Memory in Processes I

Reading for this lecture:

CMR article RURO.

Interactive CMPy lab:

Block Entropy Curves

Memory in Processes I ...

Motivation:

Previous: Measures of randomness of information source
Block entropy $H(L)$
Entropy rate h_μ

End point of next lectures:
Measures of memory & information storage

Big Picture:
Complementary properties of a source.
Need both: Measures of randomness *and* structure.

How random?

Block entropy growth: $H(L)$.

If L is large enough, we see linear rate of increase of $H(L)$:

$$H(L) \propto h_\mu L$$

which is the entropy rate:

$$h_\mu = \lim_{L \to \infty} (H(L) - H(L-1))$$

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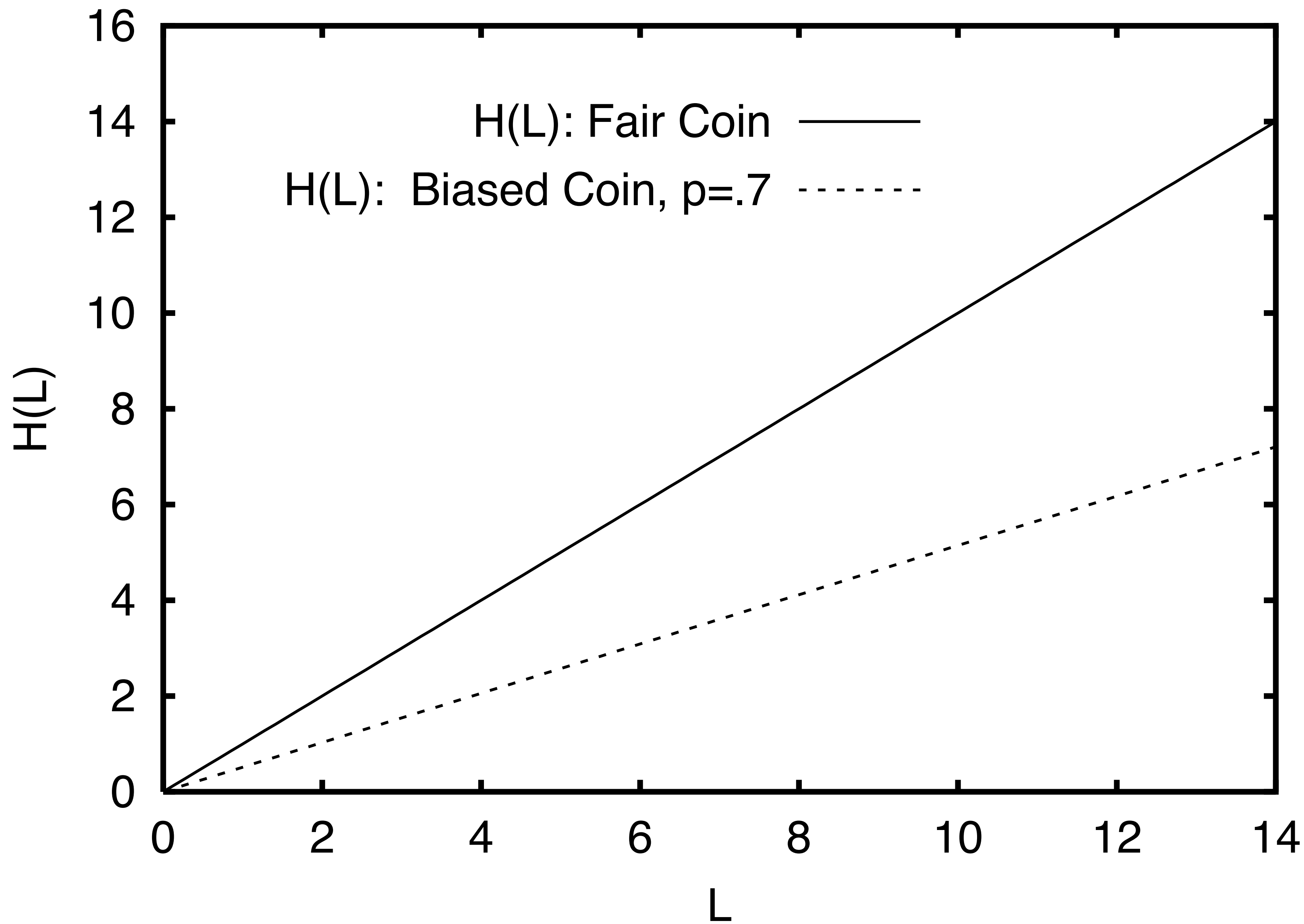
How random ...

Fair and Biased Coins:



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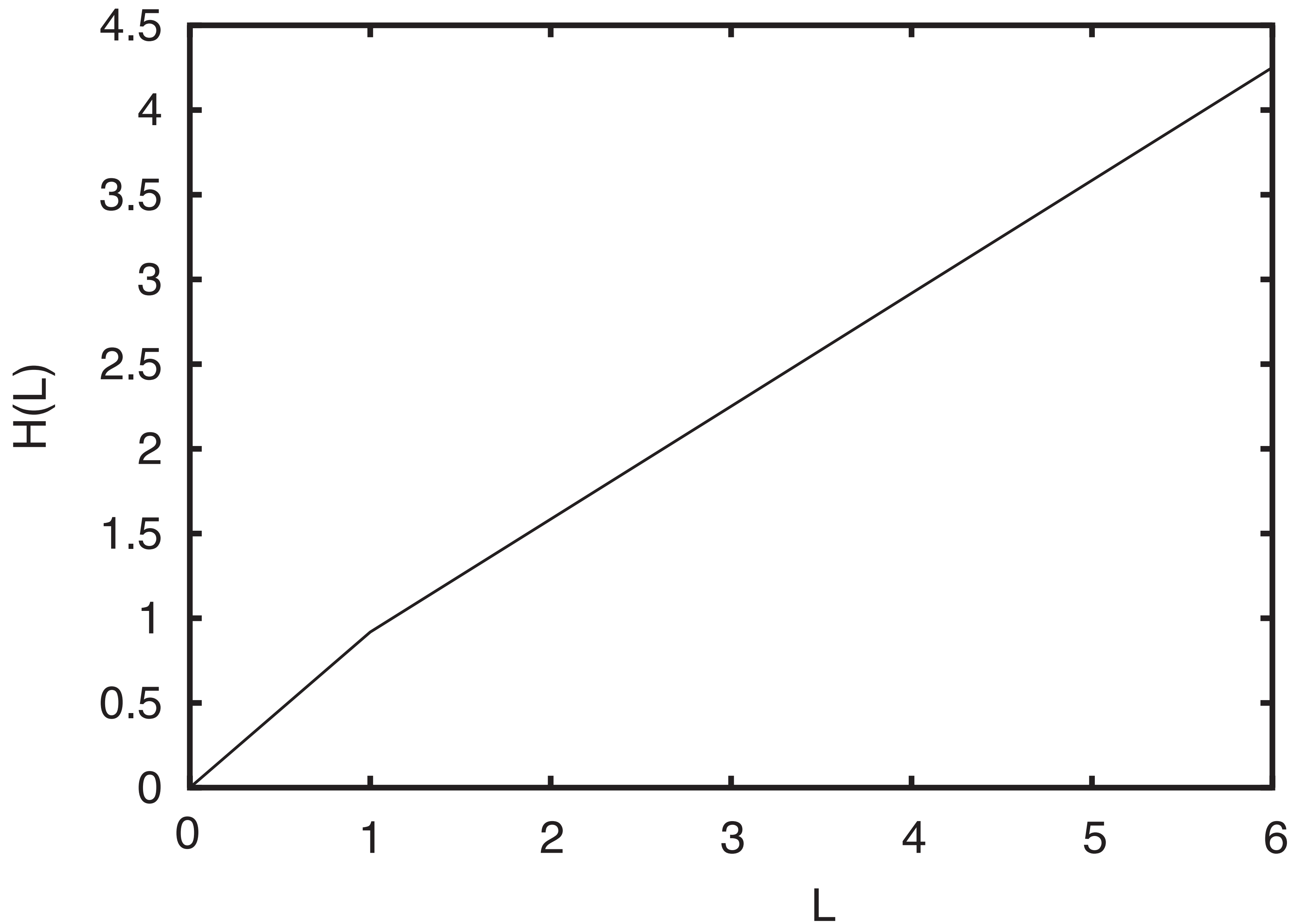
How random ...

Golden Mean Process:



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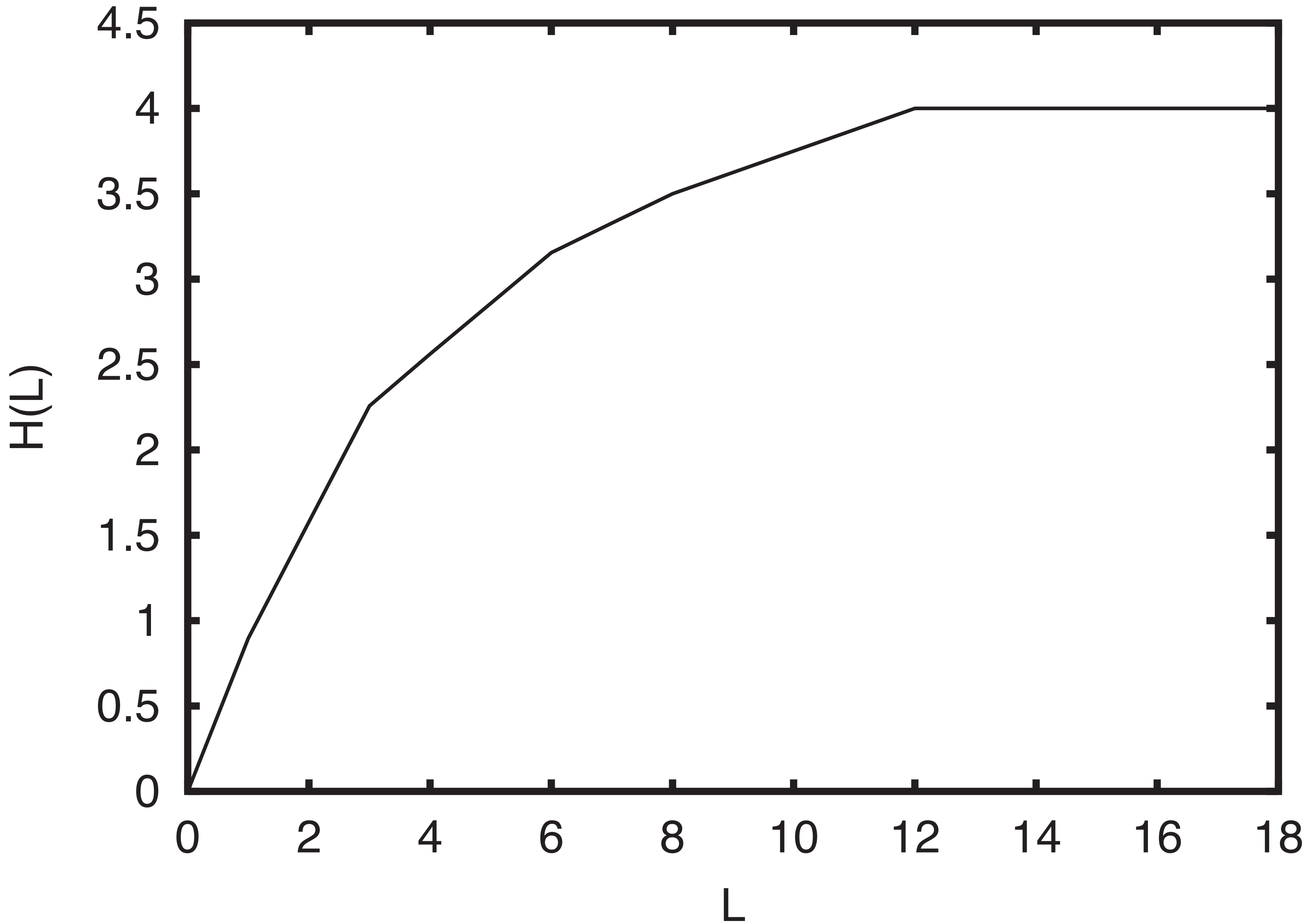
How random ...

Period-16 Process:



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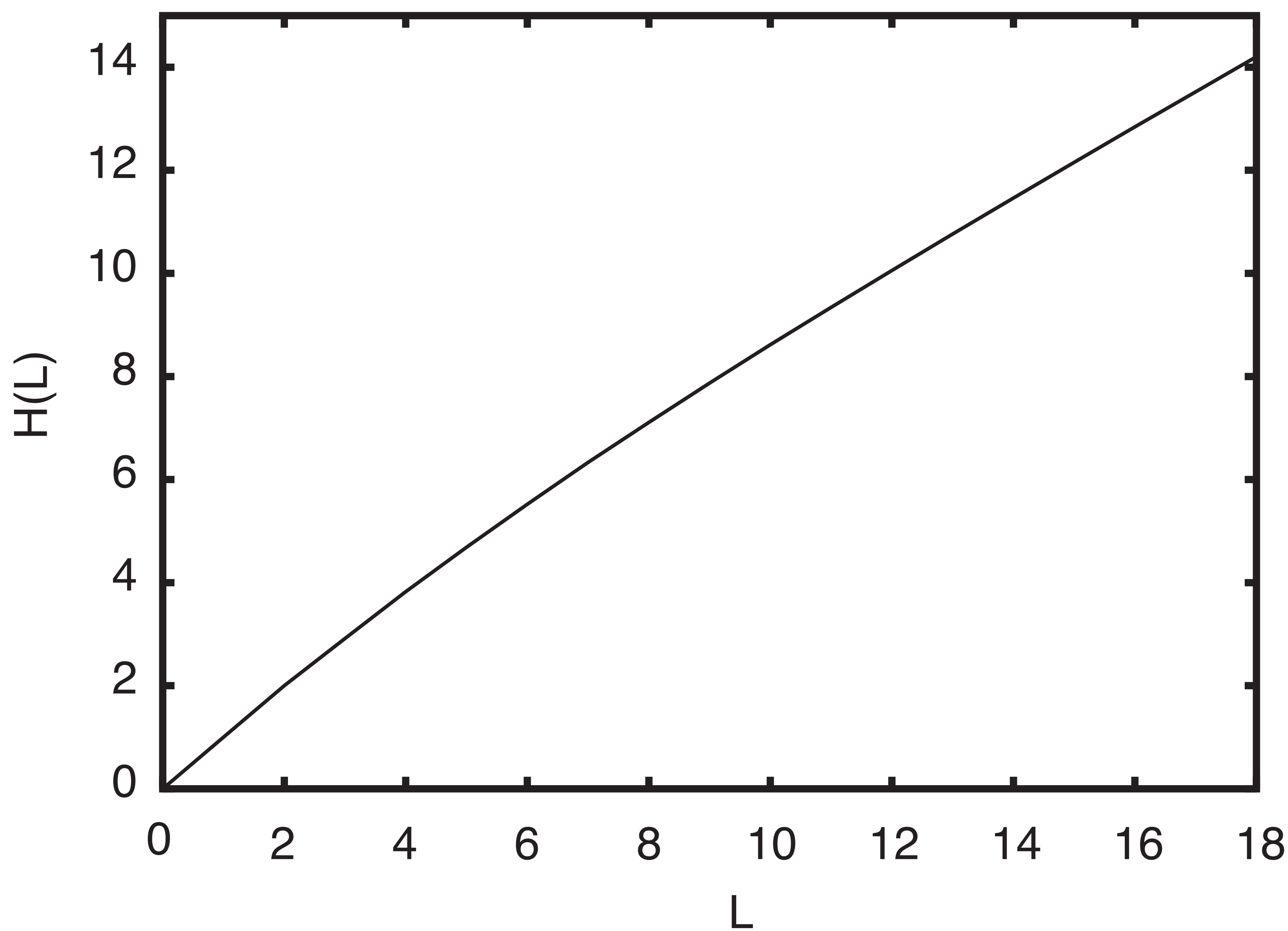
How random ...

RRXOR Process:



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How random ...

How large must L be to see the intrinsic randomness h_μ?

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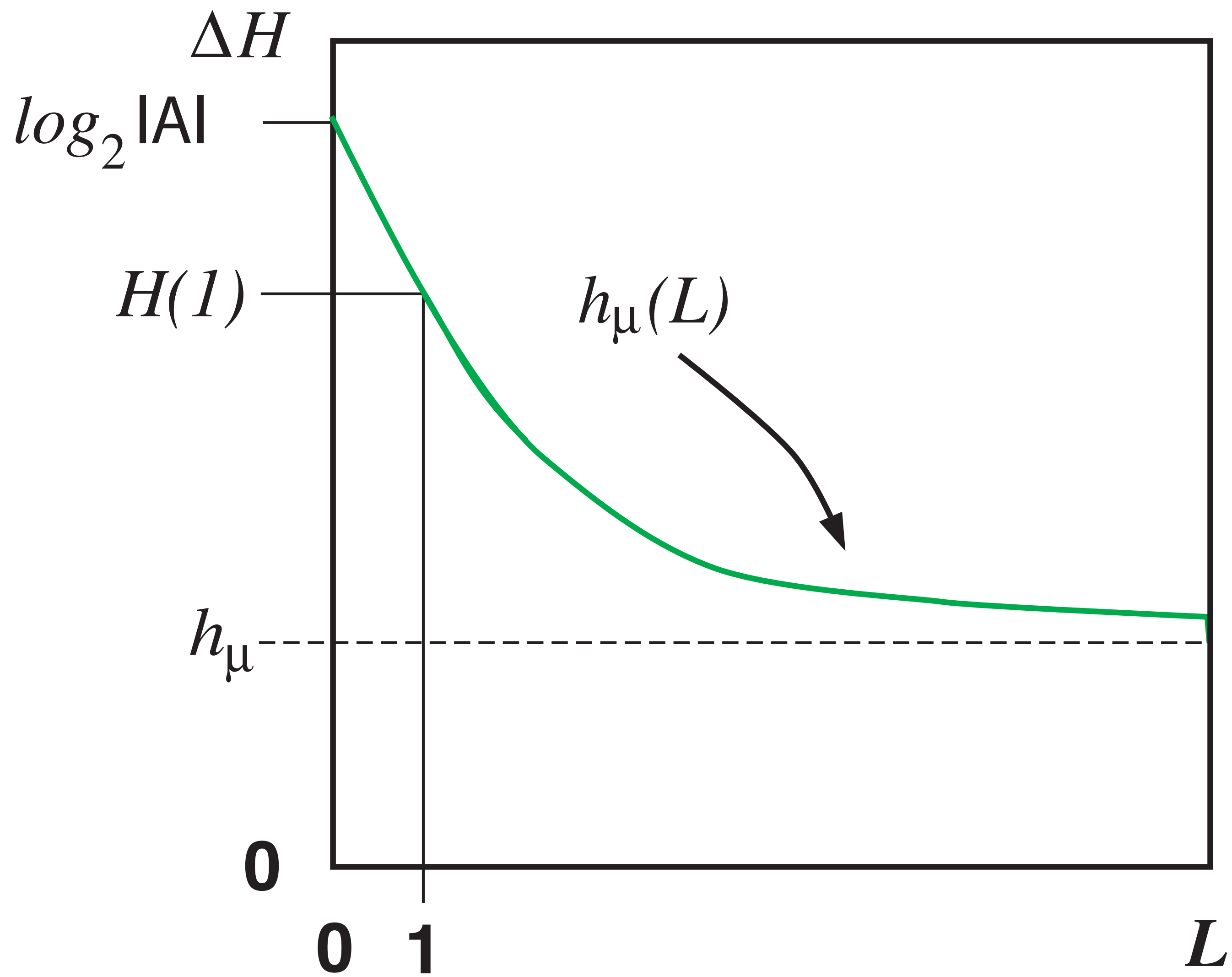
Entropy Convergence:

Length-L entropy rate estimate:

$$h_\mu(L) = H(L) - H(L-1)$$

$$h_\mu(L) = \Delta H(L)$$



Monotonic decreasing:

$$h_\mu(L) \leq h_\mu(L-1)$$

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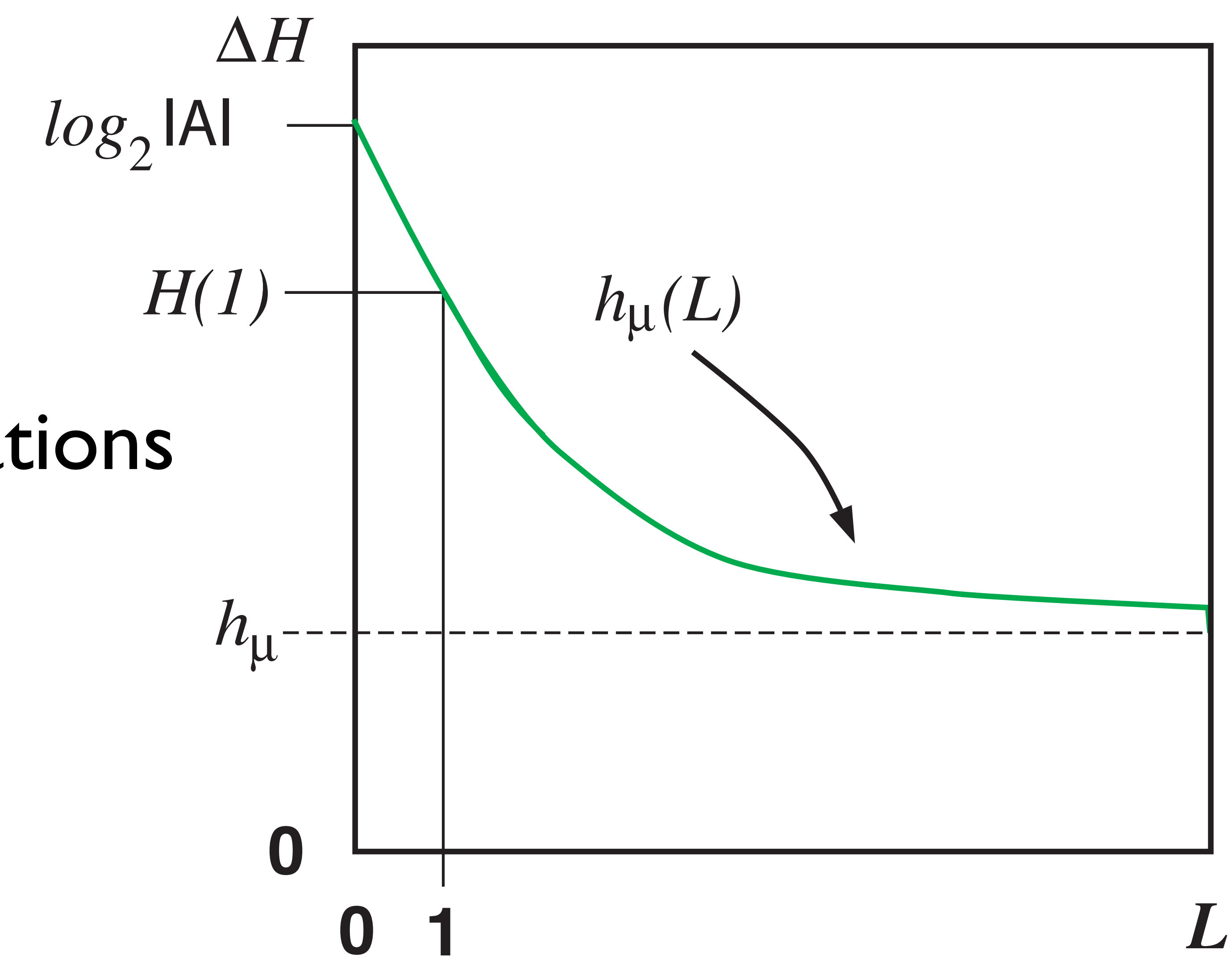
Entropy Convergence ...



Process appears less random
as account for longer correlations

Entropy (rate) Loss is an Information Gain:

$$h_\mu(L) = \mathcal{D}(\Pr(s^L) || \Pr(s^{L-1}))$$

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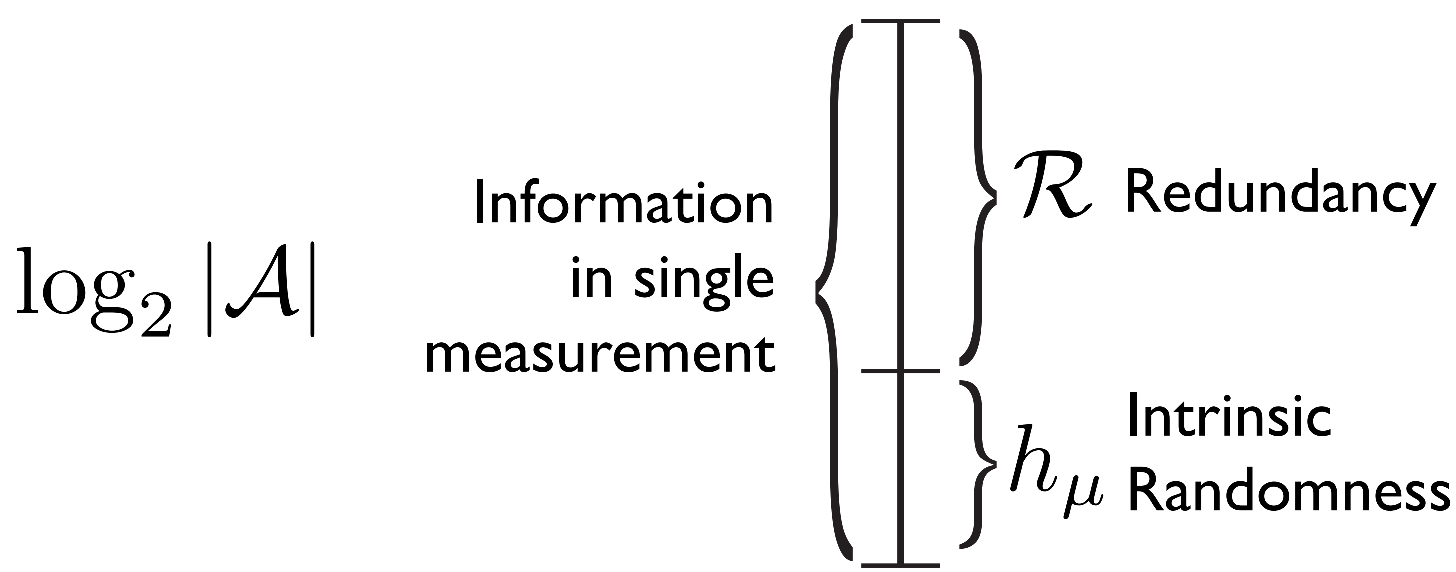
Redundancy in Processes:

$$\mathcal{R} = \log_2 |\mathcal{A}| - h_\mu$$

Anatomy of Measurement:



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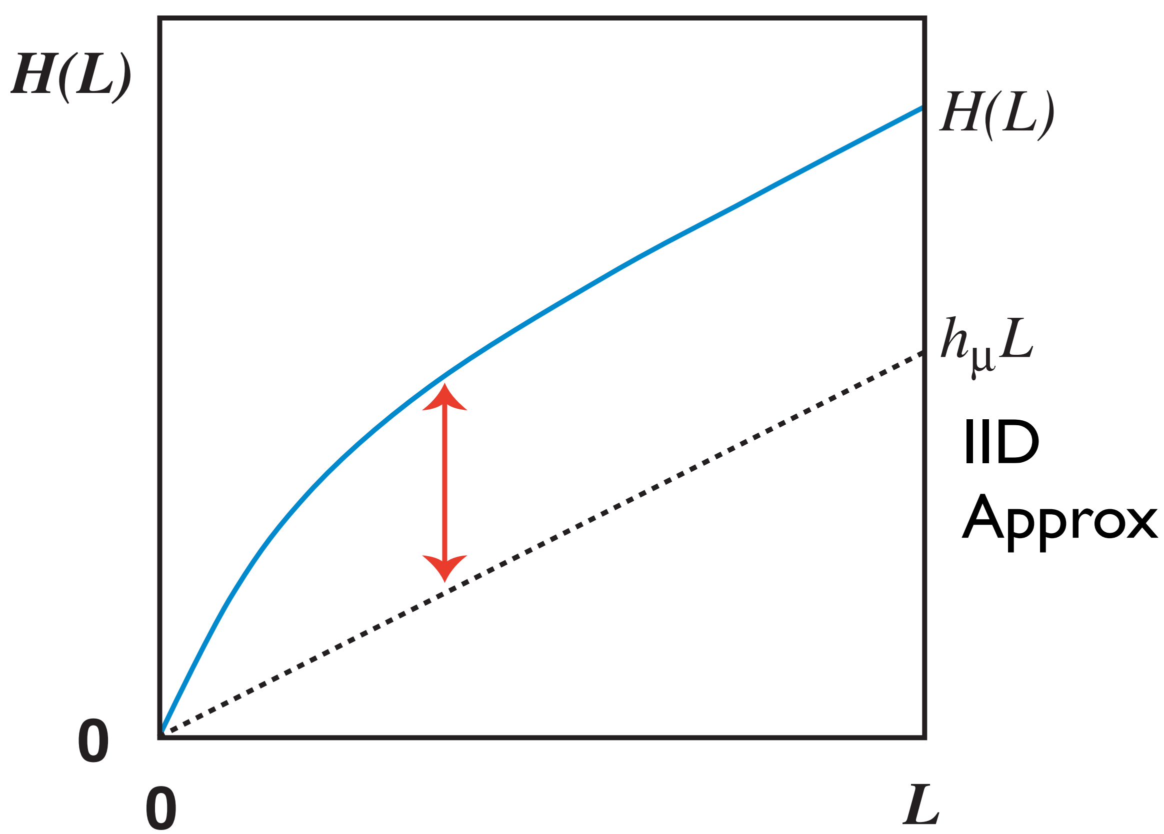
Redundancy in Processes ...

$$\mathcal{R} = \lim_{L \to \infty} \mathcal{D}(\Pr(s^L) || U(s^L))$$

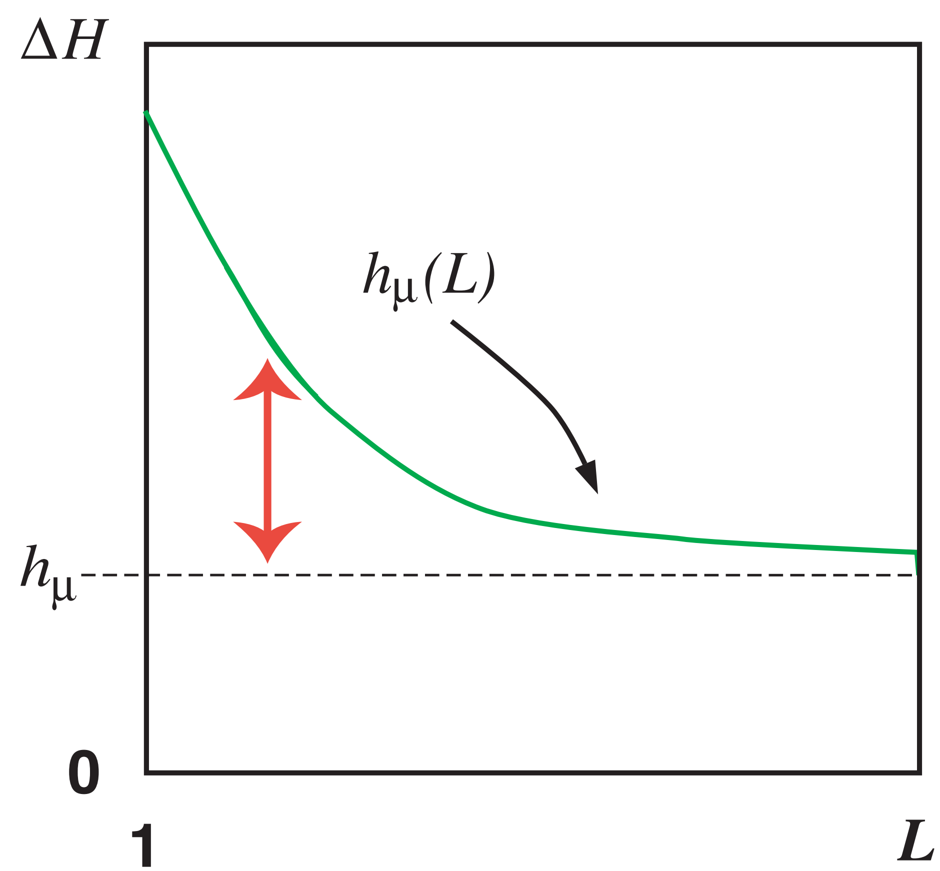
Redundancy in words:

$$\mathcal{R}(L) = H(L) - h_\mu L$$



Redundancy per symbol:

$$r(L) = \mathcal{R}(L) - \mathcal{R}(L-1) = h_\mu(L) - h_\mu$$



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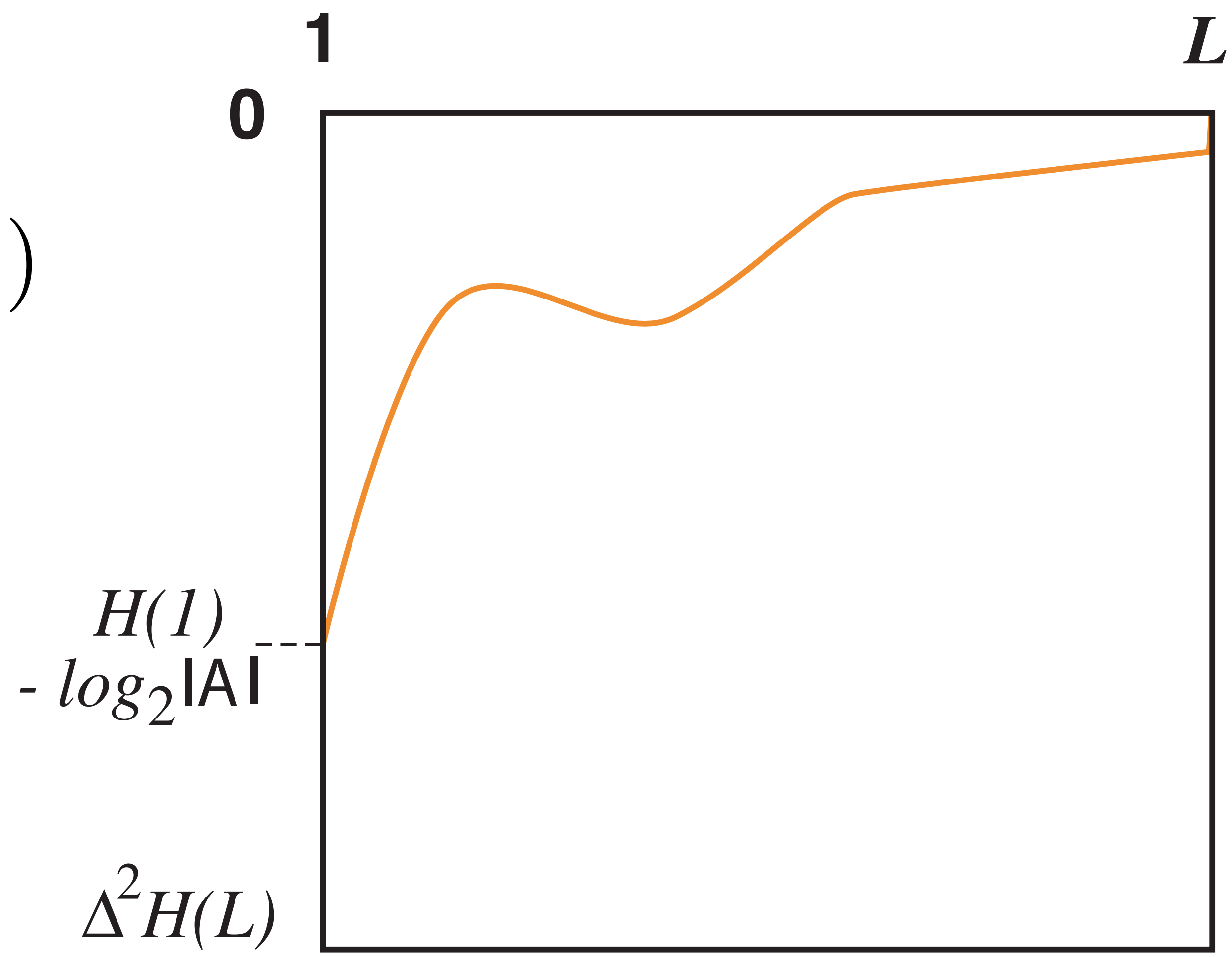
Predictability Gain:

$$\Delta^2 H(L) = h_\mu(L) - h_\mu(L-1)$$



Boundary condition:

$$\Delta^2 H(1) = H(1) - \log_2 |\mathcal{A}|$$

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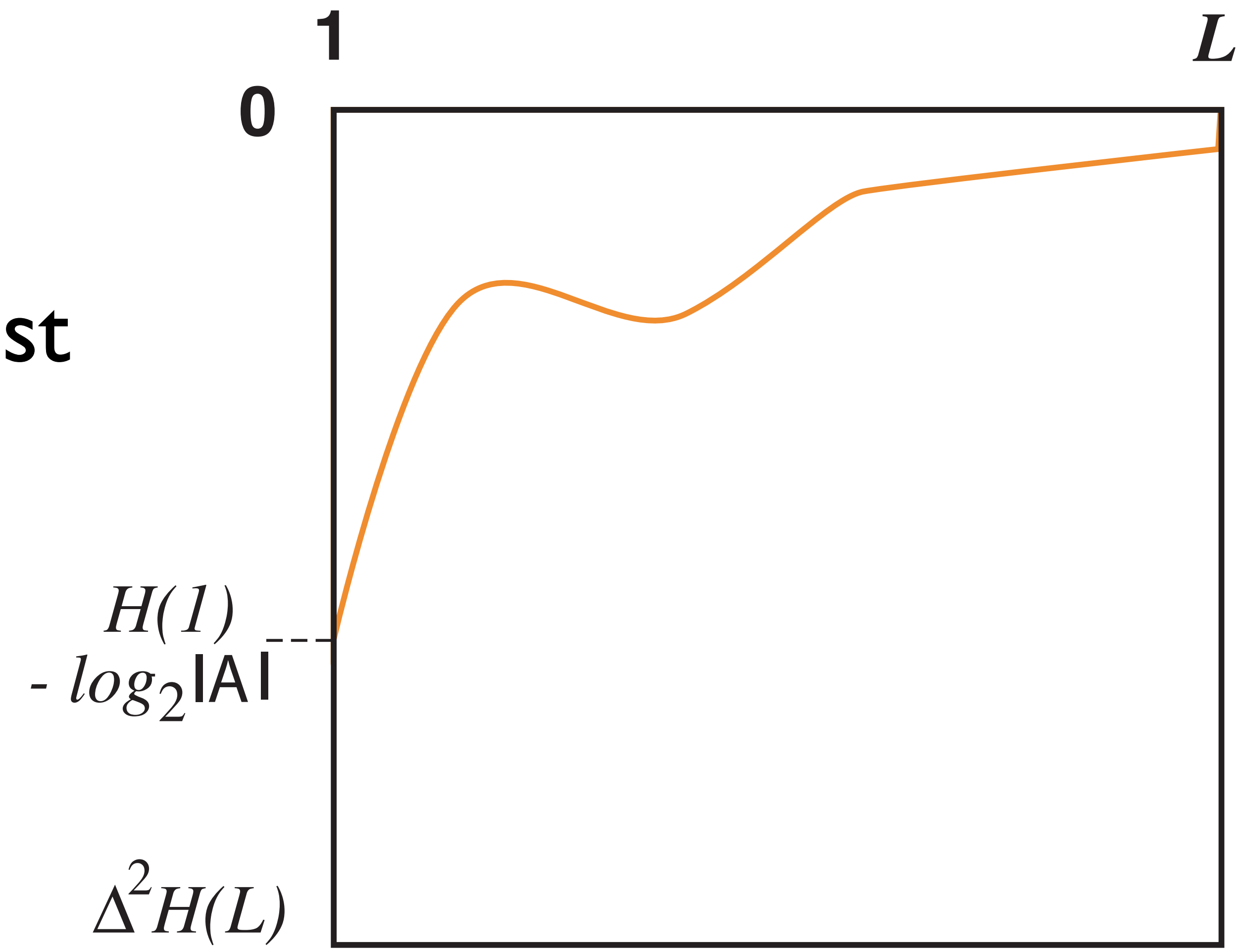
Predictability Gain ...

Rate at which unpredictability is lost



Properties:

(1) $H(L)$ Curvature:

$$\Delta^2 H(L) = H(L) - 2H(L-1) + H(L-2)$$

(2) $H(L)$ Concavity:

$$\Delta^2 H(L) \leq 0$$

(3) $|\Delta^2 H(L)| \gg 1 \Rightarrow$ L[th] measurement significant

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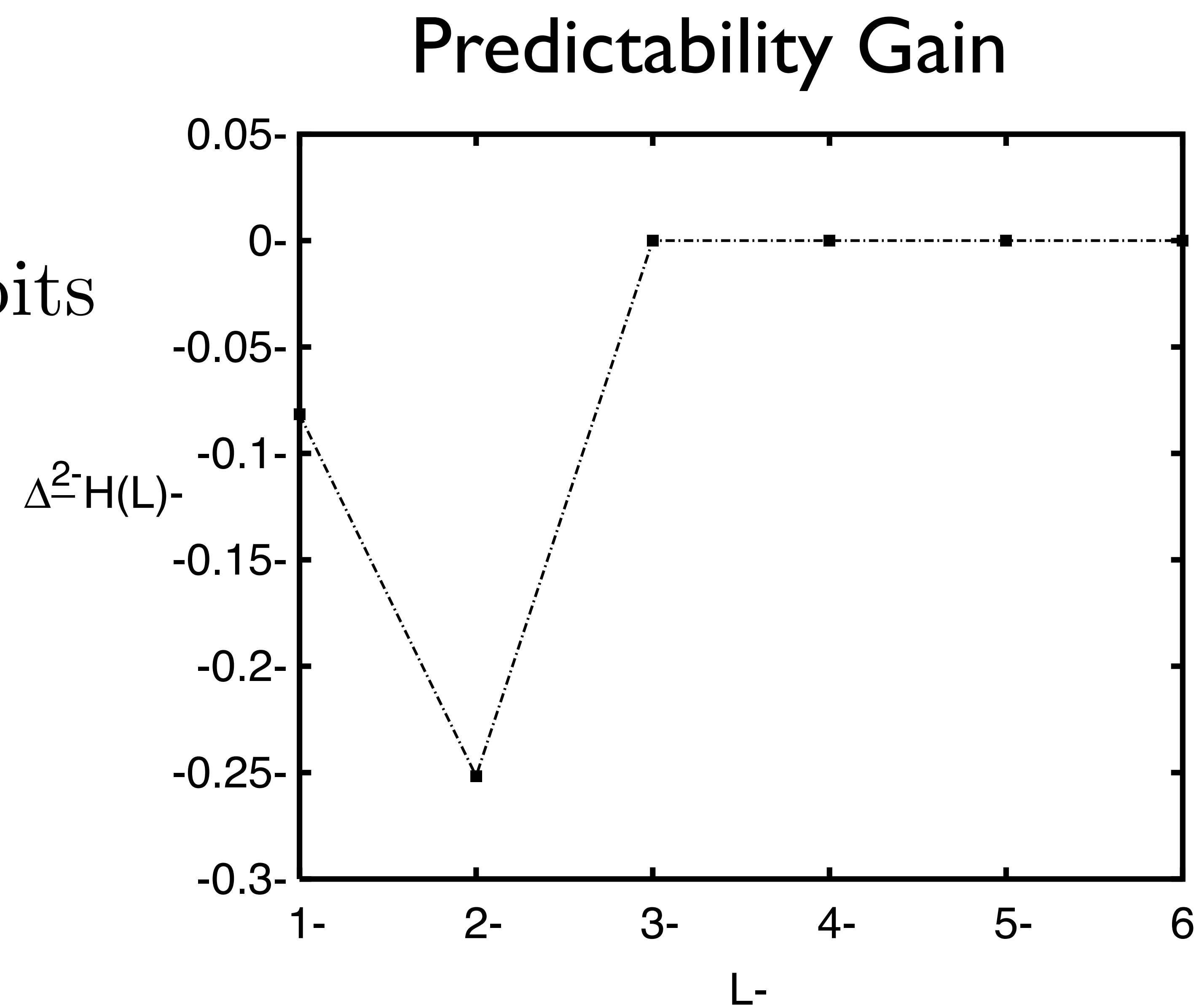
Predictability Gain ...

Golden Mean Process:

$$\Delta^2 H(2) = -0.2516 \text{ bits}$$



Second measurement is informative:
00 restriction observed

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Entropy Hierarchy:

Take derivatives:

(1) Block entropy: $H(L)$

(2) Entropy rate: $h_\mu(L) = \Delta H(L)$

(3) Predictability gain: $\Delta h_\mu(L) = \Delta^2 H(L)$

Now take integrals!

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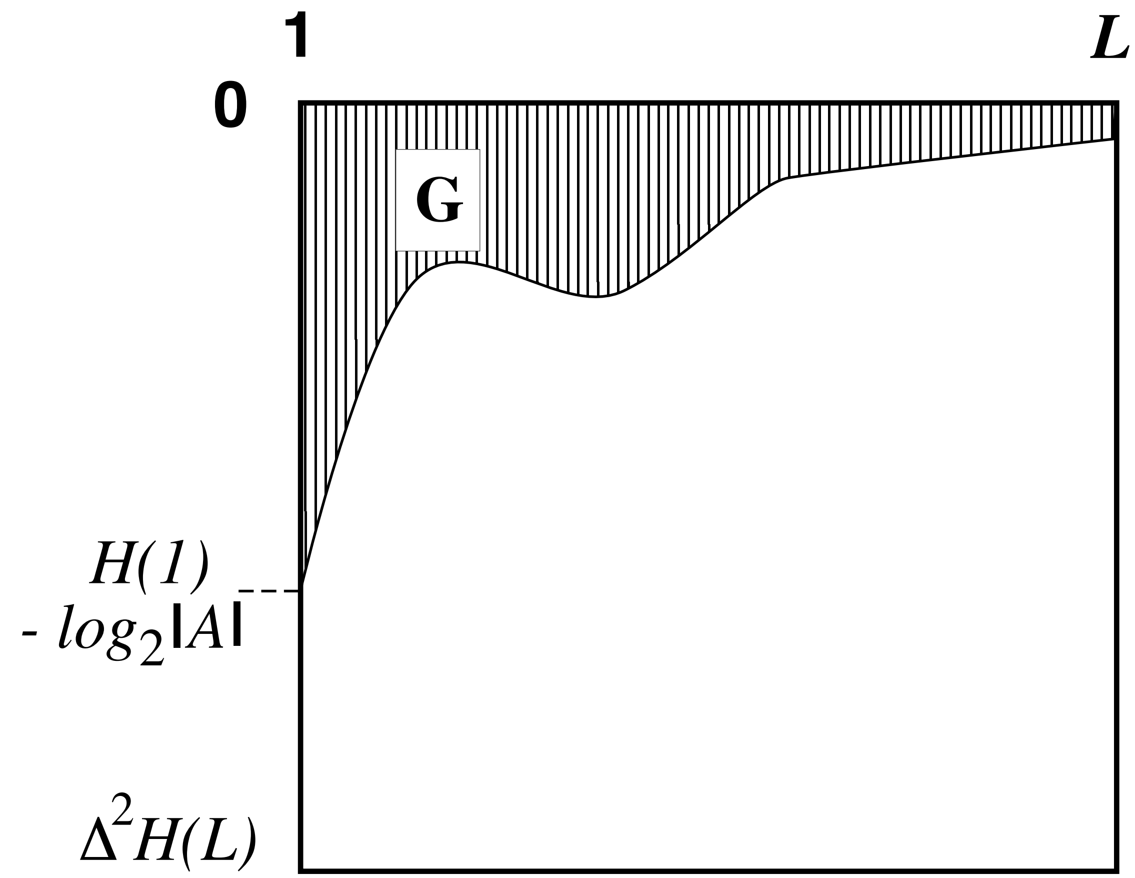
Total Predictability:

$$\mathbf{G} = \sum_{L=1}^{\infty} \Delta^2 H(L)$$



Redundancy:

$$-\mathbf{G} = \mathcal{R} = \log_2 |\mathcal{A}| - h_\mu$$

Interpretation:

(1) Account for all correlations to see intrinsic randomness

(2) Until that point, correlations appear as *excess randomness*

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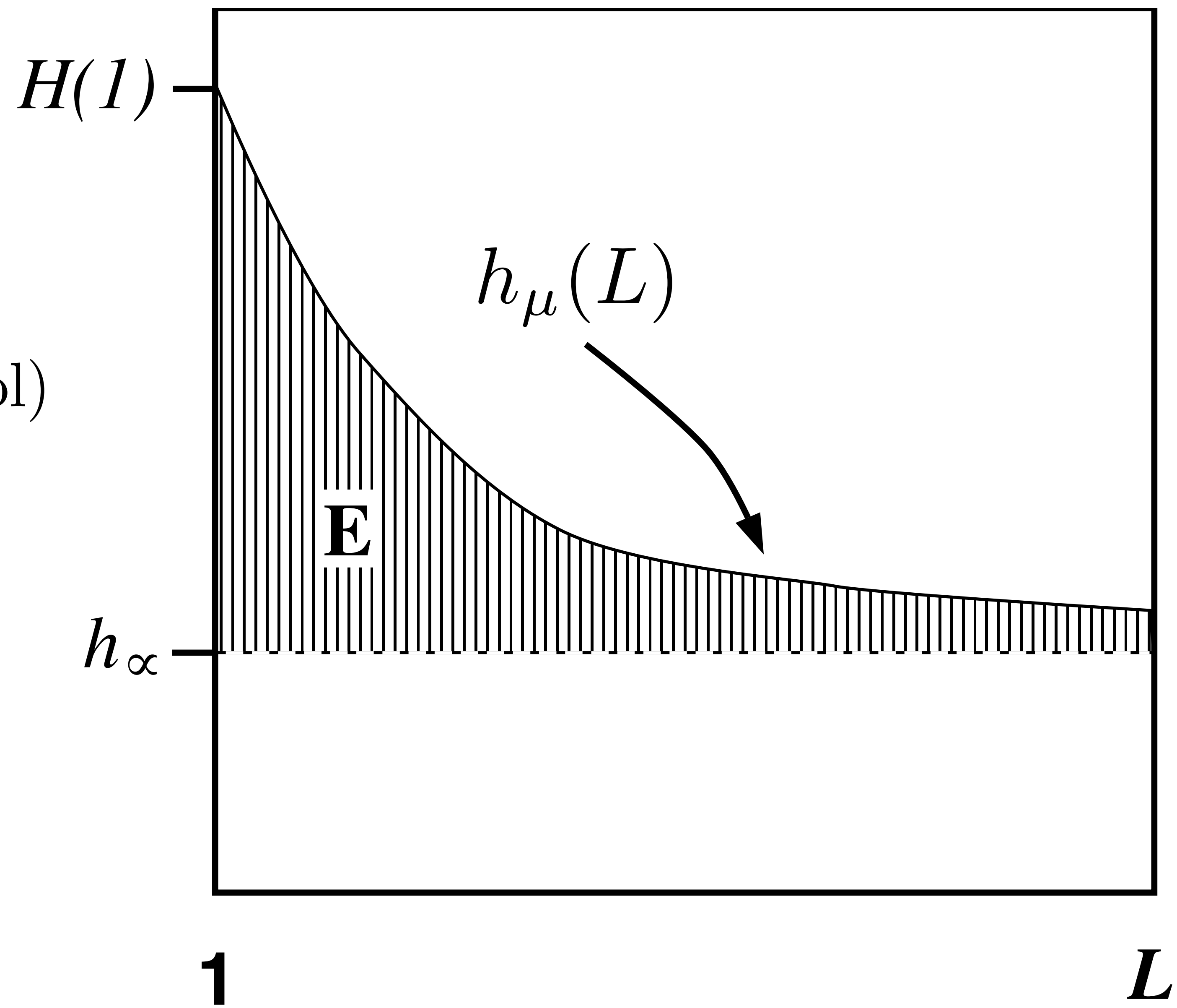
Excess Entropy:

As entropy convergence:

$$\mathbf{E} = \sum_{L=1}^{\infty} [h_\mu(L) - h_\mu]$$

$(\Delta L = 1 \text{ symbol})$



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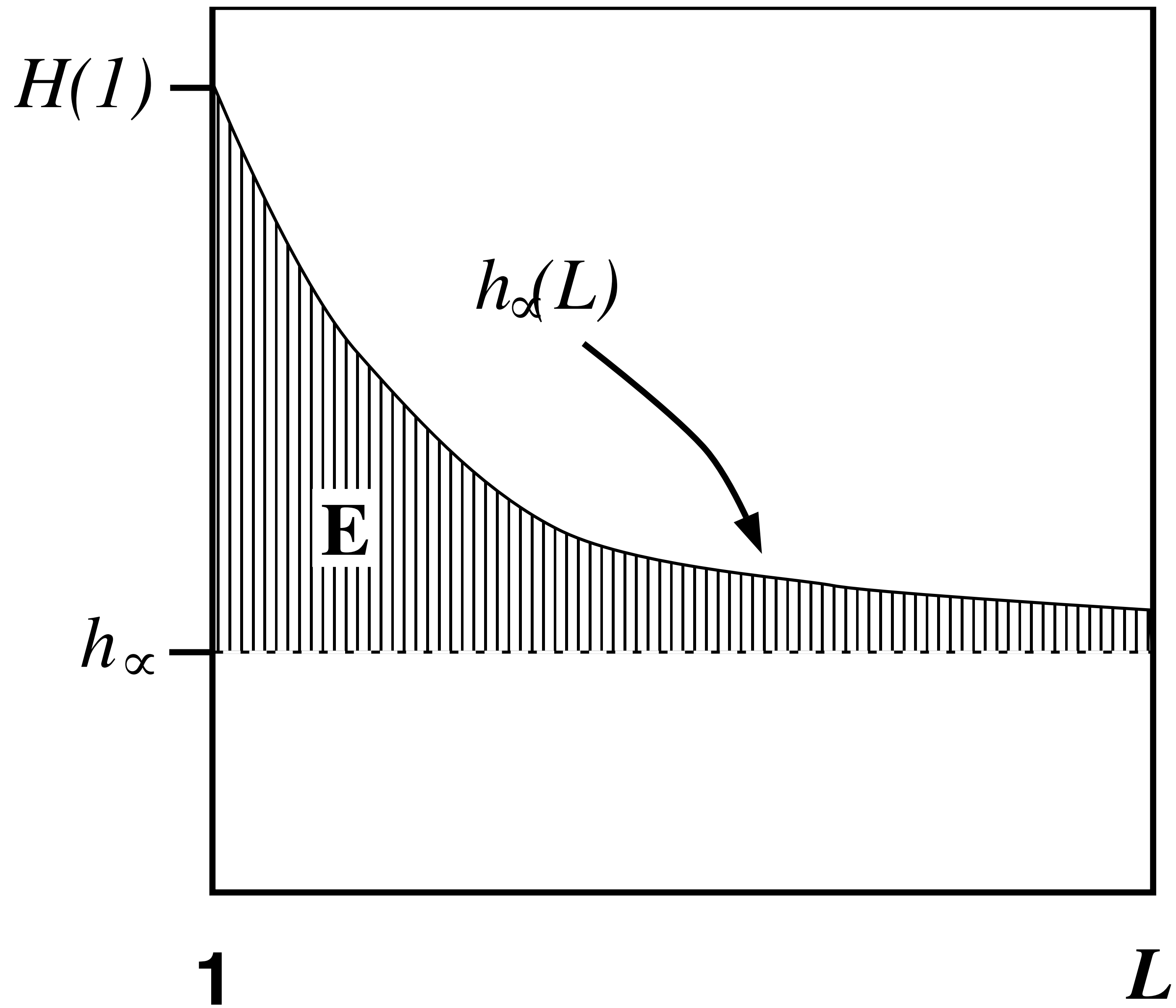
Excess Entropy ...

As intrinsic redundancy:

$$\mathbf{E} = \sum_{L=1}^{\infty} r(L)$$



Properties:

(1) Units: $\mathbf{E} = [\text{bits}]$

(2) Positive: $\mathbf{E} \geq 0$

(3) Controls convergence to actual randomness.

(4) Slow convergence ⇔ Correlations at longer words.

(5) Complementary to entropy rate.

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Excess Entropy ...

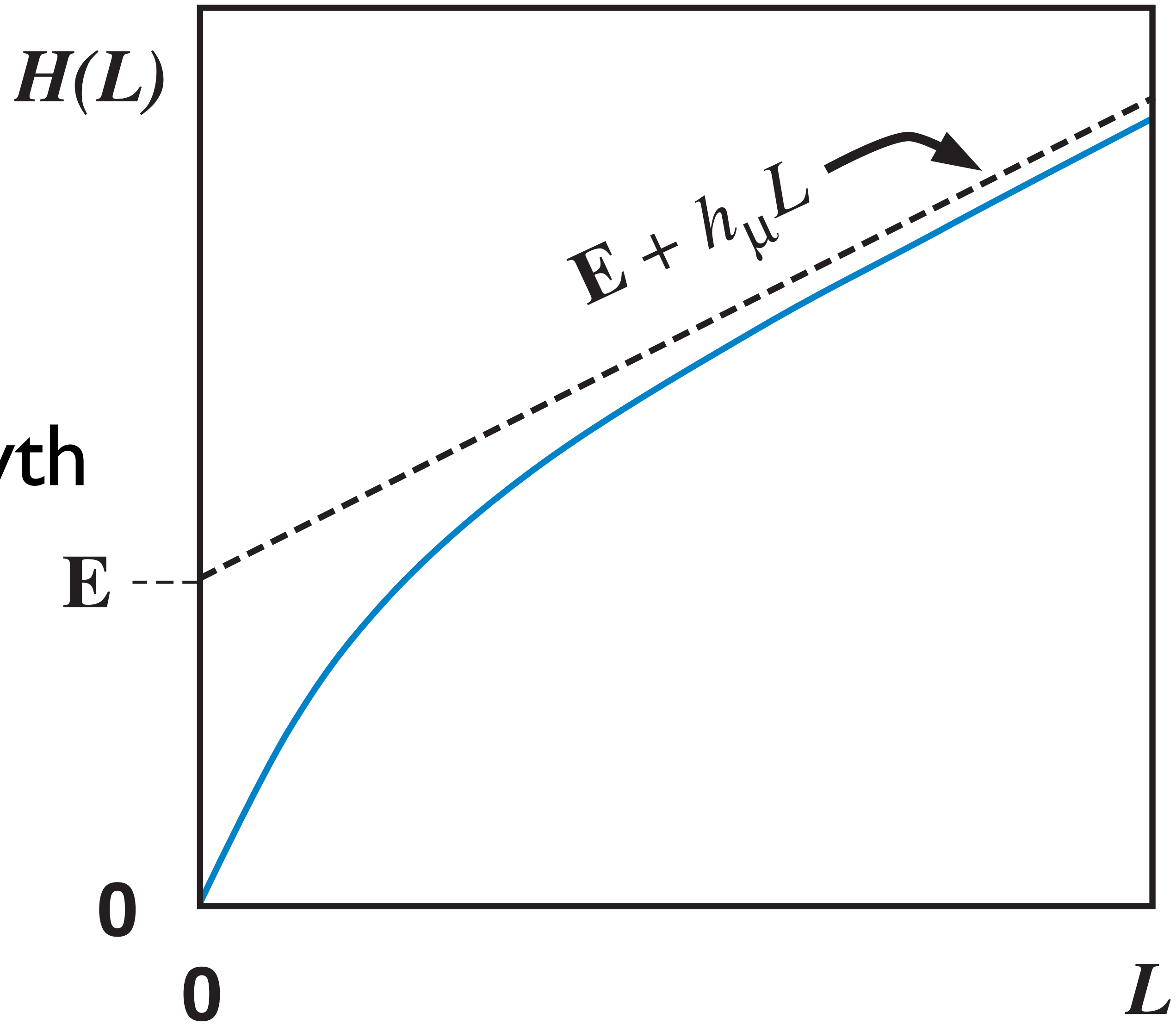
Asymptote of entropy growth:

$$\mathbf{E} = \lim_{L \to \infty} [H(L) - h_\mu L]$$

That is,

$$H(L) \propto \mathbf{E} + h_\mu L$$

Y-Intercept of entropy growth



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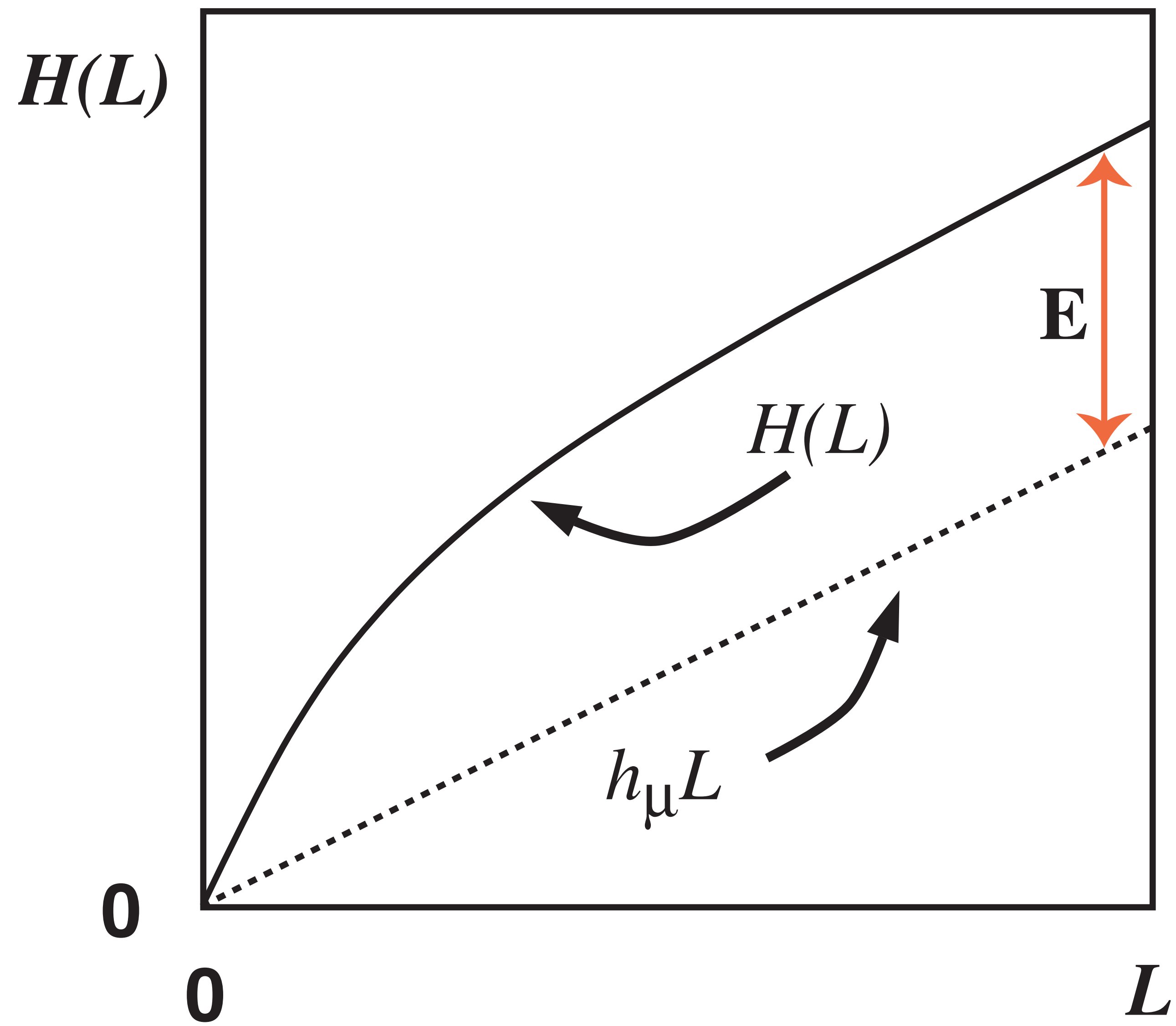
Excess Entropy ...

Cost of Amnesia:

Forget what you know:

Information needed to recover predicting with error ~h_μ



Cf. Memoryless Source: IID at same entropy rate

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Excess Entropy ...

Mutual information between past and future:

View process as a communication channel: Past to Future

$$\mathbf{E} = I(\overleftarrow{S}; \overrightarrow{S})$$

Property:

Symmetric in time

Interpretation:

Information that process communicates from past to future.

Reduction in uncertainty about the future, given the past.

Reduction in uncertainty about the past, given the future.

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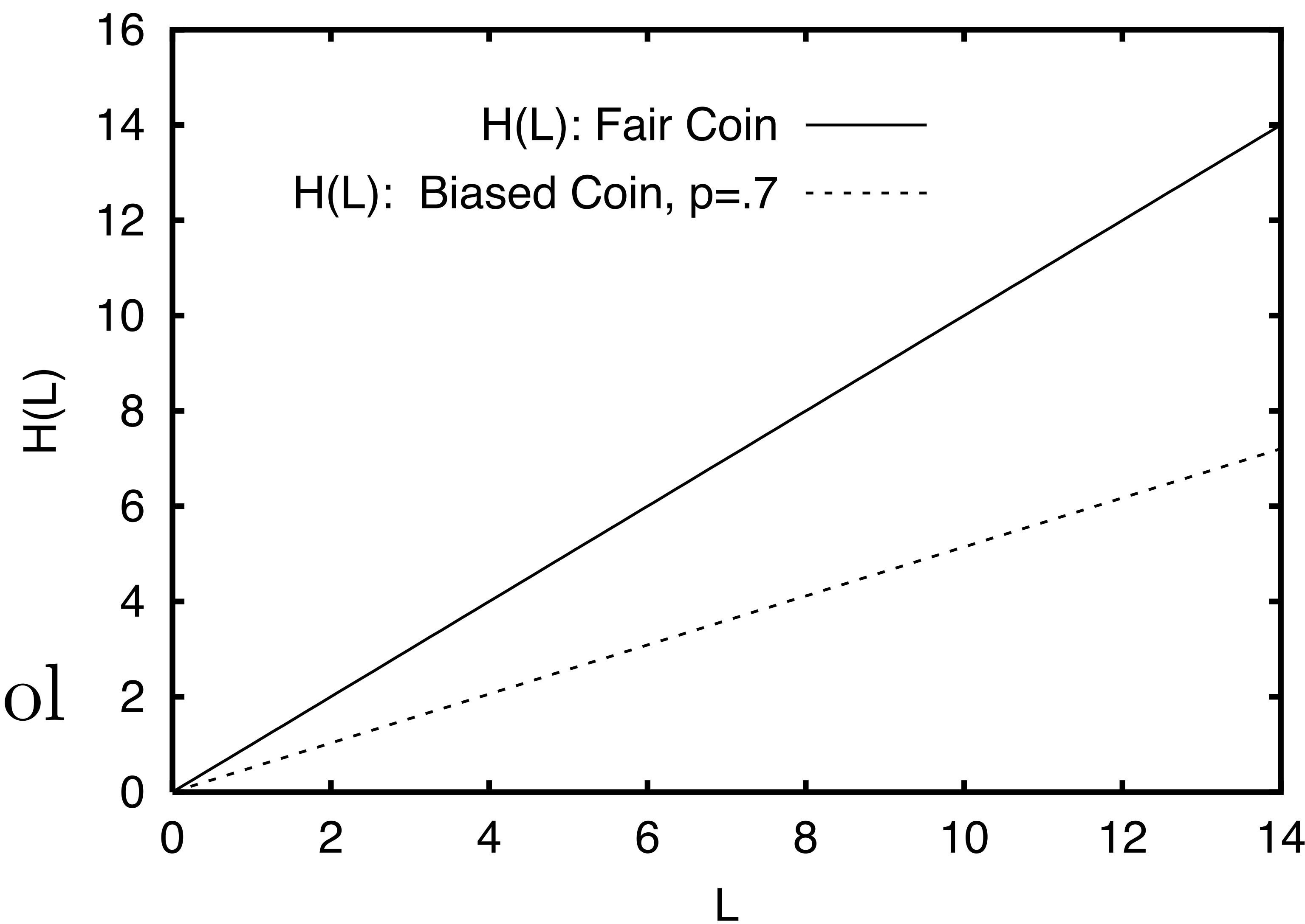
Examples of Excess Entropy:



Fair Coin:

$$h_\mu = 1 \text{ bit per symbol}$$

$$\mathbf{E} = 0 \text{ bits}$$

Biased Coin:

$$h_\mu = H(p) \text{ bits per symbol}$$

$$\mathbf{E} = 0 \text{ bits}$$

Any IID Process:

$$h_\mu = H(X) \text{ bits per symbol}$$

$$\mathbf{E} = 0 \text{ bits}$$

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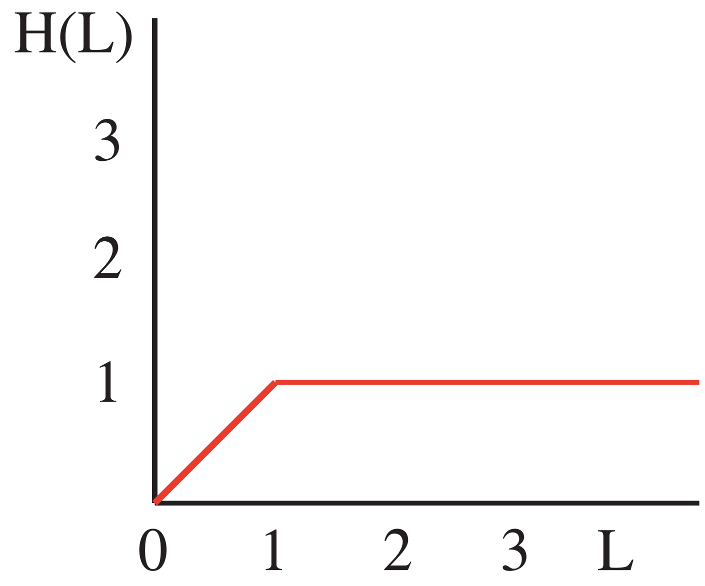
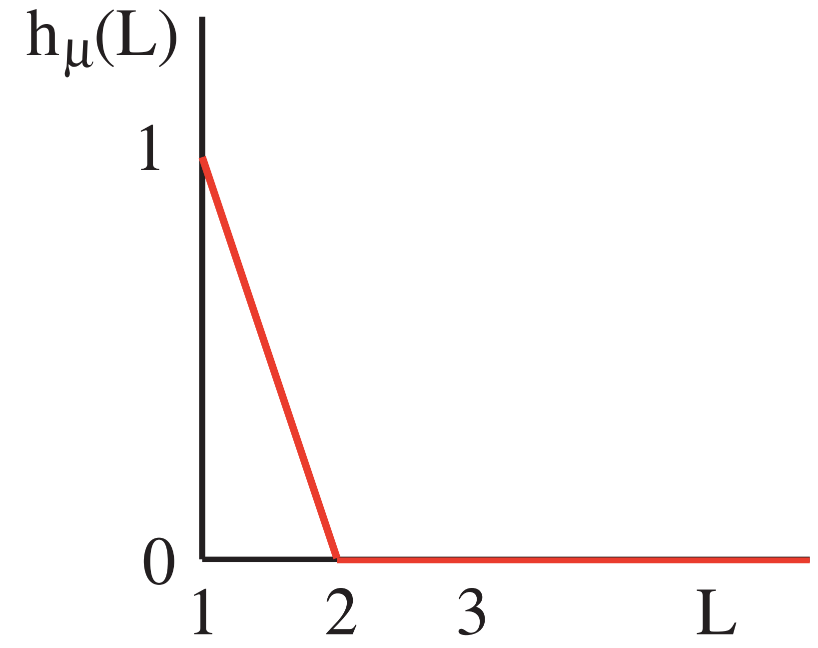
Examples of Excess Entropy ...

Period-2 Process: 010101010101

$$H(1) = 1$$
$$H(2) = 1$$
$$H(3) = 1$$



$$h_\mu(1) = 1$$
$$h_\mu(2) = 0$$
$$h_\mu(3) = 0$$



$$h_\mu = 0 \text{ bits per symbol}$$

$$\mathbf{E} = 1 \text{ bit}$$

Meaning:

1 bit of phase information

0-phase or 1-phase?

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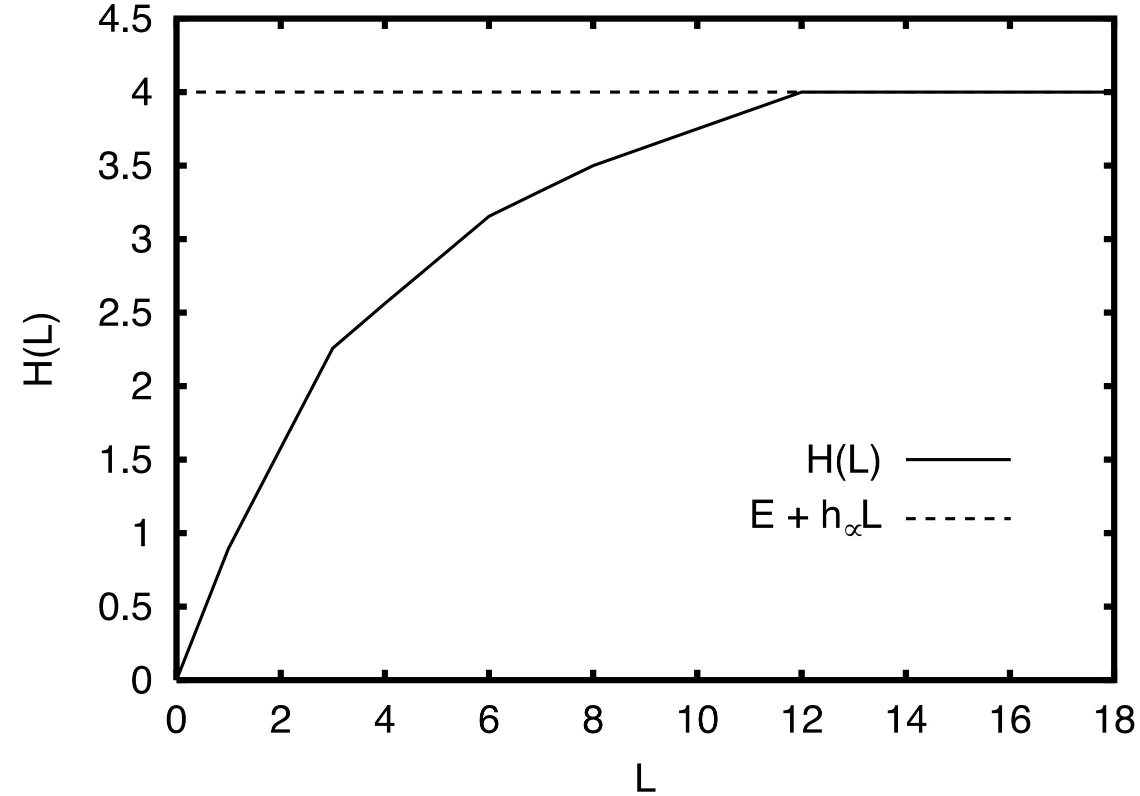
Examples of Excess Entropy ...

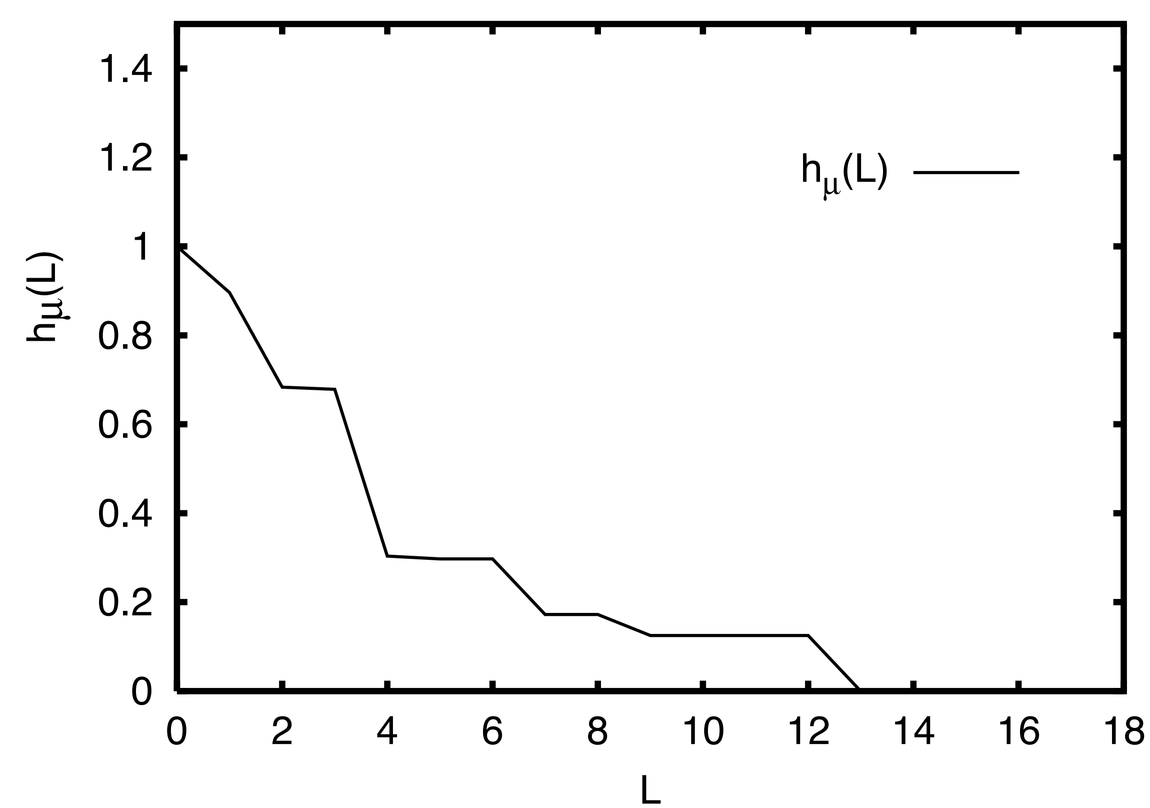
Period-16 Process:

$$(1010111011101110)^\infty$$

$$h_\mu = 0 \text{ bits per symbol}$$

$$\mathbf{E} = 4 \text{ bits}$$





Period-P Processes:

$$h_\mu = 0 \text{ bits per symbol}$$

$$\mathbf{E} = \log_2 P \text{ bits}$$

Cf., entropy rate does not distinguish periodic processes!

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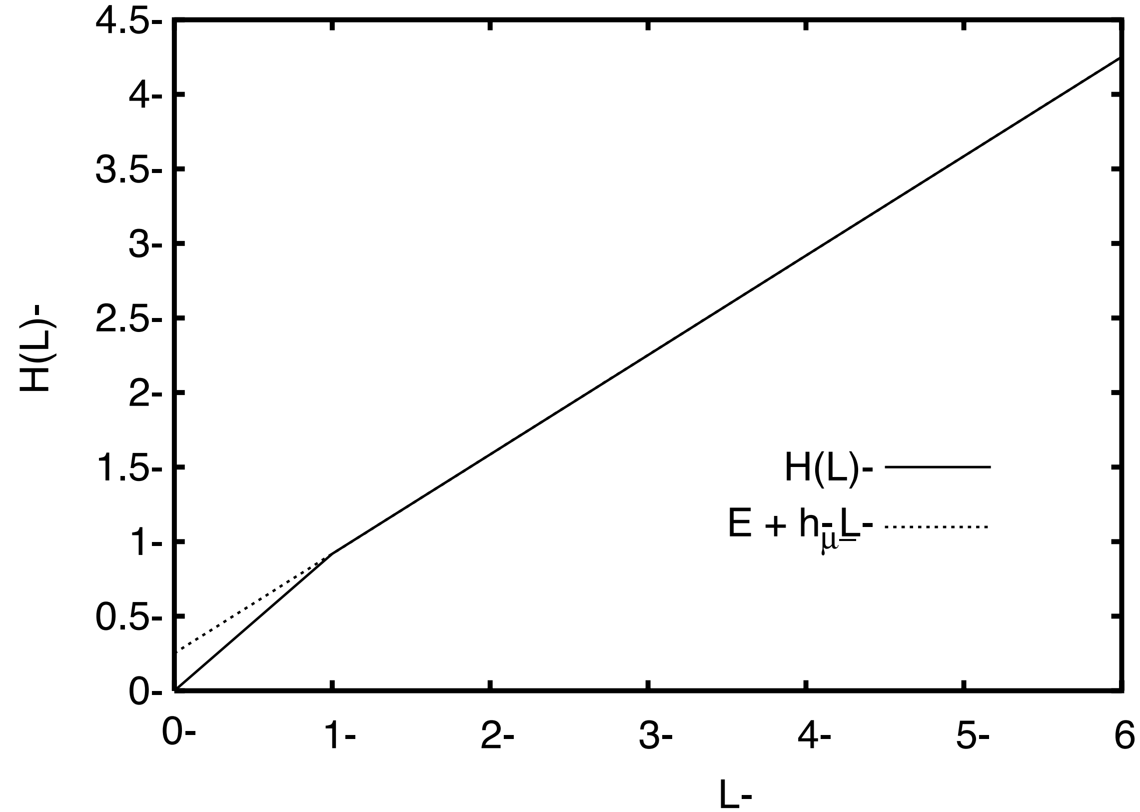
Examples of Excess Entropy ...

Golden Mean Process:

$$h_\mu = \frac{2}{3} \text{ bits per symbol}$$

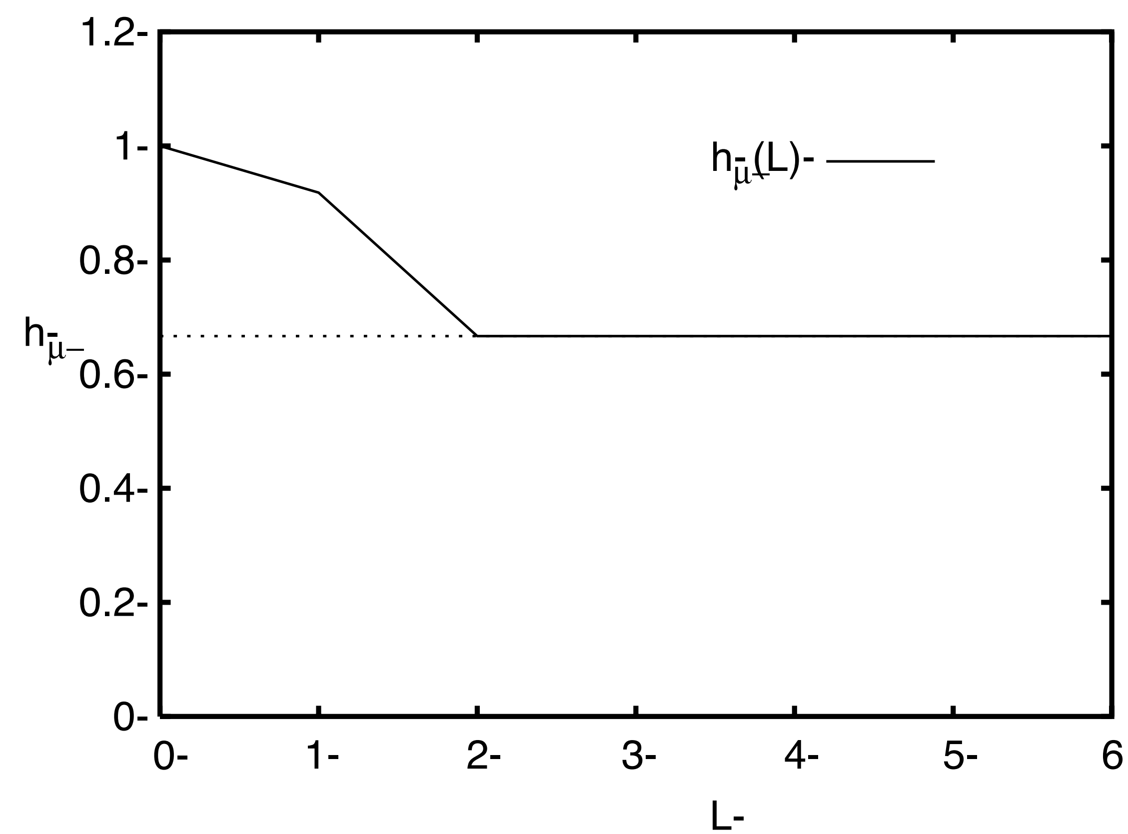
$$\mathbf{E} \approx 0.2516 \text{ bits}$$



R-Block Process:

$$\mathbf{E} = H(R) - R \cdot h_\mu$$

(Specifically, Spin-Block Process)



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Examples of Excess Entropy:

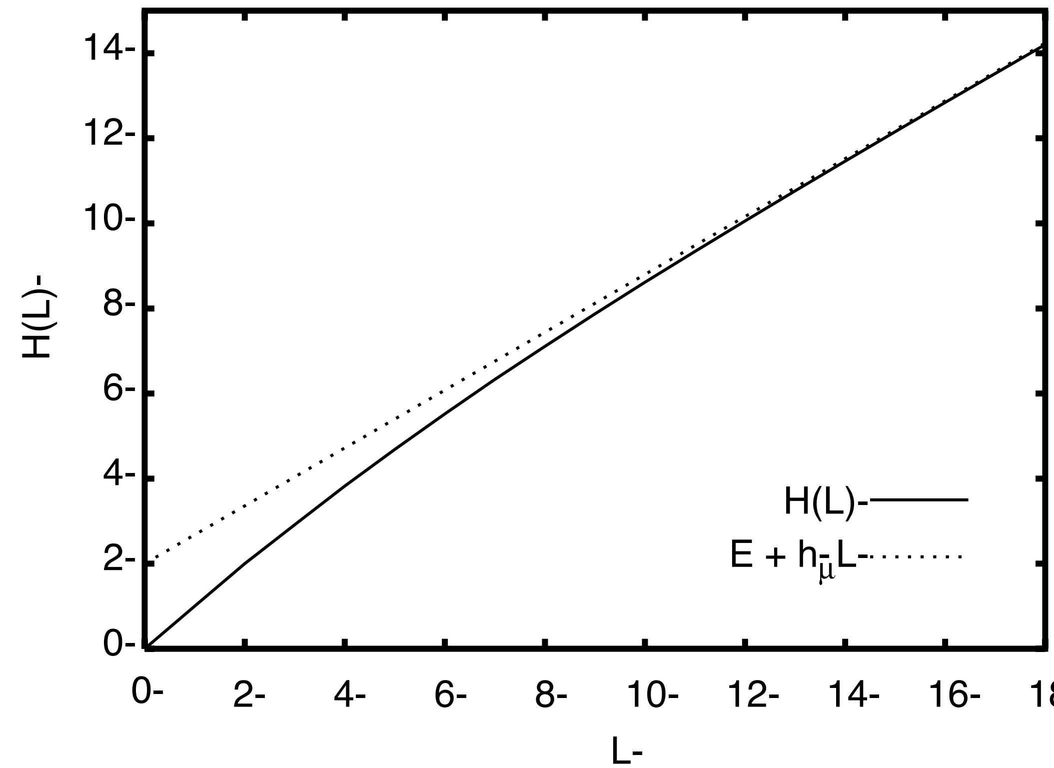
Finitary Processes: Exponential entropy convergence

Random-Random XOR (RRXOR) Process:

$$S_t = S_{t-1} \text{ XOR } S_{t-2}$$

$$h_\mu = \tfrac{2}{3} \text{ bits per symbol}$$

$$\mathbf{E} \approx 2.252 \text{ bits}$$

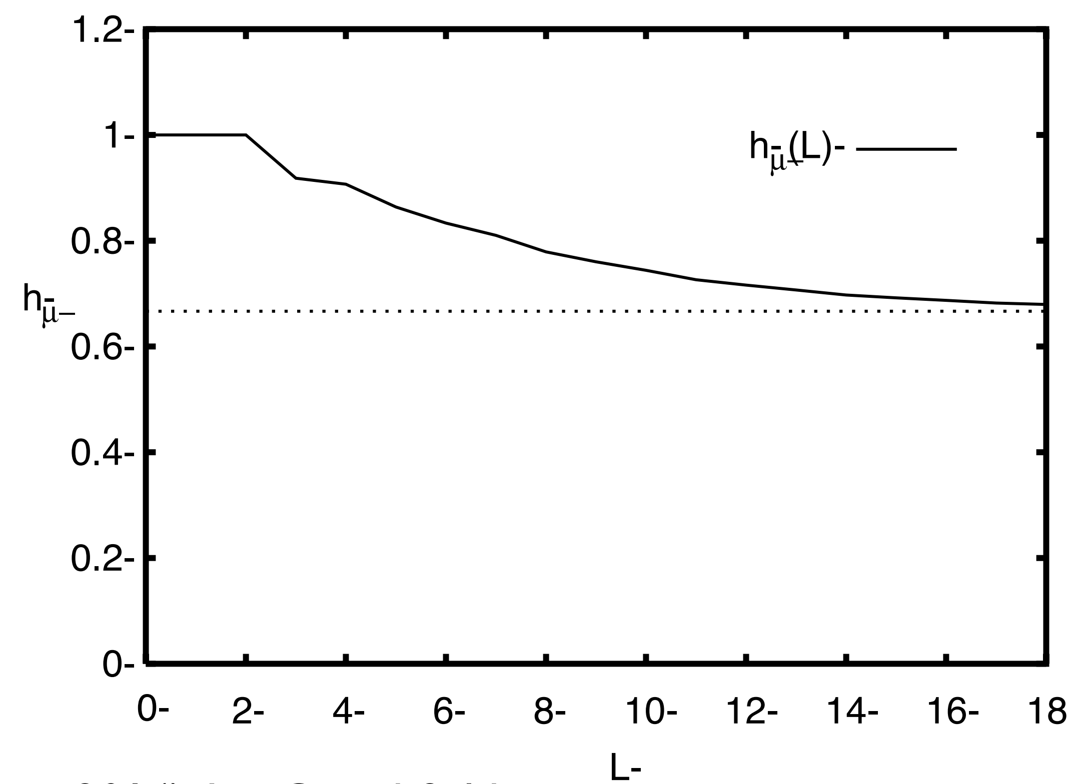


Finitary processes: Exponential convergence:

$$h_\mu(L) - h_\mu \approx 2^{-\gamma L}$$

$$\mathbf{E} = \frac{H(1) - h_\mu}{1 - 2^{-\gamma}}$$

$$\gamma \approx 0.30$$



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Examples of Excess Entropy:

Infinitary Processes:

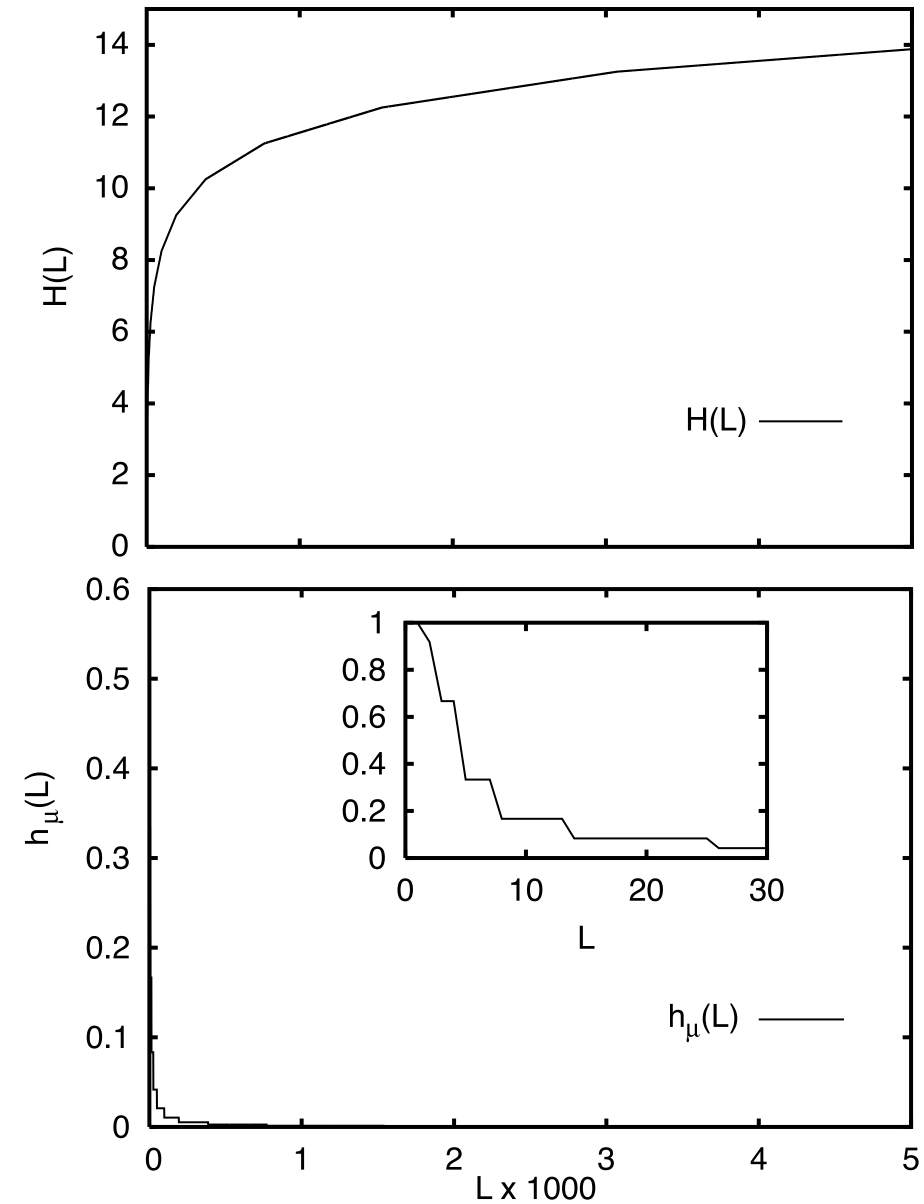
$$\mathbf{E} \to \infty$$

Excess entropy can diverge:

- Slow entropy convergence
- Long-range correlations
  - (e.g., at phase transitions)

Morse-Thue Process:

- A context-free language
- From Logistic map at onset of chaos



$$h_\mu = 0 \text{ bits per symbol}$$

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Reading for next lecture:

CMR article *RURO*.