Information Diagrams for Processes

Reading for this lecture:
CMR articles *Yeung, TBA, & PRATISP*

Projects!

Proposal due in one week.

Projects!

3. Write up your Project Proposal with the following sections. The result should be 2-3 pages long.

3a. Goal: What is your primary project goal? What you would like to learn?

3b. System: Describe how the dynamical system is nonlinear and time-dependent.

What's the state space?

What's the dynamic?

Why is the system behavior interesting?

3c. Dynamical properties: What dynamical properties are you going to investigate?

3d. Intrinsic computation properties: What information processing properties are you going to investigate?

3e. Methods: What methods will you use? Why are they appropriate?

3f. Hypothesis: What is your current guess as to what you will find?

3g. Steps: List the appropriate steps for your investigation; for example, read literature, write simulator, do mathematical analysis, estimate properties from simulation, write up report, and so on.

3h. Time: Estimate how long each step will take. Can you complete the project within one month?

Project:

1. Topic: Information processing and computation in natural or engineered dynamical system.

2. Select in consultation with us.

3. Project report presented to class at the end of the term.

4. Written report (including code and documentation) due then.

5. Website on project with the report, code, and documentation preferred.

Possible topics:

- Estimate information quantities for complex system of your choice
- Survey structural complexity versus entropy for a class of dynamical systems
- Implement CA pattern analysis
- Analyze patterns generated by variant of 1D or 2D spin systems
- Relationship between energy and information
- Maxwell's demon: Energy versus information
- Relationship between intrinsic computation and phase transitions
- Novel computation:
    - Quantum
    - DNA
    - Analog/continuous
    - Stochastic
    - Evolutionary
    - Neural

Possible topics ...

- Review current research on a complex system of your choice; such as,
    - Complex materials, self-assembly in nanotechnology
    - Chemical pattern formation
    - Biological morphogenesis
    - Bioinformatics
    - Economics: Game dynamical systems

- Simulate a self-organizing system:
    - Statistical mechanical model:
      Ising, Potts, Heisenberg, X-Y, spin glass, and the like.
      Cellular automata, map lattice, and the like.
    - Population dynamics: Ecological or evolutionary
    - Networks: Neural, Internet, WWW, social, gene expression, ...
    - Transportation networks: Traffic flow, power grid, world trade, ...

- Build an experimental chaotic or pattern-forming system:
    - Electronic circuit
    - Mechanical device
    - Chemical oscillator
    - Video feedback (see JPC articles)

Possible topics ...

- Effect of external noise on
  - Chaotic behavior
  - This or that kind of bifurcation
  - Routes to chaos

- Probability densities:
  - Time evolution of densities for 1D and 2D maps
  - Approximate invariant distributions
  - Convergence to invariant distributions
  - Intrinsic computation analysis of density evolution

- Transform-based analysis of chaos:
  - Fourier analysis
  - Wavelet analysis

- Chaotic encryption

Possible topics ...

- Philosophical review of:
    - Causality
    - Teleology
    - Randomness, including human perception of (cf. Amos Tversky papers)
    - Coincidence, including human perception of (cf. Persi Diaconis papers)
    - Prediction
    - Cybernetics (cf. Wiener biography “Dark Hero of the Information Age”)

Examples from past years:

- 2012:
  - Christina Cogdell & Paul Reichers: Nonlinear Dynamics of Passionflower Tendril Free Coils
  - Nichole Sanderson: “Killing” and “Collapsing”—How varying transition probabilities between states alters the statistical and topological properties of probabilistic ε-machines
  - Vikram Vijayaraghavan: Complexity and Critical Behavior
- 2010:
  - Charles Brummitt: Networks Of Chaotic Maps—A New Network Growth Model, Inferring Topology From Symbolic Dynamics
  - Luke Grecki: An Algebraic View of Topological ε-Machines
  - Paul Riechers: Spatiotemporal Computational Mechanics
- 2009:
  - Ryan James: Block Entropy in 1+1 Dimensions
  - Ben Johnson: An Introduction to Quantum Computation
  - Richard Watson: Information Theoretic Approaches to Spiking Models
  - Paul Smaldino: Does Learning Mean a Decrease in Entropy?
  - Nicholas Travers: Computational Mechanics of ECAs, and Machine Metrics

Information Diagrams for Processes

Reading for this lecture:
CMR articles *Yeung, TBA, & PRATISP*

Information Diagrams for Processes

Time (a)symmetry of processes:

Process: $\mathcal{P} \sim \Pr(\overleftrightarrow{X})$

$$\overleftrightarrow{X} = \ldots X_{-2} X_{-1} X_0 X_1 X_2 \ldots$$
$$\overleftarrow{X} = \ldots X_{-2} X_{-1}$$
$$\overrightarrow{X} = X_0 X_1 X_2 \ldots$$

Forward process: Same

$$\overrightarrow{\mathcal{P}} \sim \Pr(\sigma^+ \circ \overleftrightarrow{X})$$

with +t shift: $\sigma^+ \circ \overleftrightarrow{X} = \ldots X_{-2} X_{-1} X_0 X_1 X_2 \ldots$

Reverse process: Scan measurements in reverse order

$$\overleftarrow{\mathcal{P}} \sim \Pr(\sigma^- \circ \overleftrightarrow{X})$$

with -t shift: $\sigma^- \circ \overleftrightarrow{X} = \ldots X_2 X_1 X_0 X_{-1} X_{-2} \ldots$

Information Diagrams for Processes

Time (a)symmetry of processes ...

Prediction:

Forward entropy rate: $h_\mu(\overrightarrow{\mathcal{P}}) = H[X_0|\overleftarrow{X}]$

Retrodiction:

Reverse entropy rate: $h_\mu(\overleftarrow{\mathcal{P}}) = H[X_{-1}|\overrightarrow{X}]$

Which time direction is most unpredictable?

Theorem: Entropy rate is time symmetric

$$h_\mu(\overrightarrow{\mathcal{P}}) = h_\mu(\overleftarrow{\mathcal{P}})$$

Information Diagrams for Processes

Time (a)symmetry of processes ...

Proof sketch:

$$h_\mu(\overrightarrow{\mathcal{P}}) = H[X_0 | \overleftarrow{X}]$$

$$= \lim_{L \to \infty} H[X_0 | X_{-L+1}, \ldots, X_{-1}]$$

$$= \lim_{L \to \infty} [H[X_{-L+1}, \ldots, X_0] - H[X_{-L+1}, \ldots, X_{-1}]]$$

Stationarity: $H[X_{-L+1}, \ldots, X_{-1}] = H[X_{-L+2}, \ldots, X_0]$

$$h_\mu(\overrightarrow{\mathcal{P}}) = \lim_{L \to \infty} [H[X_{-L+1}, \ldots, X_0] - H[X_{-L+2}, \ldots, X_0]]$$

$$= \lim_{L \to \infty} H[X_{-L+1} | X_{-L+2}, \ldots, X_0]$$

$$= H[X_{-L+1} | \overrightarrow{X}_{-L+2}]$$

$$= H[X_{-1} | \overrightarrow{X}]$$

$$= h_\mu(\overleftarrow{\mathcal{P}})$$

Information Diagrams for Processes

Time (a)symmetry of processes ...

Theorem: Entropy rate is time symmetric

$$h_\mu(\overrightarrow{\mathcal{P}}) = h_\mu(\overleftarrow{\mathcal{P}})$$

Both directions are equally (asymptotically) unpredictable.

Information Diagrams for Processes

Time (a)symmetry of processes ...

Excess entropy is time symmetric.

$$\mathbf{E}(\overrightarrow{\mathcal{P}}) = \mathbf{E}(\overleftarrow{\mathcal{P}})$$

Proof sketch:

$$I[\overleftarrow{X}; \overrightarrow{X}] = I[\overrightarrow{X}; \overleftarrow{X}]$$

Conclusion:

Neither entropy rate or excess entropy
detect temporal asymmetry

Information Diagrams for Processes

Time (a)symmetry of processes ...

Forward $\epsilon\mathrm{M}$:

$$\overrightarrow{M} = \overrightarrow{\mathcal{P}} / \sim$$

Reverse $\epsilon\mathrm{M}$:

$$\overleftarrow{M} = \overleftarrow{\mathcal{P}} / \sim$$

Is a process differently structured in forward or reverse time?

Theorem: $\epsilon\mathrm{M}$ need not be time symmetric

$$\overleftarrow{M} \neq \overrightarrow{M}$$

$$C_\mu(\overleftarrow{M}) \neq C_\mu(\overrightarrow{M})$$

Time (a)symmetry of processes ...

Proof by example:

Misiurewicz parameter in the Logistic map:

First root, $r < 4$, where critical point is periodic and

$$f^4(\tfrac{1}{2}) = f^5(\tfrac{1}{2})$$

Find

$$r \approx 3.9277370017867516$$

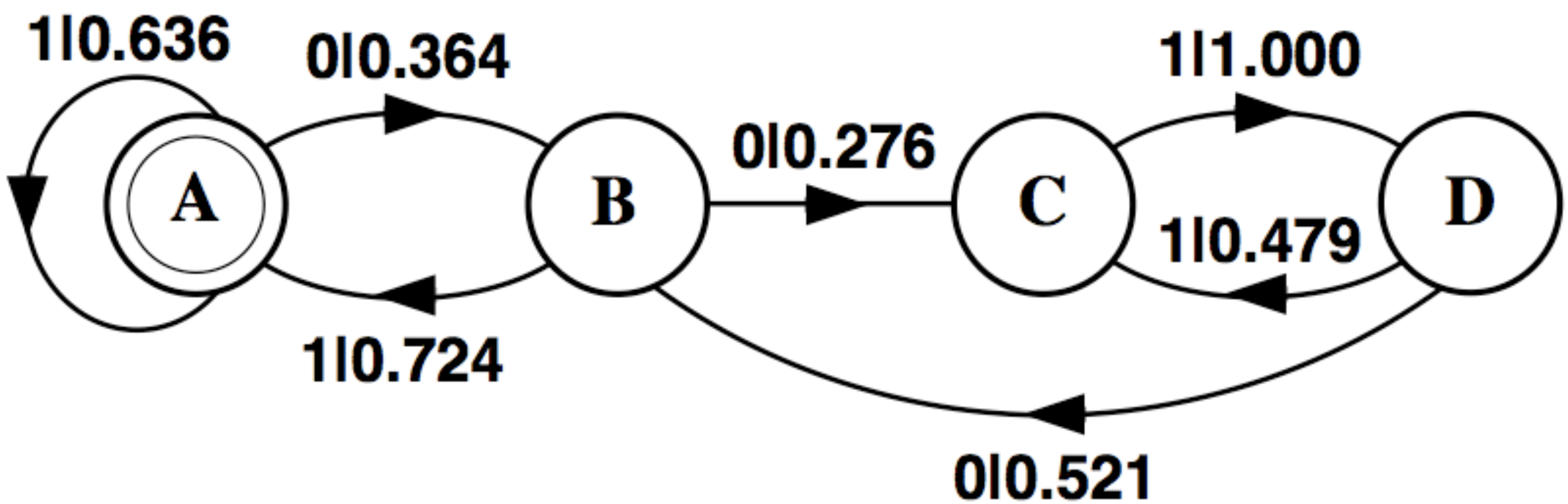
Use binary generating partition.

Information Diagrams for Processes

Time (a)symmetry of processes ...

Example: Misiurewicz parameter for Logistic map ...

Forward machine:



$$h_\mu(\overrightarrow{\mathcal{P}}) \approx 0.81 \text{ bits/symbol}$$

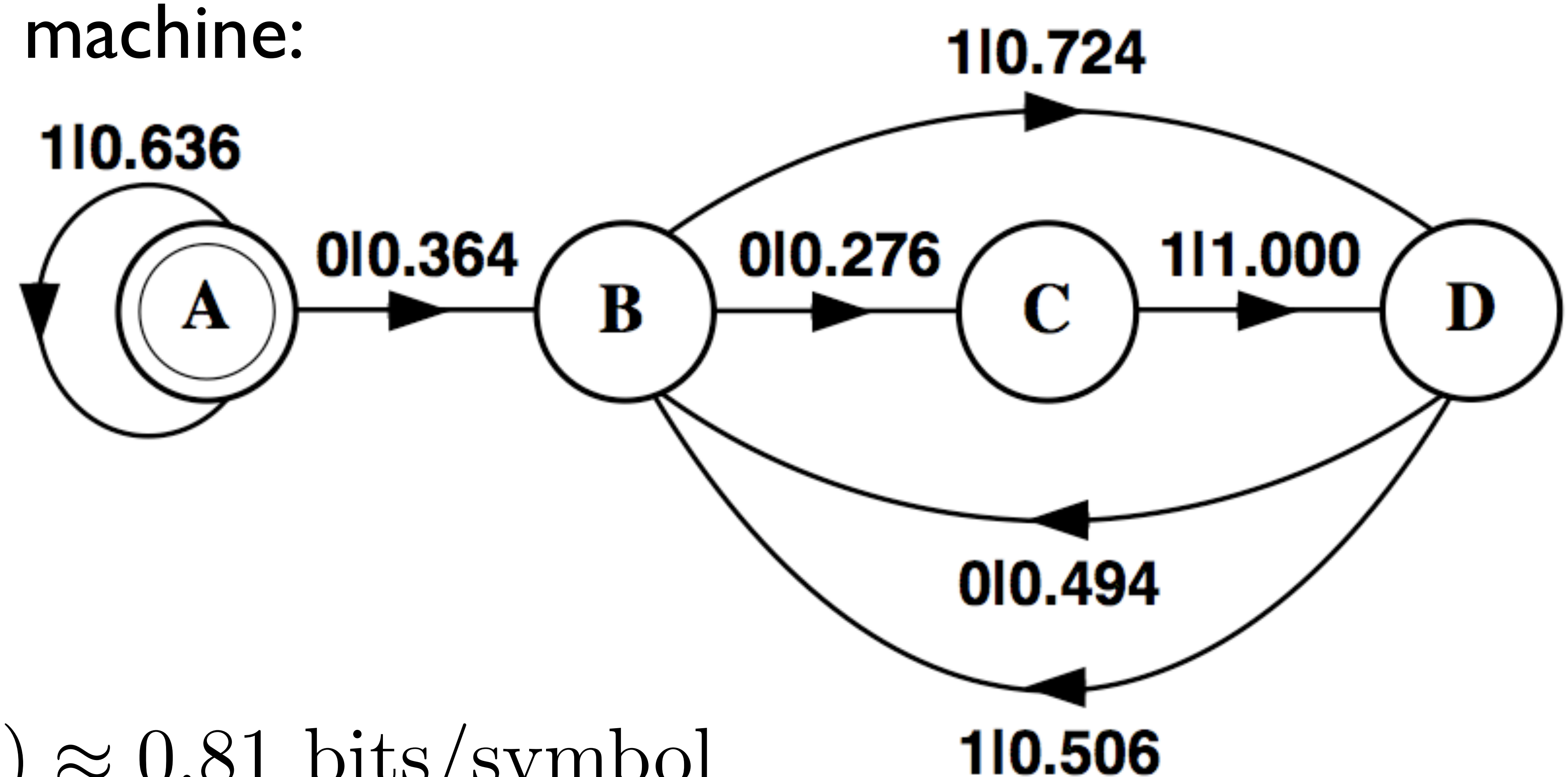
$$C_\mu(\overrightarrow{\mathcal{P}}) \approx 1.77 \text{ bits}$$

Information Diagrams for Processes

Time (a)symmetry of processes ...

Example: Misiurewicz parameter for Logistic map ...

Reverse machine:



$$h_\mu(\overleftarrow{\mathcal{P}}) \approx 0.81 \text{ bits/symbol}$$

$$C_\mu(\overleftarrow{\mathcal{P}}) \approx 1.41 \text{ bits}$$

Time (a)symmetry of processes ...

Measure of time asymmetry: Causal Irreversibility

$$\Xi(\mathcal{P}) = C_\mu(\overrightarrow{M}) - C_\mu(\overleftarrow{M})$$

Misiurewicz process irreversibility:

$$\Xi(\mathcal{P}) \approx 0.36 \text{ bits}$$

Information Diagrams for Processes

Time (a)symmetry of processes ...

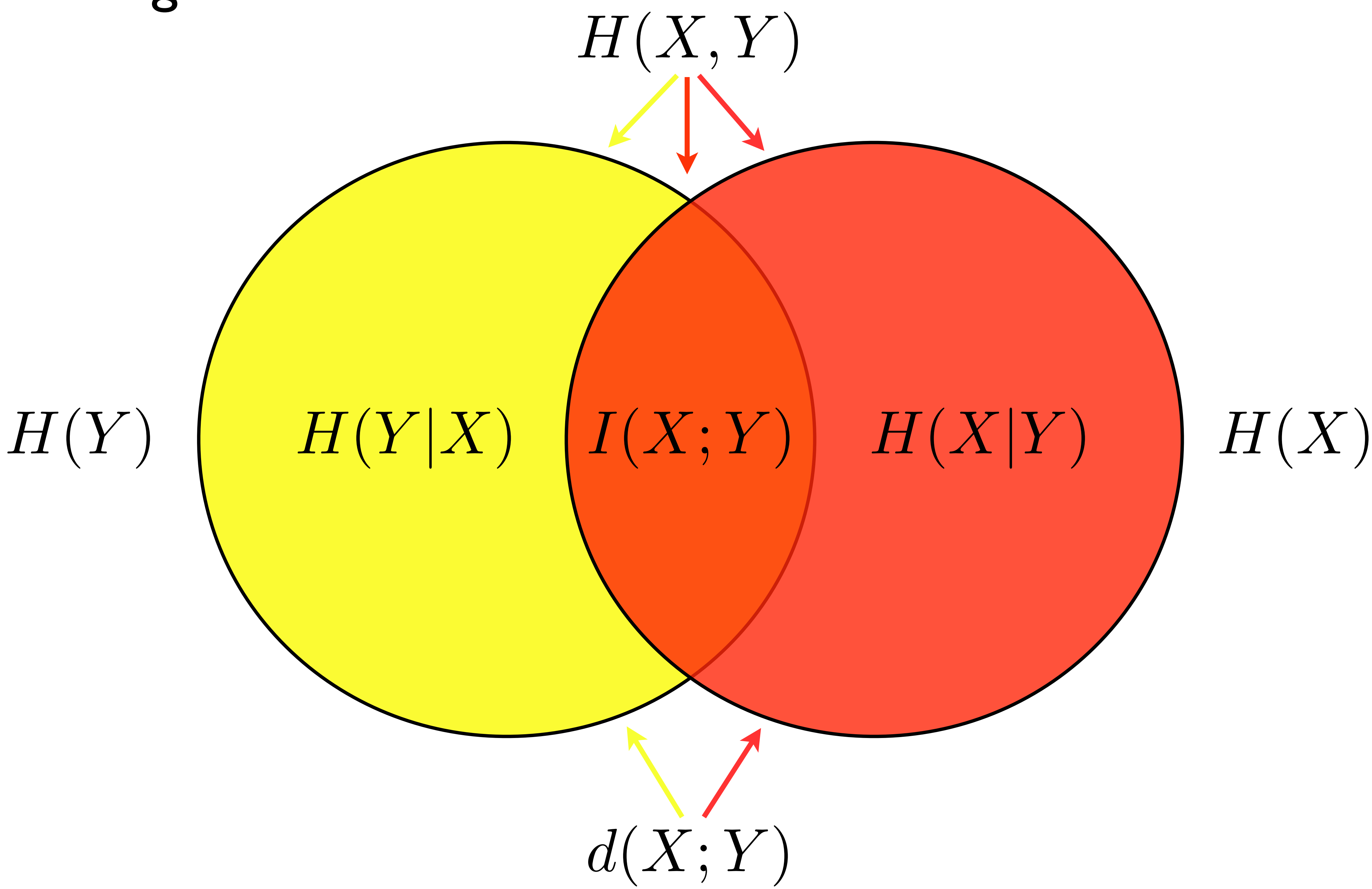
Causal Irreversibility:

Information to remember for optimally predicting and optimally retrodicting can differ.

Even though the degree of unpredictability is the same in both time directions.

Information Diagrams for Processes

Recall I-diagrams:



Information Diagrams for Processes

The I-Diagram ...

Three random variables:

$$X \sim \Pr(x)$$

$$Y \sim \Pr(y) \qquad (X, Y, Z) \sim \Pr(x, y, z)$$

$$Z \sim \Pr(z)$$

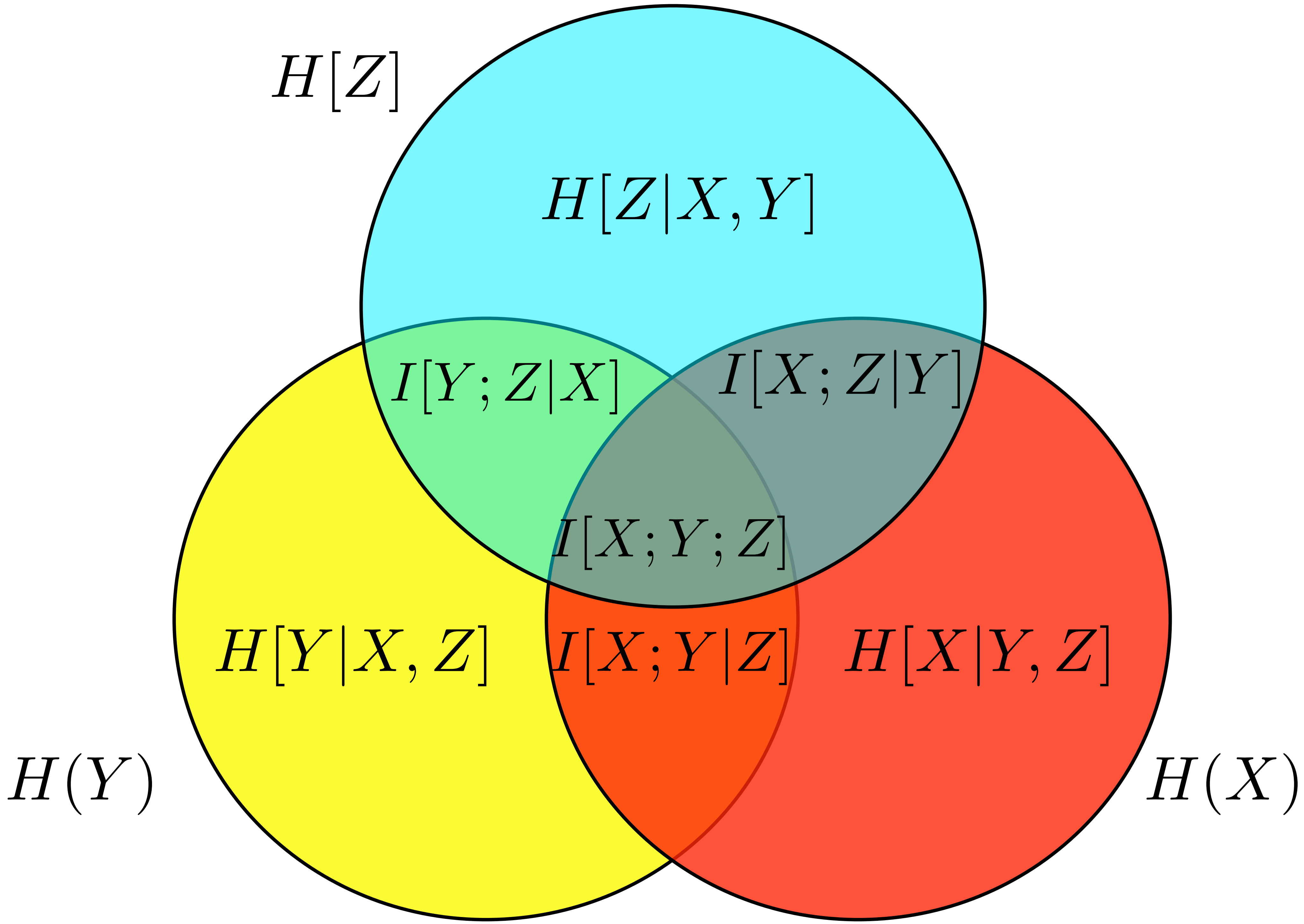
Information measures:

$$H[X] \quad H[Y] \quad H[Z] \quad \cdots \quad I[X;Y;Z] \quad \cdots \quad H[X,Y,Z]$$

7 atomic information measures.

Information Diagrams for Processes

Information diagram for three random variables:

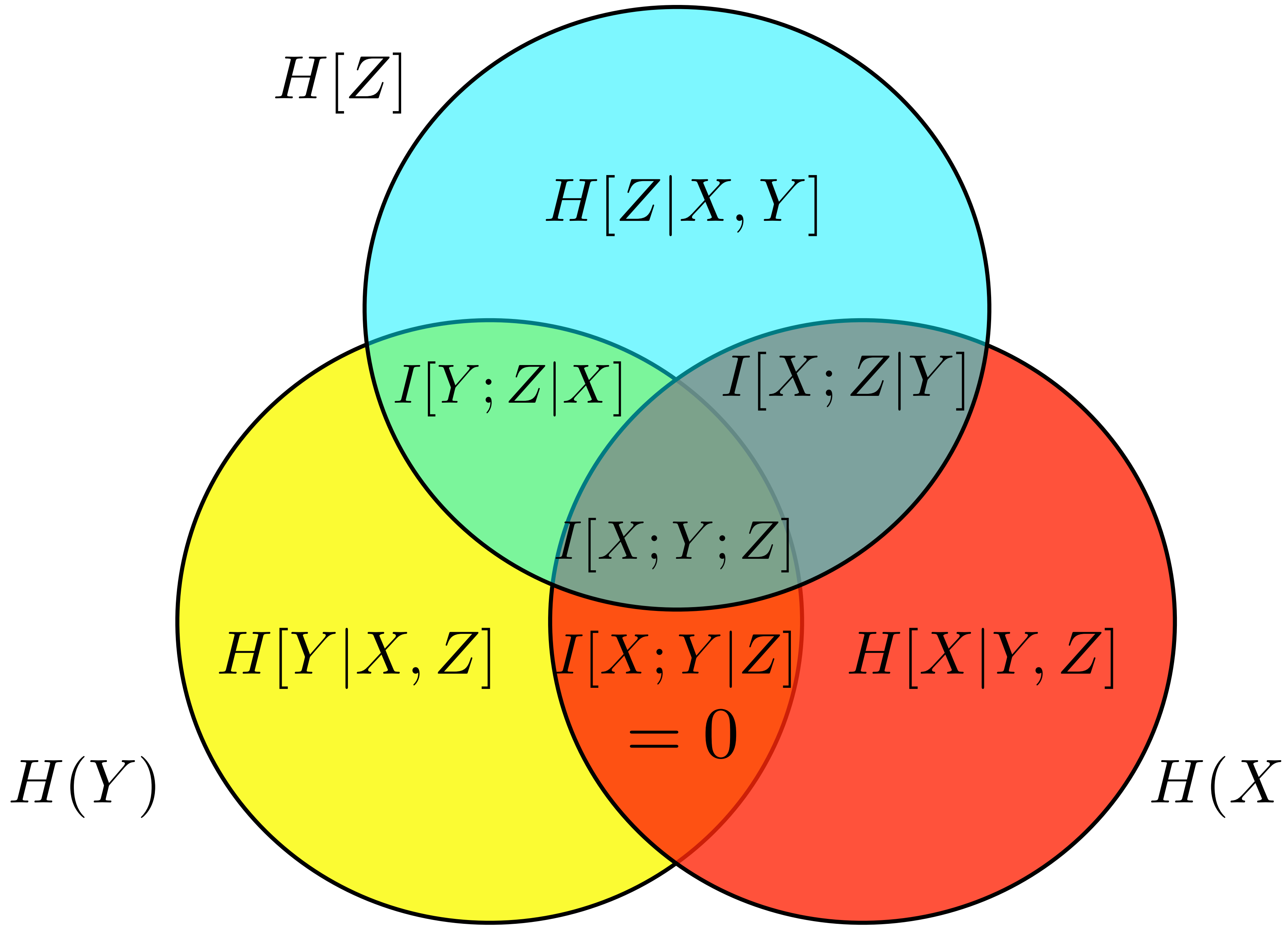


Information Diagrams for Processes

Information diagram for three random variables ...

Markov chain: $X \to Z \to Y$

Consequence: $I[X;Y|Z] = 0$



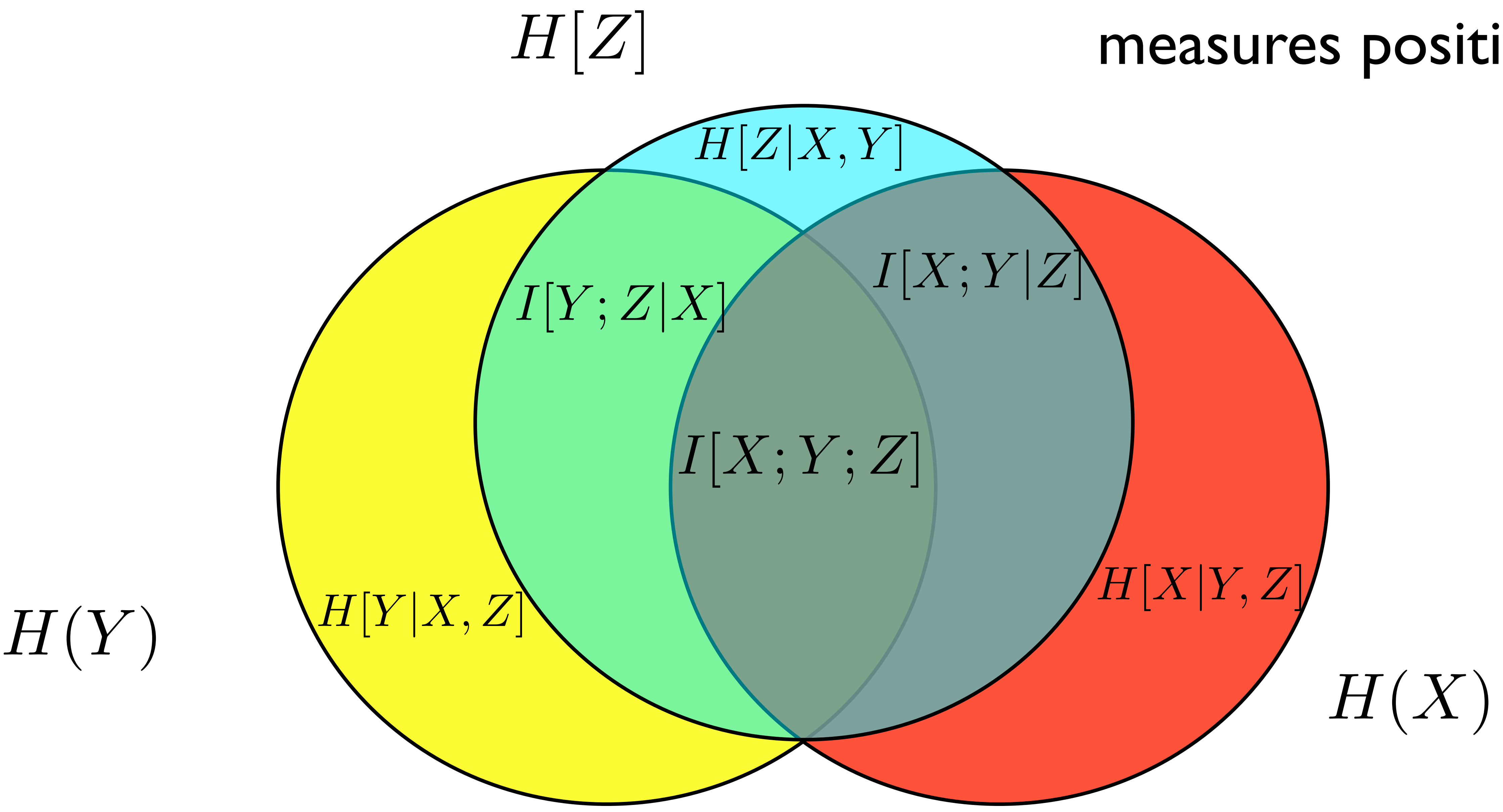
Information Diagrams for Processes

Information diagram for three random variables ...

Markov chain: $X \rightarrow Z \rightarrow Y$

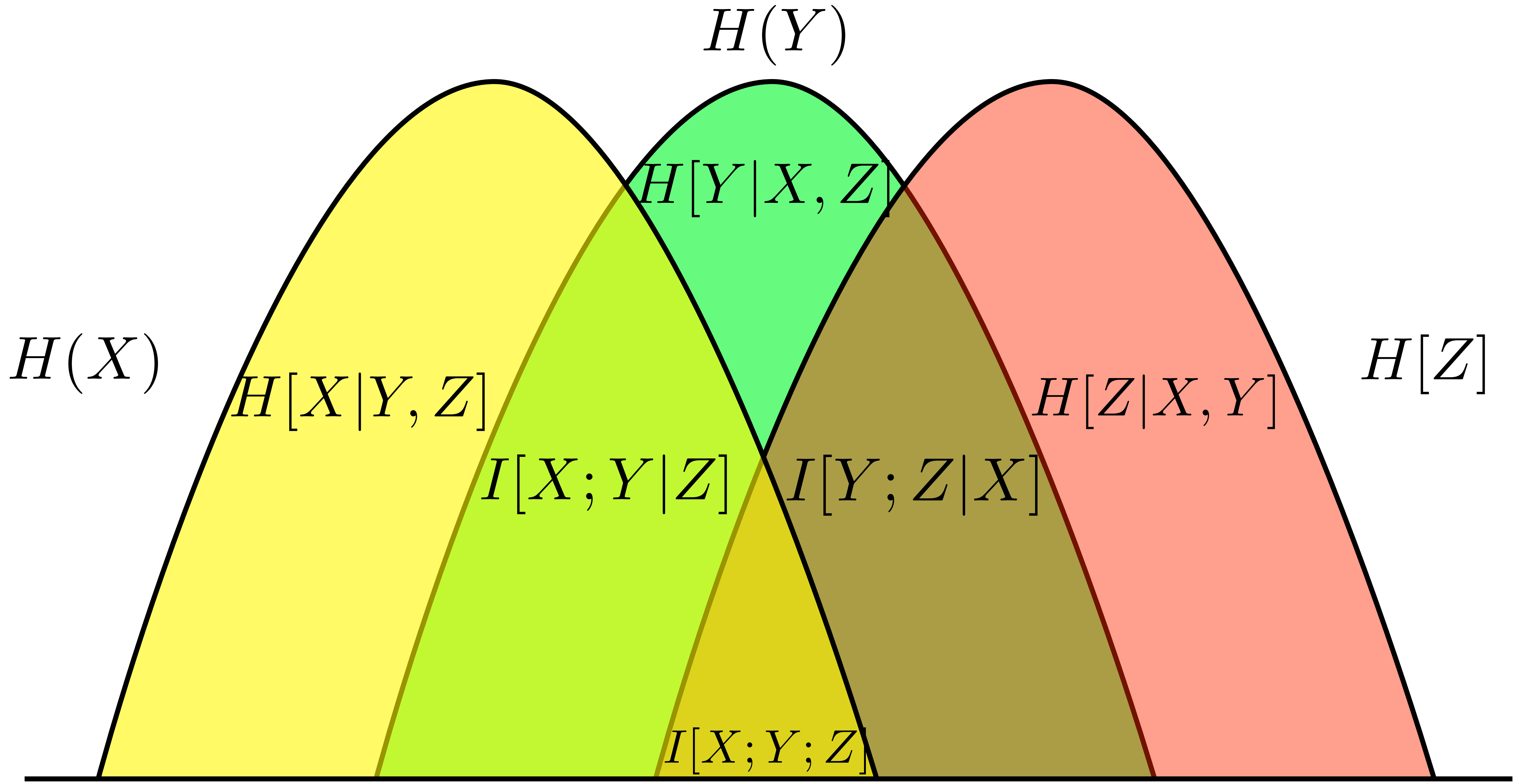
Consequence: $I[X; Y|Z] = 0$

All areas have positive measure; all info measures positive.



Information Diagrams for Processes

Markov chain: $X \to Y \to Z$



Information Diagrams for Processes

Process I-diagrams:

Process has an infinite number of RVs!

$$\Pr(\overleftrightarrow{X}) = \Pr(\ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots)$$

Rather: $\Pr\left(\overleftrightarrow{X}\right) = \Pr\left(\overleftarrow{X}\,\overrightarrow{X}\right)$

Start with 2-variable I-diagram and whittle down:

Past as composite random variable: $\overleftarrow{X}$

Future as composite random variable: $\overrightarrow{X}$

Information measures:

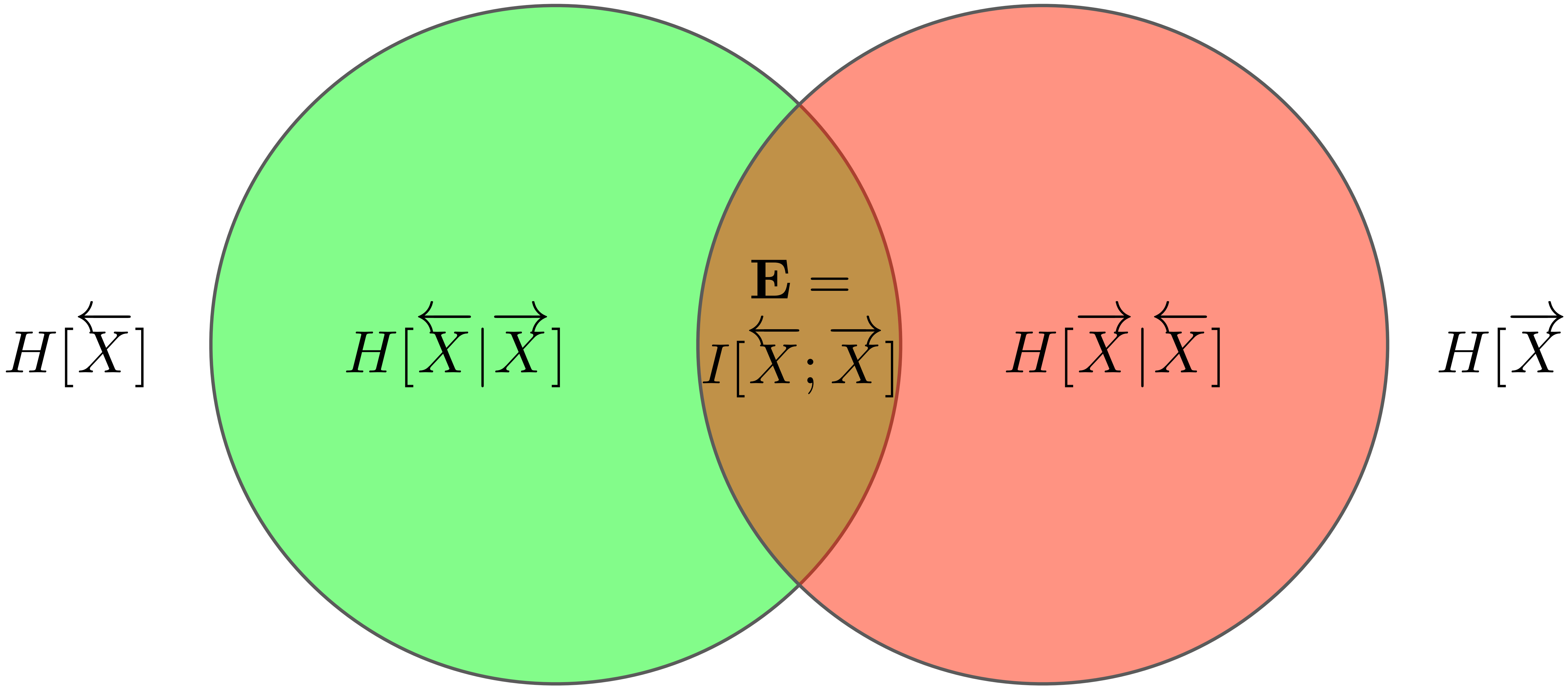
$$H[\overleftarrow{X}] \quad H[\overrightarrow{X}] \quad H[\overrightarrow{X}, \overleftarrow{X}]$$

$$H[\overleftarrow{X}|\overrightarrow{X}] \quad H[\overrightarrow{X}|\overleftarrow{X}] \quad I[\overrightarrow{X}; \overleftarrow{X}] \quad H[\overrightarrow{X}|\overleftarrow{X}] + H[\overleftarrow{X}|\overrightarrow{X}]$$

There are 3 = 2^2-1 atomic information measures:

$$H[\overrightarrow{X}|\overleftarrow{X}] \quad H[\overleftarrow{X}|\overrightarrow{X}] \quad I[\overrightarrow{X}; \overleftarrow{X}]$$

Information Diagrams for Processes



Information Diagrams for Processes

Process I-diagram using ε-machine:

Start with 3-variable I-diagram and whittle down:

Past as composite random variable: $\overleftarrow{X}$

Future as composite random variable: $\overrightarrow{X}$

Causal states: $\mathcal{S} \in \boldsymbol{\mathcal{S}}$

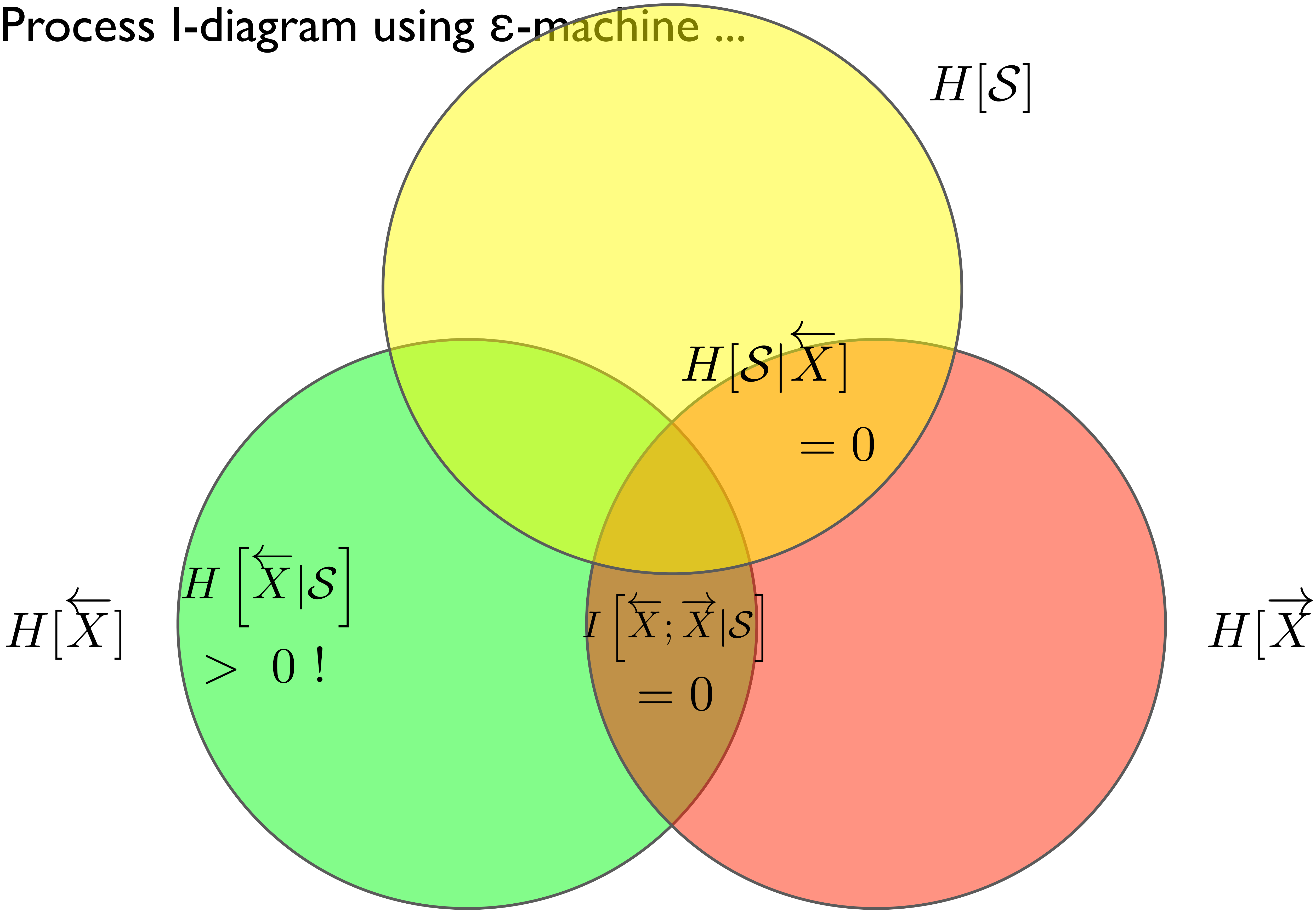
Information measures:

$$H[\overleftarrow{X}] \quad H[\overrightarrow{X}] \quad H[\mathcal{S}] \quad \cdots \quad I[\overrightarrow{X};\overleftarrow{X};\mathcal{S}] \quad \cdots \quad H[\overrightarrow{X},\overleftarrow{X},\mathcal{S}]$$

There are 7 (= 2^3 - 1) atomic information measures.

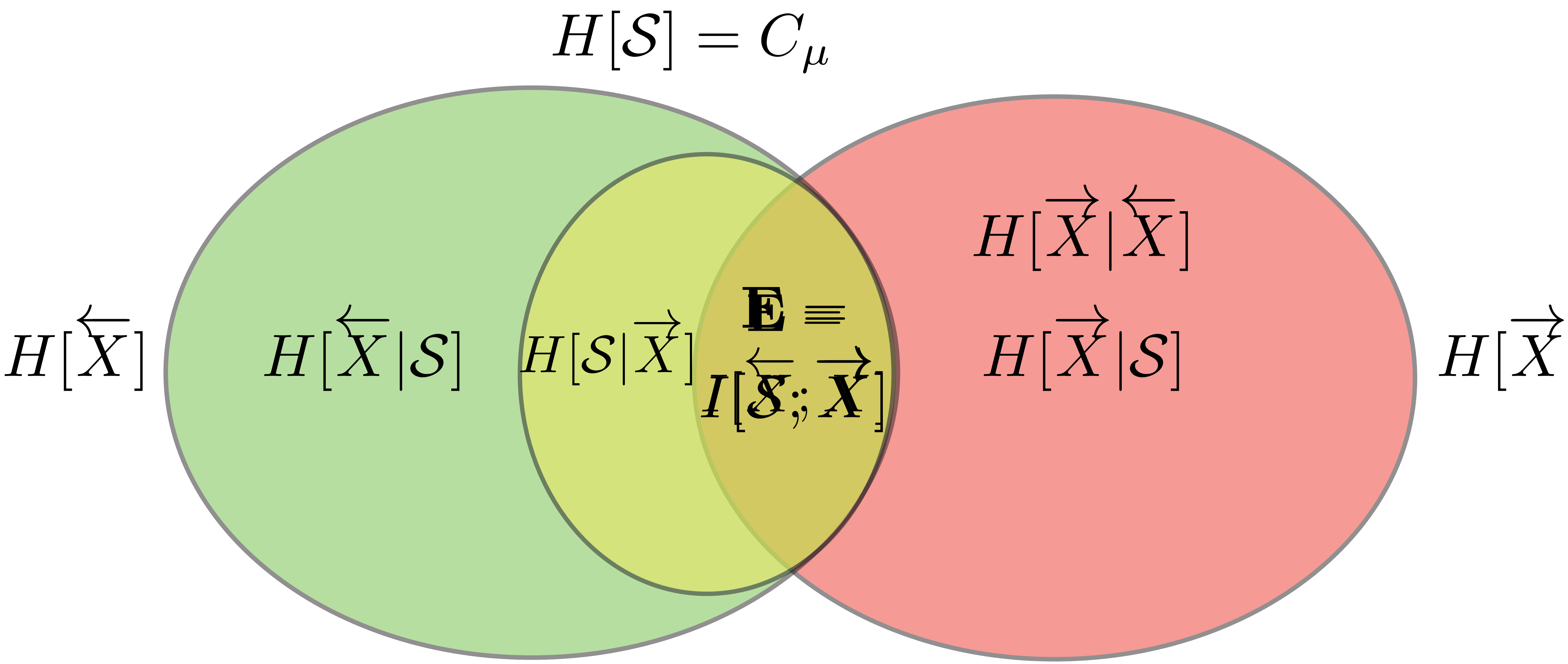
Information Diagrams for Processes

Process I-diagram using ε-machine ...



Information Diagrams for Processes

ε-Machine I-diagram:



Information Diagrams for Processes

What is $H[\overrightarrow{X}|\mathcal{S}]$?

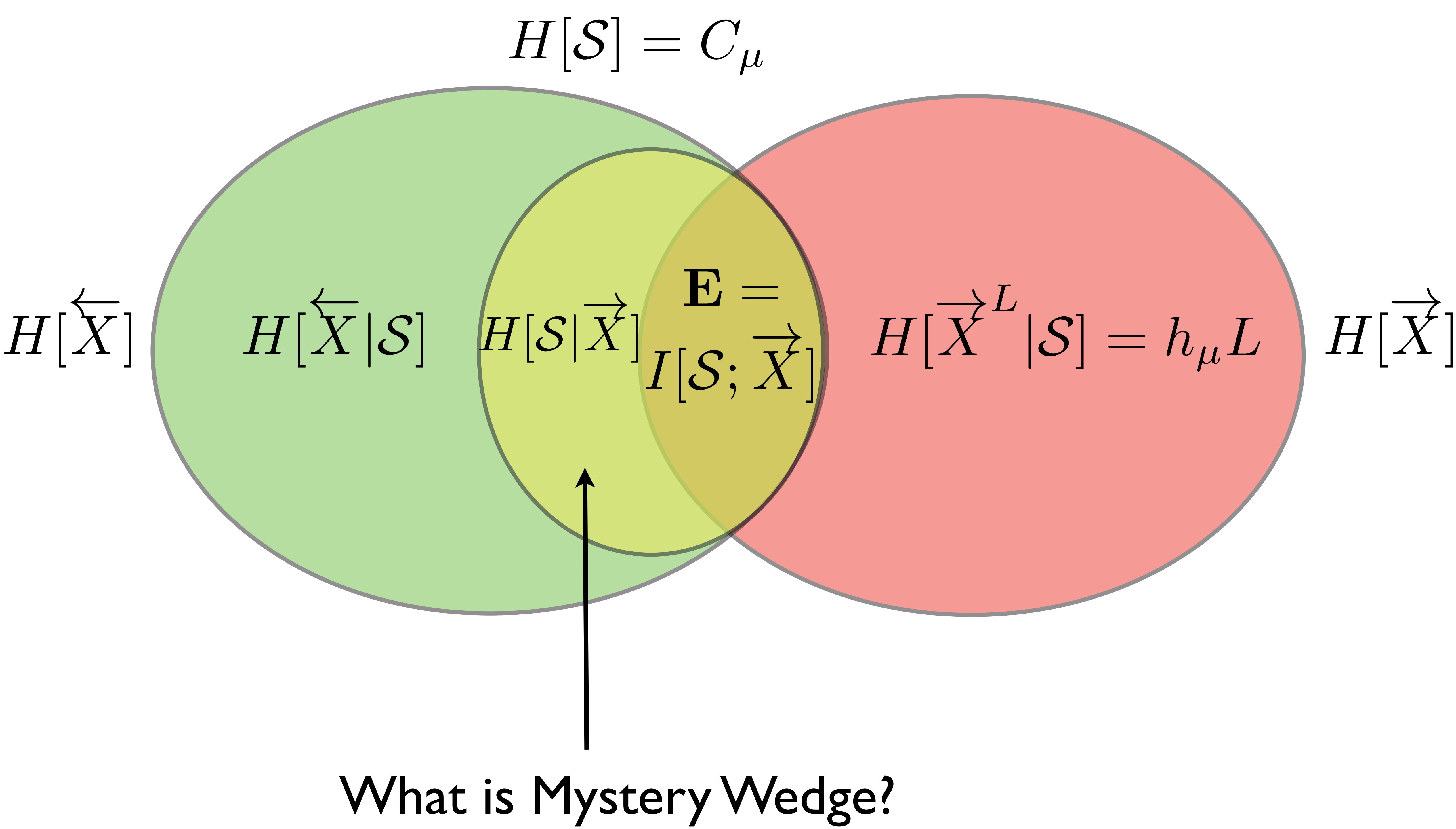
Unpredictability: $H[\overrightarrow{X}^L|\mathcal{S}] = Lh_\mu$

Proof Sketch:

$$
\begin{aligned}
H[\overrightarrow{X}^L|\mathcal{S}] &= H[\overrightarrow{X}^L|\overleftarrow{X}] \\
&= H[X_0 X_1 \ldots X_{L-1}|\overleftarrow{X}] \\
&= H[X_1 \ldots X_{L-1}|\overleftarrow{X} X_0] + H[X_0|\overleftarrow{X}] \\
&= H[X_1 \ldots X_{L-1}|\overleftarrow{X}] + H[X_0|\overleftarrow{X}] \\
&\;\vdots \\
&= H[X_{L-1}|\overleftarrow{X}] + \cdots + H[X_1|\overleftarrow{X}] + H[X_0|\overleftarrow{X}] \\
&= LH[X_0|\overleftarrow{X}] \\
&= Lh_\mu
\end{aligned}
$$

Information Diagrams for Processes

ε-Machine I-diagram:



Information Diagrams for Processes

What is Mystery Wedge? $H[\mathcal{S}|\overrightarrow{X}]$

Uncertainty of causal state given future. Implications?

Recall Bound on Excess Entropy: $\mathbf{E} \leq C_\mu$

Proof sketch:

$$
\begin{aligned}
\mathbf{E} &= I[\overleftarrow{X};\overrightarrow{X}] \\
&= H[\overrightarrow{X}] - H[\overrightarrow{X}|\overleftarrow{X}] \\
&= H[\overrightarrow{X}] - H[\overrightarrow{X}|\mathcal{S}] \\
&= I[\overrightarrow{X};\mathcal{S}] \\
&= H[\mathcal{S}] - H[\mathcal{S}|\overrightarrow{X}] \\
&\leq H[\mathcal{S}] \\
&= C_\mu \qquad \boxed{\circlearrowleft}
\end{aligned}
$$

I am the Mystery Wedge!

Wedge is the inaccessibility of hidden state information!

$$H[\mathcal{S}|\overrightarrow{X}] = C_\mu - \mathbf{E}$$

Wedge controls Internal - Observed

Information Diagrams for Processes

What is $H[\overleftarrow{X}|\overrightarrow{X}] + H[\overrightarrow{X}|\overleftarrow{X}]$?

Recall distance: $d(X,Y) = H[X|Y] + H[Y|X]$

So $H[\overrightarrow{X}|\overleftarrow{X}] + H[\overleftarrow{X}|\overrightarrow{X}]$

is the distance between the past and the future!

How related to eM?

From I-Diagram: $H[\overleftarrow{X}^L|\overrightarrow{X}^L] = H[\overleftarrow{X}^L|\mathcal{S}] + H[\mathcal{S}|\overrightarrow{X}^L]$

$$H[\overrightarrow{X}^L|\overleftarrow{X}^L] = H[\overrightarrow{X}^L|\mathcal{S}] = h_\mu L$$

So: $d(\overleftarrow{X}^L, \overrightarrow{X}^L) = H[\overleftarrow{X}^L|\overrightarrow{X}^L] + H[\overrightarrow{X}^L|\overleftarrow{X}^L]$

$$= H[\overleftarrow{X}^L|\mathcal{S}] + H[\mathcal{S}|\overrightarrow{X}^L] + h_\mu L$$

Retrodiction — Mystery Wedge — Prediction Error

Information Diagrams for Processes

Mystery wedge? $H[\mathcal{S}|\overrightarrow{X}]$

Retrodiction?

Need extended framework for analyzing:

Temporal asymmetry

Reverse time view of process

Next week:

Mixed states!

A key tool.

Information Diagrams for Processes

Reading for next lecture: CMR articles

TBA

PRATISP

IACP

IACPLCOCS