & B & C \end{matrix}}{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}}$$

$$E \equiv \nu(100) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$F \equiv \nu(10) = \begin{pmatrix} 0 & \frac{p}{p+q(1-p)} & \frac{q(1-p)}{p+q(1-p)} \end{pmatrix}$$

Directional ε-Machines

Mixed-state presentation: $M^- = \mathcal{U}(\widetilde{M}^+)$

Transition probabilities:

Consider $\nu(00) \to \nu(100)$

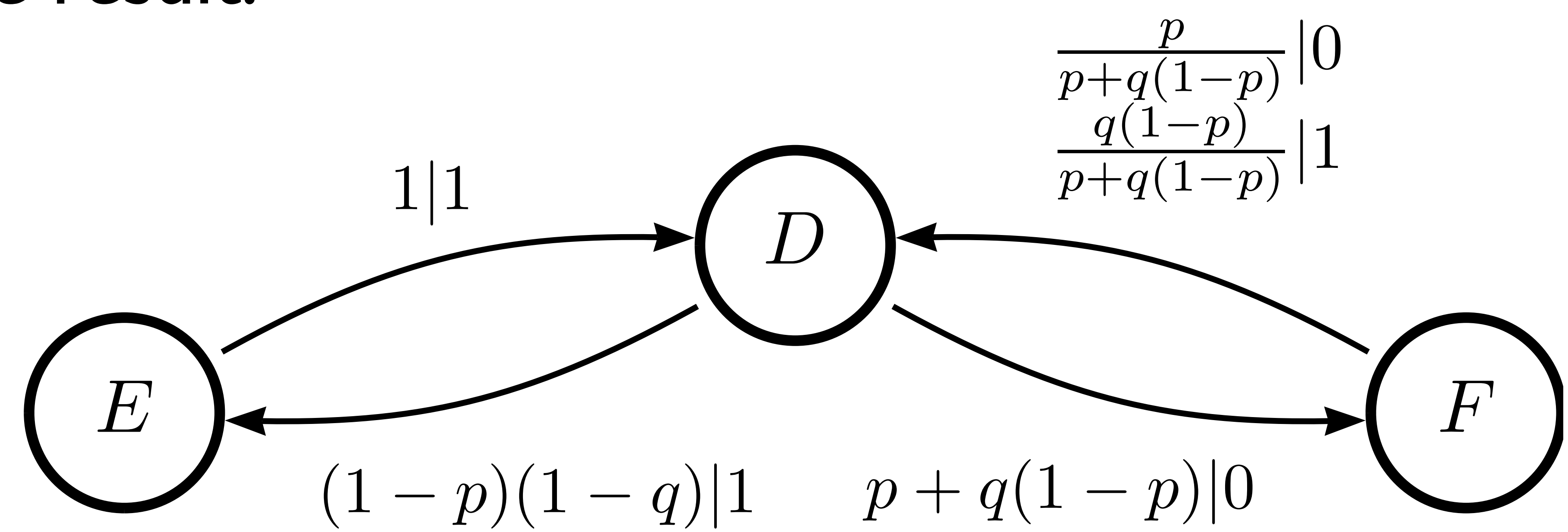
$$\begin{aligned} \Pr\big(1, \nu(100) | \nu(00)\big) &= \Pr(1|00) \\ &= \nu(00)\widetilde{T}^{(1)}\mathbf{1} \\ &= \frac{1-p}{1+p+q-pq} \end{aligned}$$

Do this for all possible state-to-state transitions.

Directional ε-Machines

Mixed-state presentation: $M^- = \mathcal{U}(\widetilde{M}^+)$

The result:

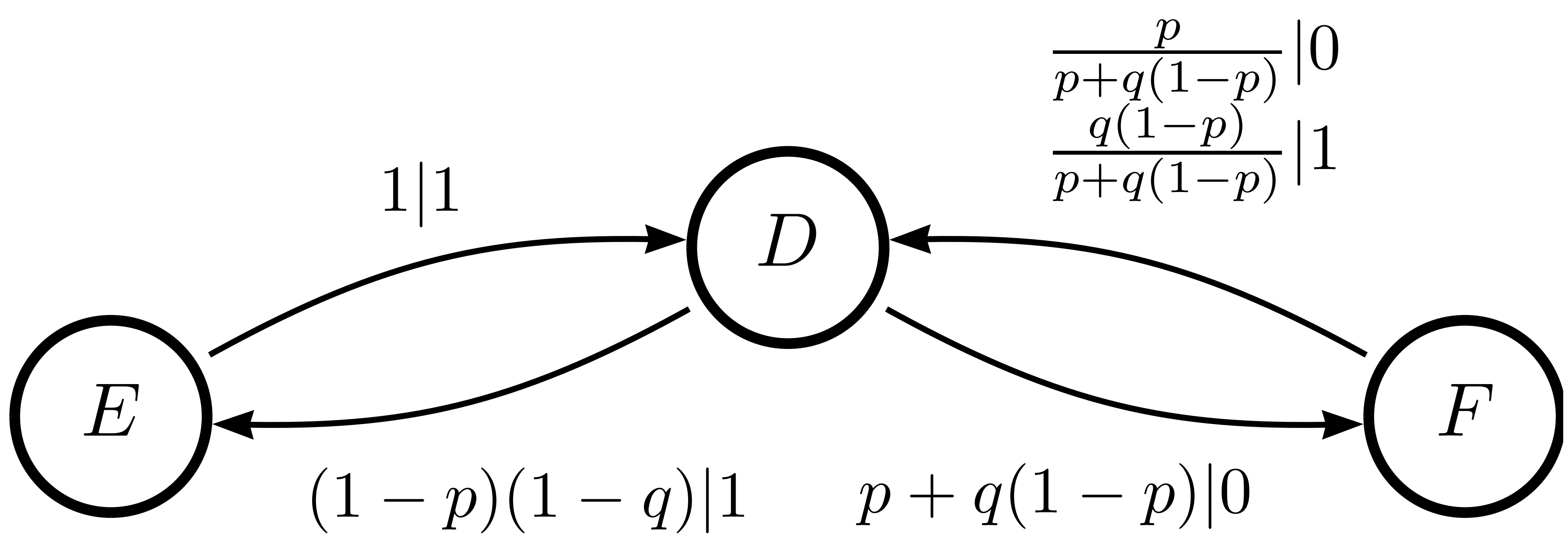


Note:

This is unifilar.

Directional ε-Machines

Minimize mixed-state presentation to get ε-machine:



For RnC, the MSP turned out to be minimal.

So, this is the εM of the reverse process: $M^- = M(\mathcal{P}^-)$

Directional ε-Machines

Minimizing MSP to get ε-machine:

Recall MSP gives a unifilar presentation

Do recurrent portion of MSP
Recurrent states are a refinement of causal states.

So minimizing simply groups these.

Gives recurrent causal states of ε-machine.

Trick for general case: Strip transients and MSP again!

Minimization conjecture: Reverse ⇒ MSP ⇒ Reverse ⇒ MSP.

Directional ε-Machines

How to calculate $\Pr(\mathcal{S}^-, \mathcal{S}^+)$?

Track states through

$$\begin{array}{ccc} M^+ & \xleftarrow{\ \mathcal{U}\ } & \widetilde{M}^- \\ \mathcal{T} \big\downarrow & & \big\uparrow \mathcal{T} \\ \widetilde{M}^+ & \xrightarrow[\ \mathcal{U}\ ]{} & M^- \end{array}$$

Time-reversed presentation

$$\widetilde{M}^+ = \mathcal{T}(M^+)$$

Mixed-state presentation

$$M^- = \mathcal{U}(\widetilde{M}^+)$$

Directional ε-Machines

How to calculate $\Pr(\mathcal{S}^-, \mathcal{S}^+)$?

Consider only the recurrent causal states: $\Pr(\mathcal{S}^-) > 0$

$$\boldsymbol{\mathcal{S}}^- = \{D, E, F\}$$

$$D = \nu(1001) = \begin{matrix} A & B & C \\ \big(0 & 0 & 1 \big) \end{matrix} = \Pr(\mathcal{S}^+ | \mathcal{S}^- = D)$$

$$E = \nu(100) = \begin{matrix} A & B & C \\ \big(1 & 0 & 0 \big) \end{matrix} = \Pr(\mathcal{S}^+ | \mathcal{S}^- = E)$$

$$F = \nu(10) = \begin{matrix} A & B & C \\ \left(0 \right. & \frac{p}{p+q(1-p)} & \left. \frac{q(1-p)}{p+q(1-p)} \right) \end{matrix}$$

$$= \Pr(\mathcal{S}^+ | \mathcal{S}^- = F)$$

Directional ε-Machines

How to calculate $\Pr(\mathcal{S}^-, \mathcal{S}^+)$?

These MSP states give

$$\Pr(\mathcal{S}^+|\mathcal{S}^-)$$

With $\Pr(\mathcal{S}^-)$

get

$$\Pr(\mathcal{S}^+, \mathcal{S}^-) = \Pr(\mathcal{S}^+|\mathcal{S}^-) \Pr(\mathcal{S}^-)$$

Reading for next lecture: CMR articles

TBA

PRATISP

MSP

IACP, IACPLCOCS